Universal Journal of Mathematics and Applications, 4 (1) (2021) 9-25
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.751868

Generalized Rayleigh-Quotient Formulas for the Real Parts, Imaginary Parts, and Moduli of the Eigenvalues of General Matrices

Ludwig Kohaupt

Department of Mathematics, Beuth University of Technology Berlin, Berlin, Germany

Article Info

Keywords: *Asymptotic stability of dynamical systems, Circular damped eigenfrequencies, Moduli of eigenvalues, Rayleigh quotient, Real and imaginary parts of eigenvalues, Weighted norm*
2010 AMS: *11E39, 15A18, 15B57, 15B99, 65F35, 65J05*
Received: *12 June 2020*
Accepted: *13 January 2021*
Available online: *22 March 2021*

Abstract

In the present paper, generalized Rayleigh-quotient formulas for the real parts, imaginary parts, and moduli of the eigenvalues of general (not necessarily diagonalizable) matrices are derived by using quotients of the form $(Au,v)/(u,v)$ instead of $(Au,u)/(u,u)$. These formulas are new and correspond to similar formulas for diagonalizable matrices obtained recently. Numerical examples underpin the theoretical findings. We point out that, in the case of general matrices, the principal vectors of largest stage of matrix A^* take over the role of the eigenvectors in the case of diagonalizable matrices. So, even though the formulas in both cases look very similar, the result is somehow unexpected and surprising.

1. Introduction

For self-adjoint matrices, there are formulas for the eigenvalues in the form of generalized Rayleigh quotients; more precisely, max-, min-, min-max-, and max-min-formulas have been derived by the author in [8].
Recently, corresponding formulas could be carried over to formulas for the real parts, imaginary parts, and moduli of diagonalizable matrices in [10].
The aim of the present paper is to extend these results to general matrices.
We mention also that the presentation of this paper parallels that of [8] and [10]. So, similarities in the formulation do not happen by accident, but are intended in order to underline the similarities. As a consequence, many verbatim passages in the formulations are taken from there.
As has already been said in [9], first, the obtained formulas are of interest on their own in Linear Algebra. Second, these are also of potential interest, for example, in the theory of linear dynamical systems. The reason for this is as follows. The real parts of the eigenvalues multiplied by the time are equal to the arguments of the exponential functions that describe the decay behavior of the solution (see, e.g., [7, Section 7.1, p.2011, Formulas (89), (90)]). Further, the system is asymptotically stable if the real parts of all eigenvalues are negative. Moreover, when the eigenvalues are pairwise conjugate-complex, then the moduli of the imaginary parts are the circular damped eigenfrequencies of the system (see, e.g., [7, Section 7.1, p. 2011, (89)]).
The paper is structured as follows.
In Sections 2 - 4, the new generalized Rayleigh-quotient formulas for the real parts, imaginary parts, and moduli for general matrices are stated, as the case may be. In Section 5, the special case of general matrices with real eigenvalues is treated. Section 6 contains an application and Section 7 the definitions of new generalized numerical ranges. In Sections 8 and 9, numerical examples are presented that underpin the obtained findings. In the first example, matrix A is taken as the non-diagonalizable system matrix of a linear dynamical problem. In the second example, we choose a non-diagonalizable matrix with real eigenvalues. Finally, Section 10 contains the conclusion. The References are restricted to those that are cited in this paper augmented by those used in [8] and [10], the latter being [2], [3], [12], [13], and [14].

Email address and ORCID number: **kohaupt@bht-berlin.de, 0000-0003-1781-0600, (L. Kohaupt)**

2. Generalized Rayleigh-Quotient Formulas for the Real Parts of the Eigenvalues of a General Matrix

In this section, we want to derive formulas for the representation of the real parts of the eigenvalues of a general matrix $A \in \mathbb{C}^{n\times n}$ by Rayleigh quotients that generalize existing ones. More precisely, max-, min-, min-max-, and max-min-representations are obtained in the form of more general Rayleigh quotients corresponding to associated formulas for the eigenvalues of diagonalizable matrices assembled in [10]. The difference to the results obtained in [9] is that here we use the scalar product $(\cdot,\cdot)$ in $\mathbb{C}^n$ instead of a weighted scalar product $(\cdot,\cdot)_R$.

