Available online: May 25, 2020

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 981–992 (2020)
DOI: 10.31801/cfsuasmas.450209
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr

ON THE STABILITY ANALYSIS OF THE TIME-FRACTIONAL VARIABLE ORDER KLEIN-GORDON EQUATION AND SOME NUMERICAL SIMULATIONS

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ABSTRACT. In this paper, the Klein - Gordon equation is generalized using the concept of the variational order derivative. We try to construct the Crank-Nicholson scheme for numerical solutions of the modified Klein- Gordon equation. Stability analysis of the Crank-Nicholson scheme is examined and analyzed to prove the proposed method is stable for solving the time-fractional variable order Klein- Gordon equation. Numerical examples are also given for illustration.

1. Introduction

In recent years, fractional calculus and especially fractional differential equations (FDEs) have been extensively used for many different fields of mathematical physics such as relaxation processes,control theory of dynamical systems, viscoelasticity, diffusion and so on [1–5]. The main reason why they are so important is that a realistic modeling of many physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional derivatives. Besides, quite a number of different methods have been enhanced to analyze many different types of fractional differential equations for showing the importance of the fractional calculus [6–11]. On the other hand, stability analysis of fractional differential equations has attracted much attention over the past decade. Atangana has analyzed the stability of numerical solutions for many different types of FDEs such as groundwater flow equation [12], Schrödinger equation [13] and telegraph equation [14]. In [15], Zhang et. al. have examined the stability of FDEs, including linear FDEs, nonlinear FDEs and the FDEs with time-delay.

Received by the editors: August 01, 2018; Accepted: May 21, 2020.

2010 *Mathematics Subject Classification.* 81Q05, 26A33, 65N12.

Key words and phrases. Klein-Gordon equation, fractional variable order derivative, Crank-Nicholson scheme, stability.

As it is well known, partial differential equations are encountered frequently in many fields of applied physics [16–23]. One of them is Klein - Gordon equation which models many problems in quantum mechanics, condensed matter physics, etc. A Josephson junction, the motion of rigid pendula attached to a stretched wire can be described by sine Klein-Gordon equation and a non-local version of them are properly modeled by the fractional version of them [24]. In [25], Sweilam et al. has constructed a new and effective numerical scheme, namely weighted average nonstandard finite difference method, for analyzing the time variable-order fractional of nonlinear Klein-Gordon equation and so on.

In this paper, we investigate the stability of the linear time-fractional variable order Klein-Gordon equation:

$$D_{tt}^{\alpha(x,t)} y(x,t) - y_{xx}(x,t) + \mu y(x,t) = 0,\ 1 < \alpha(x,t) \le 2, \mu > 0, \tag{1}$$

with the conditions

$$y(x,0) = \delta(x), y_t(x,0) = 0;\ 0 \le t \le T, 0 \le x \le L \tag{2}$$

where $\delta(x)$ is a real-valued continuous function.

2. Some basic information for the variable order fractional derivative

In this section, we give some basic definitions that we need for our analysis. For much more details about fractional analysis we refer to the books and papers in [26–28].

Definition 2.1. Let $0 < \alpha(x,t) < 1$ for all $(x,t) \in [a,b]$ and $f \in L_1[a,b]$. Then

$$_aI_t^{\alpha(.,.)}\left(f(t)\right) = \int_a^t \frac{1}{\Gamma\left[\alpha(t,x)\right]} (t-x)^{\alpha(t,x)-1} f(x)dx\ (t > a) \tag{3}$$

and

$$_bI_t^{\alpha(.,.)}\left(f(t)\right) = \int_t^b \frac{1}{\Gamma\left[\alpha(t,x)\right]} (x-t)^{\alpha(x,t)-1} f(x)dx\ (t > b) \tag{4}$$

are called the left and right Riemann-Liouville integral of variable fractional order $\alpha(.,.)$ respectively.