First, we formulate the following conditions $(C1')$ - $(C4')$:

$(C1')$ $A \in \mathbb{C}^{n\times n}$
$(C2')$ $\lambda_i,\ i=1,\cdots,r$ are the eigenvalues of A corresponding to the Jordan blocks $J_i(\lambda_i) \in \mathbb{C}^{m_i\times m_i},\ i=1,\cdots,r$ with the chains of principal vectors $p_1^{(i)},\cdots,p_{m_i}^{(i)},\ i=1,\cdots,r$
$(C3')$ $u_1^{(i)^*},\cdots,u_{m_i}^{(i)^*},\ i=1,\cdots,r$ are the principal vectors of A^* corresponding to the eigenvalues $\overline{\lambda_i},\ i=1,\cdots,r$ of the Jordan blocks $J_i(\overline{\lambda_i}) \in \mathbb{C}^{m_i\times m_i},\ i=1,\cdots,r$
$(C4')$ $\lambda_i \neq \lambda_j,\ i\neq j,\ i,j=1,\cdots,r$

We mention that, even though condition $(C4')$ may be omitted (see [6, Theorem 4]), it is nevertheless useful here since it will turn out to be fulfilled in the numerical examples in Sections 8 and 9 and since the biorthogonal system in Theorem 2.1 can be constructed more easily than without this condition. One has the following theorem.

Theorem 2.1. (Biorthogonality relations for principal vectors)
Let the conditions $(C1')$-$(C4')$ be fulfilled. Then, the systems $\{p_1^{(1)},\cdots,p_{m_1}^{(1)};\cdots;p_1^{(r)},\cdots,p_{m_r}^{(r)}\}$ and $\{u_1^{(1)^*},\cdots,u_{m_1}^{(1)^*};\cdots;u_1^{(r)^*},\cdots,u_{m_r}^{(r)^*}\}$ can be constructed such that the following biorthogonality relations hold:

$$(p_k^{(i)},u_l^{(i)^*}) = \begin{cases} 1, & l = m_i-k+1 \\ 0, & l \neq m_i-k+1 \end{cases} \tag{2.1}$$

$k=1,\cdots,m_i,\ i=1,\cdots,r$ and

$$(p_k^{(i)},u_l^{(j)^*}) = 0,\ i\neq j, \tag{2.2}$$

$k=1,\cdots,m_i,\ l=1,\cdots,m_j,\ i,j=1,\cdots,r$.
So, with

$$v_l^{(i)^*} := u_{m_i-l+1}^{(i)}{}^*, \tag{2.3}$$

$l=1,\cdots,m_i,\ i=1,\cdots,r$ one has the biorthogonality relations

$$(p_k^{(i)},v_l^{(i)^*}) = \delta_{kl}, \tag{2.4}$$

$k,l=1,\cdots,m_i,\ i=1,\cdots,r$, and

$$(p_k^{(i)},v_l^{(j)^*}) = 0,\ i\neq j, \tag{2.5}$$

$k=1,\cdots,m_i,\ l=1,\cdots,m_j,\ i,j=1,\cdots,r$.

Proof. See proof of [5, Theorem 2] or [6, Theorem 4]. □

Next, we want to derive a relation corresponding to that of [10, Formula (12)]. This is done in the following Formula (2.19).
First, with the identity matrix E, we introduce the abbreviation

$$N_{\lambda_j(A)} := \{u\in\mathbb{C}^n \,|\, (A-\lambda_j(A)E)u=0\},\ j=1,\ldots,r \tag{2.6}$$

for the geometric eigenspaces so that

$$N_{\lambda_j(A)} := [p_1^{(j)}] = [p_j],\ j=1,\ldots,r.$$

Herewith, we define

$$N_{\sigma(A)} := \bigoplus_{j=1}^{r} N_{\lambda_j(A)}. \tag{2.7}$$

Further, we define the following subspaces of $\mathbb{C}^n$. For every $k=1,\cdots,r$, let

$$N_{p,k} := \left\{u\in\mathbb{C}^n \,|\, u=\sum_{j=1}^{k}\alpha_j p_j \text{ with } \alpha_j\in\mathbb{C},\ j=1,\cdots,k\right\} =: [p_1,\cdots,p_k] \tag{2.8}$$

and

$$N_{p,k,\mathbb{R}} := \left\{u\in\mathbb{C}^n \,|\, u=\sum_{j=1}^{k}\beta_j p_j \text{ with } \beta_j\in\mathbb{R},\ j=1,\cdots,k\right\} =: [p_1,\cdots,p_k]_{\mathbb{R}} \tag{2.9}$$

as well as

$$N_p := N_{p,r} := \left\{ u \in \mathbb{C}^n \,|\, u = \sum_{j=1}^{r} \alpha_j p_j \; with \; \alpha_j \in \mathbb{C}, \; j = 1, \cdots, r \right\} =: [p_1, \cdots, p_r] \tag{2.10}$$

and

$$N_{p,\mathbb{R}} := N_{p,r,\mathbb{R}} := \left\{ u \in \mathbb{C}^n \,|\, u = \sum_{j=1}^{r} \beta_j p_j \; with \; \beta_j \in \mathbb{R}, \; j = 1, \cdots, r \right\} =: [p_1, \cdots, p_r]_{\mathbb{R}} \tag{2.11}$$

where $N_{p,\mathbb{R}}$ is apparently isomorphic to $\mathbb{R}^r$ and N_p is isomorphic to $\mathbb{C}^r$.
We mention that all these spaces (2.8) - (2.11) are subspaces of the geometric eigenspace $N_{\sigma(A)}$.
In [10], we have defined the further spaces $N_{u^*,k}$, $N_{u^*,k,\mathbb{R}}$, N_{u^*}, and $N_{u^*,\mathbb{R}}$ which are subspaces of the geometric eigenspace $N_{\sigma(A^*)}$. Here, however, we need different spaces. For this, we begin with the abbreviations

$$v_j^* := v_1^{(j)^*} = u_{m_j-1+1}^{(j)}{}^* = u_{m_j}^{(j)^*}, \tag{2.12}$$

$j = 1, \cdots, r$ that are principal vectors of stage m_j pertinent to the eigenvalue $\lambda_j(A^*) = \overline{\lambda_j(A)}, \; j = 1, \cdots, r$. Herewith, for every $k = 1, \cdots, r$, we define

$$N_{v^*,k} := \left\{ u \in \mathbb{C}^n \,|\, u = \sum_{j=1}^{k} \alpha_j v_j^* \; with \; \alpha_j \in \mathbb{C}, \; j = 1, \cdots, k \right\} =: [v_1^*, \cdots, v_k^*] \tag{2.13}$$

and

$$N_{v^*,k,\mathbb{R}} := \left\{ u \in \mathbb{C}^n \,|\, u = \sum_{j=1}^{k} \beta_j v_j^* \; with \; \beta_j \in \mathbb{R}, \; j = 1, \cdots, k \right\} =: [v_1^*, \cdots, v_k^*]_{\mathbb{R}} \tag{2.14}$$

as well as

$$N_{v^*} := N_{v^*,r} := \left\{ u \in \mathbb{C}^n \,|\, u = \sum_{j=1}^{r} \alpha_j v_j^* \; with \; \alpha_j \in \mathbb{C}, \; j = 1, \cdots, n \right\} =: [v_1^*, \cdots, v_r^*] \tag{2.15}$$

and

$$N_{v^*,\mathbb{R}} := N_{v^*,r,\mathbb{R}} := \left\{ u \in \mathbb{C}^n \,|\, u = \sum_{j=1}^{r} \beta_j v_j^* \; with \; \beta_j \in \mathbb{R}, \; j = 1, \cdots, r \right\} =: [v_1^*, \cdots, v_r^*]_{\mathbb{R}} \tag{2.16}$$

where $N_{v^*,\mathbb{R}}$ is apparently isomorphic to $\mathbb{R}^r$ and N_{v^*} is isomorphic to $\mathbb{C}^r$.
After these preparations, we are able to prove the following lemma.