Definition 2.2. Let $aI_t^{1-\alpha(.,.)} f \in C[a,b]$ and $0 < \alpha(x,t) < 1$ for all $(x,t) \in [a,b]$. Then

$$_aD_t^{\alpha(.,.)}\left(f(t)\right) = \frac{d}{dt}\int_a^t \frac{1}{\Gamma\left[1-\alpha(t,x)\right]} (t-x)^{-\alpha(t,x)} f(x)dx\ (t > a) \tag{5}$$

and

$$ {}_{b}D_t^{\alpha(.,.)}\left(f(t)\right) = \frac{d}{dt}\int_t^b \frac{1}{\Gamma\left[1-\alpha(x,t)\right]}(x-t)^{-\alpha(t,x)} f(x)dx \ (t<b) \tag{6} $$

are called the left and right Riemann-Liouville derivative of variable fractional order $\alpha(.,.)$ respectively.

Definition 2.3. Let f be a real valued differentiable function and $\alpha(x) \in C(0,1]$. Then the Caputo variable order differential operator is given by

$$ D_0^{\alpha(x)}\left(f(x)\right) = \frac{1}{\Gamma\left[1-\alpha(x)\right]}\int_0^x \frac{df(t)}{dt}(x-t)^{-\alpha(t)}dt. \tag{7} $$

3. Crank-Nicholson Scheme for numerical solutions

The numbers of the works for numerical solutions of different types of fractional differential equations have begun to increase considerably in recent years. A few of the most important ones of them can be found in [13, 14, 29–31].

In this section, we construct the Crank-Nicholson scheme for the fractional Klein-Gordon equation by taking $x_l = lh, t_j = j\tau, Mh = L, N\tau = T, 0 \le l \le M, 0 \le j \le N$ where M, N are grid points, h, τ are step size and time respectively. Under these assumptions, Crank-Nicholson scheme can be presented by giving the following discretizations:

$$ y = \frac{1}{2}\left(y(x_l, t_{j+1}) + y(x_l, t_j)\right) \tag{8} $$

$$ \begin{aligned} y_{xx} = \frac{\partial^2 y}{\partial x^2} = {} & \frac{1}{2}\left(\frac{y(x_{l+1}, t_{j+1}) - 2y(x_l, t_{j+1}) + y(x_{l-1}, t_{j+1})}{h^2}\right) + \\ & \frac{1}{2}\left(\frac{y(x_{l+1}, t_j) - 2y(x_l, t_j) + y(x_{l-1}, t_j)}{h^2}\right) + O(h^2) \end{aligned} \tag{9} $$

$$ \begin{aligned} & D_{tt}^{\alpha(x,t)} y = \frac{\partial^{{\alpha_l}^{j+1}} y(x_l, t_{j+1})}{\partial t^{{\alpha_l}^{j+1}}} = \frac{\tau^{-\alpha^{j+1}}}{\Gamma(2-\alpha_l^{j+1})} \times \\ & \left[\begin{array}{l} y(x_l, t_{j+1}) - y(x_l, t_j) + \\ \sum\limits_{n=1}^{j}\left(y(x_l, t_{j-n+1}) - y(x_l, t_{j-n})\right)\left((n+1)^{(1-\alpha_l^{j+1})} - n^{(1-\alpha_l^{j+1})}\right) \end{array}\right] \end{aligned} \tag{10} $$

Substituting (8), (9),(10) into the fractional Klein-Gordon equation (1) yields

$$\frac{\tau^{-\alpha^{j+1}}}{\Gamma(2-\alpha_l^{j+1})}\left[\begin{array}{l} y(x_l,t_{j+1})-y(x_l,t_j)+ \\ \sum_{n=1}^{j}\left(y(x_l,t_{j-n+1})-y(x_l,t_{j-n})\right)\left((n+1)^{(1-\alpha_l^{j+1})}-n^{(1-\alpha_l^{j+1})}\right) \end{array}\right]$$
$$-\left(\begin{array}{l} \frac{1}{2}\left(\frac{y(x_{l+1},t_{j+1})-2y(x_l,t_{j+1})+y(x_{l-1},t_{j+1})}{h^2}\right)+ \\ \frac{1}{2}\left(\frac{y(x_{l+1},t_j)-2y(x_l,t_j)+y(x_{l-1},t_j)}{h^2}\right) \end{array}\right)$$
$$+\mu\left(\frac{1}{2}\left(y(x_l,t_{j+1})+y(x_l,t_j)\right)\right)=0 \tag{11}$$