Lemma 2.2. *Let the conditions (C1′)-(C4′) be fulfilled.*
Then, with the denotations of Theorem 2.1 and (2.12),

$$(A\,u, v) = \sum_{j=1}^{r} \lambda_j(A) \sum_{k=1}^{m_j} (u, v_k^{(j)^*})(p_k^{(j)}, v) + \sum_{j=1}^{r} \sum_{k=2}^{m_j} (u, v_k^{(j)^*})(p_{k-1}^{(j)}, v), \; u, v \in \mathbb{C}^n \tag{2.17}$$

leading to

$$(A\,u, v) = \sum_{j=1}^{r} \lambda_j(A)\,(u, v_j^*)(p_j, v), \; u \in N_{\sigma(A)}, \; v \in \mathbb{C}^n \tag{2.18}$$

and thus to

$$Re(A\,u, v) = \sum_{j=1}^{r} Re\lambda_j(A)\,(u, v_j^*)(p_j, v), \; u \in N_{p,\mathbb{R}}, \; v \in N_{v^*,\mathbb{R}}. \tag{2.19}$$

Proof. First, we prove (2.17). For this, let $u \in \mathbb{C}^n$. Then, with the denotations of Theorem 2.1,

$$u = \sum_{j=1}^{r} \sum_{k=1}^{m_j} (u, v_k^{(j)^*})\, p_k^{(j)} \tag{2.20}$$

leading to

$$\begin{aligned} Au &= \sum_{j=1}^{r} \sum_{k=1}^{m_j} (u, v_k^{(j)^*})\, A p_k^{(j)} \\ &= \sum_{j=1}^{r} \sum_{k=1}^{m_j} (u, v_k^{(j)^*})\, [\lambda_j p_k^{(j)} + p_{k-1}^{(j)}] \\ &= \sum_{j=1}^{r} \lambda_j \sum_{k=1}^{m_j} (u, v_k^{(j)^*})\, p_k^{(j)} + \sum_{j=1}^{r} \sum_{k=2}^{m_j} (u, v_k^{(j)^*})\, p_{k-1}^{(j)} \end{aligned} \tag{2.21}$$

since $p_0^{(j)} = 0,\ j = 1,\ldots,r$.
Further, for every $v \in \mathbb{C}^n$,

$$v = \sum_{l=1}^{r}\sum_{s=1}^{m_l}(v, p_s^{(l)})\, v_s^{(l)^*}. \tag{2.22}$$

This leads to

$$\begin{aligned}(Au,v) &= \sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u, v_k^{(j)^*})\,(p_k^{(j)}, v) + \sum_{j=1}^{r}\sum_{k=2}^{m_j}(u, v_k^{(j)^*})\,(p_{k-1}^{(j)}, v)\\ &= \sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u, v_k^{(j)^*})\sum_{l=1}^{r}\sum_{s=1}^{m_l}\overline{(v, p_s^{(l)})}\,\underbrace{(p_k^{(j)}, v_s^{(l)^*})}_{\delta_{lj}\delta_{sk}} + \sum_{j=1}^{r}\sum_{k=2}^{m_j}(u, v_k^{(j)^*})\sum_{l=1}^{r}\sum_{s=1}^{m_l}\overline{(v, p_s^{(l)})}\,\underbrace{(p_{k-1}^{(j)}, v_s^{(l)^*})}_{\delta_{s,k-1}\delta_{lj}}\end{aligned}$$

implying

$$(Au,v) = \sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u, v_k^{(j)^*})\,(p_k^{(j)}, v) + \sum_{j=1}^{r}\sum_{k=2}^{m_j}(u, v_k^{(j)^*})\,(p_{k-1}^{(j)}, v)$$

so that (2.17) follows.
Now, for $u \in N_{\sigma(A)}$, we have

$$(u, v_k^{(j)^*}) = 0,\ k = 2,\cdots,m_j$$

since $(p_s, v_k^{(j)^*}) = (p_1^{(s)}, v_k^{(j)^*}) = 0$ if $k \neq 1, ; s, j = 1,\cdots,r$. Thus, from (17), we deduce that

$$(Au,v) = \sum_{j=1}^{r}\lambda_j (u, v_1^{(j)^*})\,(p_1^{(j)}, v). \tag{2.23}$$