Multiplying both sides of (11) with

$$\frac{\Gamma(2-\alpha_l^{j+1})}{\tau^{-\alpha^{j+1}}}=\tau^{\alpha^{j+1}}\Gamma(2-\alpha_l^{j+1})$$

we get

$$\begin{array}{l} y(x_l,t_{j+1})-y(x_l,t_j)+ \\ \sum_{n=1}^{j}\left(y(x_l,t_{j-n+1})-y(x_l,t_{j-n})\right)\left((n+1)^{(1-\alpha_l^{j+1})}-n^{(1-\alpha_l^{j+1})}\right) \\ -\frac{\tau^{\alpha^{j+1}}\Gamma(2-\alpha_l^{j+1})}{2h^2}\left(\begin{array}{l} y(x_{l+1},t_{j+1})-2y(x_l,t_{j+1})+y(x_{l-1},t_{j+1})+ \\ y(x_{l+1},t_j)-2y(x_l,t_j)+y(x_{l-1},t_j) \end{array}\right) \\ +\frac{\mu\tau^{\alpha^{j+1}}\Gamma(2-\alpha_l^{j+1})}{2}\left(y(x_l,t_{j+1})+y(x_l,t_j)\right)=0 \end{array} \tag{12}$$

and by making the following change of variables

$$\begin{array}{l} y(x_l,t_j)=y_l^j,\quad R_l^{j+1}=\frac{\tau^{\alpha^{j+1}}\Gamma(2-\alpha_l^{j+1})}{2h^2},\quad S_l^{j+1}=\frac{\mu\tau^{\alpha^{j+1}}\Gamma(2-\alpha_l^{j+1})}{2} \\ c_n^{l,j+1}=(n+1)^{(1-\alpha_l^{j+1})}-n^{(1-\alpha_l^{j+1})},\quad d_n^{l,j+1}=c_{n-1}^{l,j+1}-c_n^{l,j+1} \end{array} \tag{13}$$

Eq. (11) becomes

$$\begin{array}{l} R_l^{j+1}\left(\ y_{l+1}^{j+1}-2y_l^{j+1}+y_{l-1}^{j+1}+y_{l+1}^{j}-2y_l^{j}+y_{l-1}^{j}\ \right)- \\ \sum_{n=1}^{j}\left[y_l^{j-n+1}-y_l^{j-n}\right]c_n^{l,j+1}+S_l^{j+1}\left(y_l^{j+1}+y_l^{j}\right)+y_l^{j+1}-y_l^{j}=0. \end{array} \tag{14}$$

4. Stability analysis for Crank-Nicholson scheme

Stability analysis is a very important concept in solving many types of linear or nonlinear differential equations [32–34]. In order to examine the stability analysis of the Crank-Nicholson scheme defined above, we now take that $\varepsilon_l^j = y_l^j - Y_l^j$ where Y_l^j is the approximate numerical solution at the point (x_l, t_j) and

$$\varepsilon^j = \left[\varepsilon_1^j, \varepsilon_2^j, ..., \varepsilon_{M-1}^j\right]^T \tag{15}$$

with

$$\varepsilon^j(x) = \begin{cases} \varepsilon_l^j & \text{if} \quad x_l - h/2 < x \leq x_l + h/2, l = 1, 2, ..., M-1 \\ 0 & \text{if} \quad L - h/2 < x \leq L \end{cases} \tag{16}$$

for $l = 1, 2, ..., M-1$, $j = 1, 2, ..., N$. Thereby, one can use the Fourier series to state the function $\varepsilon^j(x)$ as:

$$\varepsilon^j(x) = \sum_{m=-\infty}^{m=\infty} \delta_m(m) exp\left[2i\pi mj/L\right] \tag{17}$$

where

$$\delta_j(x) = \frac{1}{L}\int_0^L \rho^j exp\left[2i\pi mx/L\right] dx. \tag{18}$$

Before going through a detailed analysis, we give the following remarks which will be necessary for stability conditions.

Remarks 4.1. One can set up the following properties for all $l = 1, 2, .., M-1$.

$$\begin{aligned} &i.\ R_l^{j+1}, S_l^{j+1} > 0 \\ &ii.\ 0 \leq d_n^{l,j} \leq d_{n-1}^{l,j} \\ &iii.\ 0 \leq c_n^{l,j} \leq 1, \sum_{n=0}^{j-1} c_{n+1}^{l,j+1} = 1 - d_n^{l,j+1}. \end{aligned} \tag{19}$$

Using the previous notations, one can present the error done while applying the Crank-Nicholson scheme to solve the given fractional Klein-Gordon equation (1) as:

$$\begin{aligned} &R_l^{j+1}\left(\ \varepsilon_{l+1}^{j+1} - 2\varepsilon_l^{j+1} + \varepsilon_{l-1}^{j+1} + \varepsilon_{l+1}^j - 2\varepsilon_l^j + \varepsilon_{l-1}^j\ \right) - \\ &\sum_{n=1}^{j}\left[\varepsilon_l^{j-n+1} - \varepsilon_l^{j-n}\right] c_n^{l,j+1} + S_l^{j+1}\left(\varepsilon_l^{j+1} + \varepsilon_l^j\right) + \varepsilon_l^{j+1} - \varepsilon_l^j. \end{aligned} \tag{20}$$

In order to show the equation (20) more briefly, the term ε_l^j can be represented in the delta-exponential form as:

$$\varepsilon_l^j = \delta_j exp\left[i\theta lj\right]. \tag{21}$$

where θ represents a real spatial wave number. Using (21) for $j=0$, we get

$$R_l^1\left(\ \varepsilon_{l+1}^1-2\varepsilon_l^1+\varepsilon_{l-1}^1+\varepsilon_{l+1}^0-2\varepsilon_l^0+\varepsilon_{l-1}^0\ \right)+ \\ \sum_{n=1}^{0}\left[\varepsilon_l^{1-n}-\varepsilon_l^{-n}\right]c_n^{l,1}+S_l^1\left(\varepsilon_l^1+\varepsilon_l\right)+\varepsilon_l^1-\varepsilon_l^0=0. \tag{22}$$

Eq. (22) can be arranged as:

$$\delta_1=\delta_0\frac{1+4R_l^1\sin^2\left(\frac{h\theta}{2}\right)-2S_l^1\sin^2\left(\frac{h\theta}{2}\right)}{1+4R_l^1\sin^2\left(\frac{h\theta}{2}\right)+2S_l^1\sin^2\left(\frac{h\theta}{2}\right)} \tag{23}$$

and one can similarly obtain

$$\delta_{j+1}=\frac{\delta_j\left(1+4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)-2S_l^{k+1}sin^2\left(\frac{h\theta}{2}\right)\right)-\sum_{n=0}^{j-1}d_{n+1}^{1,j+1}\delta_{j-n}+d_j^{1,j+1}\delta_0}{1+4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)+2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)} \tag{24}$$

for $j=0,1,2,....$ We must now prove that the inequality $|\delta_j|\leq|\delta_0|$ holds for all $j=1,2,...$ to accomplish the proof of the stability of numerical solutions.It is easy to see that the inequality is true for $j=1$, because

$$|\delta_1|=|\delta_0|\left|\frac{1+4R_l^1\sin^2\left(\frac{h\theta}{2}\right)-2S_l^1\sin^2\left(\frac{h\theta}{2}\right)}{1+4R_l^1\sin^2\left(\frac{h\theta}{2}\right)+2S_l^1\sin^2\left(\frac{h\theta}{2}\right)}\right|\leq \\ |\delta_0|\left|\frac{1+4R_l^1\sin^2\left(\frac{h\theta}{2}\right)+2S_l^1\sin^2\left(\frac{h\theta}{2}\right)}{1+4R_l^1\sin^2\left(\frac{h\theta}{2}\right)+2S_l^1\sin^2\left(\frac{h\theta}{2}\right)}\right|=|\delta_0|. \tag{25}$$