□

With the abbreviation (2.12), we obtain (2.18).
Relation (2.19) is a direct consequence of equation (2.18) since $(u, v_j^*) \in \mathbb{R}$, $u \in N_{p,\mathbb{R}}$ and $(p_j, v) \in \mathbb{R}$, $v \in N_{v^*,\mathbb{R}}$.
Next, as in [10], we define the vector spaces $M_{p,k,\mathbb{R}}$, namely:

$$\begin{aligned}&M_{p,1,\mathbb{R}} := N_{p,\mathbb{R}} = [p_1,\cdots,p_r]_{\mathbb{R}},\\ &M_{p,k,\mathbb{R}} := \{u \in N_{p,\mathbb{R}}\,|\,(u, u_j^*) = 0,\ j = 1,2,\cdots,k-1\},\ k = 2,\cdots,r.\end{aligned} \tag{2.24}$$

Instead of the spaces $M_{u^*,k,\mathbb{R}}$ in [10], we need the spaces $M_{v^*,k,\mathbb{R}}$, i.e.,

$$\begin{aligned}&M_{v^*,1,\mathbb{R}} := N_{v^*,\mathbb{R}} = [v_1^*,\cdots,v_r^*]_{\mathbb{R}},\\ &M_{v^*,k,\mathbb{R}} := \{u \in N_{v^*,\mathbb{R}}\,|\,(u, p_j) = 0,\ j = 1,2,\cdots,k-1\},\ k = 2,\cdots,r.\end{aligned} \tag{2.25}$$

The next lemma characterizes these spaces.

Lemma 2.3. *Let the conditions (C1′)-(C4′) be fulfilled as well as $\{p_1,\cdots,p_r\}$ be the eigenvectors of A and $\{v_1^*,\cdots,v_r^*\}$ be principal vectors of A^* defined by (2.12) with the property*

$$(p_i, v_j^*) = \delta_{ij},\ i,j = 1,\cdots,r.$$

Then,

$$M_{p,k,\mathbb{R}} = [p_k, p_{k+1},\cdots,p_n]_{\mathbb{R}},\ k = 1,\cdots,r \tag{2.26}$$

and

$$M_{v^*,k,\mathbb{R}} = [v_k^*, v_{k+1}^*,\cdots,v_r^*]_{\mathbb{R}},\ k = 1,\cdots,r. \tag{2.27}$$

Proof. The proof is similar to that of [10, Lemma 3]. □

Similarly to [10, (21)], we suppose that the eigenvalues $\lambda_1(A),\cdots,\lambda_r(A)$ of matrix A are arranged such that

$$Re\lambda_1(A) \geq Re\lambda_2(A) \geq \cdots \geq Re\lambda_r(A). \tag{2.28}$$

Further, let $u \in N_{p,\mathbb{R}}$ with $u = \sum_{k=1}^{r}\alpha_k p_k$ and $v \in N_{v^*,\mathbb{R}}$ with $v = \sum_{k=1}^{r}\beta_k v_k^*$. Then, due to Theorem 2.1, as in [10],

$$(u,v) = \sum_{k=1}^{r}\alpha_k\beta_k. \tag{2.29}$$

In order to facilitate the manner of speaking, we say that *the scalar product (u,v) of u and v is strongly positive* if $\alpha_k\beta_k \geq 0$, $k = 1,\cdots,r$ and $\sum_{k=1}^{r}\alpha_k\beta_k > 0$. For short, we write $(u,v) \gg 0$.

Remark 2.4. *One has* $\alpha_k = (u, v_k^*)$, $u \in N_{p,\mathbb{R}}$ *and* $\beta_k = (p_k, v)$, $v \in N_{v^*,\mathbb{R}}$ *for* $k = 1, \cdots, r$. *Therefore,* $(u,v) \gg 0$ *means* $(u, v_k^*)(p_k, v) \geq 0$, $k = 1, \cdots, r$ *and* $(u,v) = \sum_{k=1}^{r} (u, v_k^*)(p_k, v) > 0$.