On the basis of induction, we now suppose that

$$|\delta_{j+1}|=\left|\frac{\delta_j\left(1+4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)-2S_l^{k+1}sin^2\left(\frac{h\theta}{2}\right)\right)-\sum_{n=0}^{j-1}d_{n+1}^{1,j+1}\delta_{j-n}+d_j^{1,j+1}\delta_0}{1+4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)+2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)}\right| \tag{26}$$

for $m=2,3,...j$. Implementing the triangle inequality, the equality (26) turns into

$$|\delta_{j+1}|\leq\frac{|\delta_j|\left(\left|1+4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)-2S_l^{k+1}sin^2\left(\frac{h\theta}{2}\right)\right|\right)+\sum_{n=0}^{j-1}|d_{n+1}^{1,j+1}||\delta_{j-n}|+|d_j^{1,j+1}\delta_0|}{\left|1+4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)+2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right|}. \tag{27}$$

Using the induction hypothesis, we get

$$|\delta_{j+1}| \leq |\delta_0| \left[\frac{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) - 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right| + \sum_{n=0}^{j-1} |d_{n+1}^{1,j+1}| + |d_j^{1,j+1}|}{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) + 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right|} \right]. \tag{28}$$

By taking advantage of Remark 1, we finally obtain the inequality

$$\begin{aligned} |\delta_{j+1}| &\leq |\delta_0| \left[\frac{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) - 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right|}{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) + 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right|} \right] \\ &\leq |\delta_0| \left[\frac{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) + 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right|}{\left|1 + 4R_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right) + 2S_l^{k+1}\sin^2\left(\frac{h\theta}{2}\right)\right|} \right] = |\delta_0| \end{aligned} \tag{29}$$

thus,

$$|\delta_{j+1}| \leq |\delta_0|,$$

and the proof is completed.

5. Numerical examples

In this section, we give some numerical simulations for the approximate solution of the time-fractional variable order Klein-Gordon equation.

Example 1. Consider the problem (1) with $\mu = 0.9$, $\alpha(x,t) = 0.04tanh(x^3+t) - sin^2(5x^4t - 9x^2)$ and $\delta(x) = 0.08\cos(x^3)$. The error surface figures of approximate solutions are depicted for different N's and for $h = 0.0002$. As can be seen from the figures 1 and 2, the larger the N, the smaller the error.

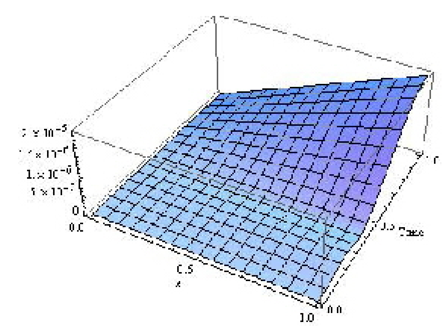

Figure 1. The error surface figures for $N = 40$

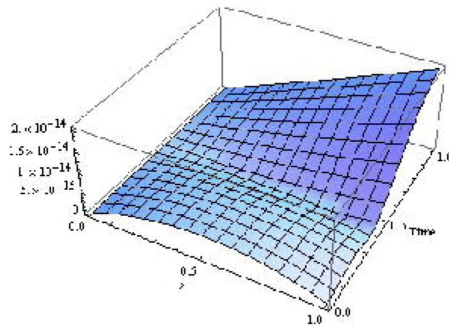

FIGURE 2. The error surface figures for $N = 80$

Example 2. As a second example, let us consider the problem (1) with $\mu = 0.8$, $\alpha(x, t) = 2 - sin^2(x^5 t + t^7)$ and $\delta(x) = x + \sec(x^{0.7})$. The error surface figures of approximate solutions are displayed for different N's and for $h = 0.00012$. Again, it is clear from the figures 3 and 4, we have smaller errors for the larger the N.

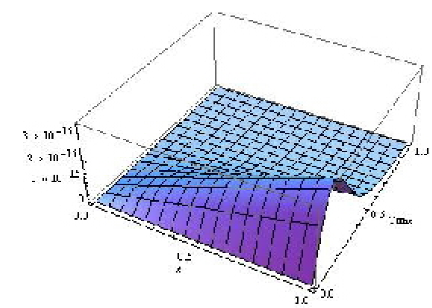

FIGURE 3. The error surface figures for $N = 80$

Example 3. As a final example, let us now consider the problem (1) with $\mu = 0.5$, $\alpha(x, t) = 1 - cos^2(x + t^3)$ and $\delta(x) = \sin(x)$. Figures of the approximate solutions are sketched for different N's and for $h = 0.0005$. A slight difference between these solutions can be seen from the simulations from Fig. 5 to 8 for $N = 10$ to $N = 70$. In addition to that, the error surface figure of approximate solution for $N = 80$ is demonstrated in Figure 9.

6. RESULTS AND DISCUSSION

We have modified the time-fractional variable order Klein-Gordon equation to analyze the concept of the variable order derivative. We apply the Crank-Nicholson method to solve the new modified equation numerically. Stability of this method is

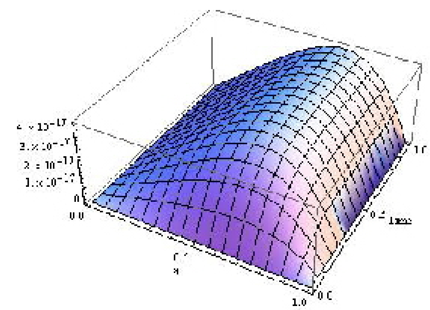

FIGURE 4. The error surface figures for $N = 85$

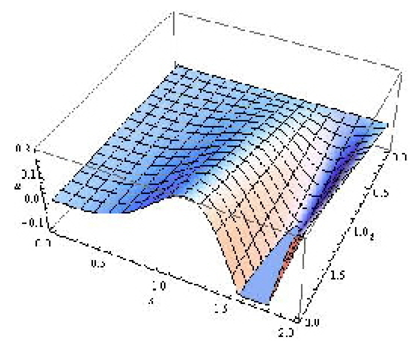

FIGURE 5. Numerical solution to problem (1) for $N = 10$

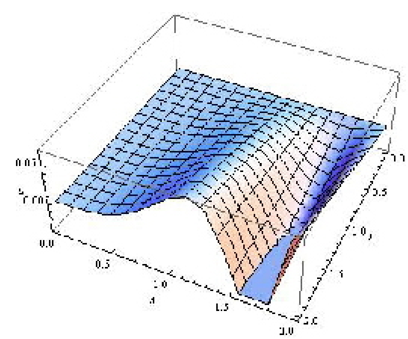

FIGURE 6. Numerical solution to problem (1) for $N = 30$

studied and reached by proving some inequalities. Some numerical examples have been also given for illustration. It can be concluded that Crank-Nicholson method

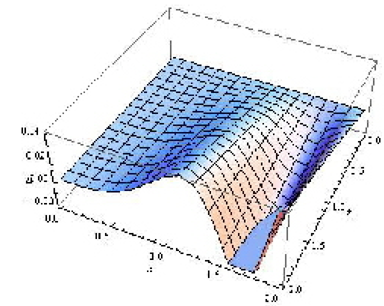

FIGURE 7. Numerical solution to problem (1) for $N = 50$

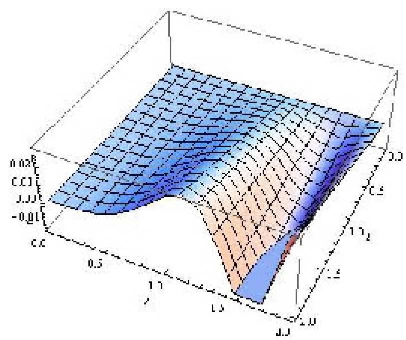

FIGURE 8. Numerical solution to problem (1) for $N = 70$

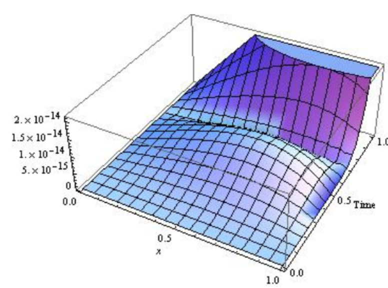

FIGURE 9. The error surface figures for $N = 80$

can be safely implemented to solve the time-fractional variable order Klein-Gordon equation.

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