Remark 2.5. *More generally, in the sequel, one could admit linear combinations* $u = \sum_{k=1}^{r} \alpha_k p_k$ *and* $v = \sum_{k=1}^{r} \beta_k v_k^*$ *with* $\alpha_k, \beta_k \in \mathbb{C}$ *such that* $\alpha_k \overline{\beta_k} = |\alpha_k \beta_k|$ *and* $\sum_{k=1}^{r} |\alpha_k \beta_k| > 0$. *For example, all elements* $\alpha_k, \beta_k \in \mathbb{C}$ *with* $\alpha_k = |\alpha_k| e^{i\varphi_k}$ *and* $\beta_k = |\beta_k| e^{i\varphi_k}$ *where* φ_k *is in* $0 \leq \varphi_k < 2\pi$ *for* $k = 1, \cdots, r$ *would be acceptable. But, we do not want to pursue this aspect in more detail.*

Comparing relation (2.19) with [10, (12)], it is clear that one can obtain similar generalized max-, min-, min-max-, and max-min-representations for the real parts, imaginary parts, and moduli as in the case of diagonalizable matrices. Therefore, we state them without proofs.

Theorem 2.6. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (2.28). Moreover, let the vector spaces* $M_{p,k,\mathbb{R}}$ *and* $M_{v^*,k,\mathbb{R}}$ *for* $k = 1, \cdots, n$ *be defined by (2.24), (2.25) or (2.26),(2.27).*
Then,

$$Re\lambda_k(A) = \max_{\substack{(u,v)\gg 0 \\ u \in M_{p,k,\mathbb{R}}, v \in M_{v^*,k,\mathbb{R}}}} \frac{Re(Au,v)}{(u,v)},\ k = 1, 2, \cdots, r. \tag{2.30}$$

The maximum is attained for $u = p_k$, $v = v_k^*$.

Theorem 2.7. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (2.28).*
Then, for every $j = 1, \cdots, r$ *and every subspace* $M_p \subset N_{p,\mathbb{R}}$ *and* $M_{v^*} \subset N_{v^*,\mathbb{R}}$ *with* $dim M_p = dim M_{v^*} = m = r + 1 - j$, *the following inequalities are valid:*

$$Re\lambda_j(A) \leq \max_{\substack{(u,v)\gg 0 \\ u \in M_p, v \in M_{v^*}}} \frac{Re(Au,v)}{(u,v)} \leq Re\lambda_1(A), \tag{2.31}$$

and the following representation formulas hold:

$$Re\lambda_j(A) = \min_{\substack{dim M_p = m \\ dim M_{v^*} = m}} \max_{\substack{(u,v)\gg 0 \\ u \in M_p, v \in M_{v^*}}} \frac{Re(Au,v)}{(u,v)}. \tag{2.32}$$

Remark 2.8. *From (2.31), it follows*

$$\frac{Re(Au,v)}{(u,v)} \leq \nu[A] = \max_{j=1,\cdots,r} Re\lambda_j(A),\ (u,v) \gg 0,\ u \in N_{p,\mathbb{R}},\ v \in N_{v^*,\mathbb{R}}.$$

Theorem 2.9. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (2.28). Moreover, let the vector spaces* $N_{p,k,\mathbb{R}}$ *and* $N_{v^*,k,\mathbb{R}}$ *for* $k = 1, \cdots, r$ *be defined by (2.9) and (2.14).*
Then,

$$Re\lambda_k(A) = \min_{\substack{(u,v)\gg 0 \\ u \in N_{p,k,\mathbb{R}}, v \in N_{v^*,k,\mathbb{R}}}} \frac{Re(Au,v)}{(u,v)},\ k = 1, 2, \cdots, r. \tag{2.33}$$

The minimum is attained for $u = p_k$, $v = v_k^*$.

Theorem 2.10. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (2.28).*
Then, for every $j = 1, \cdots, r$ *and all subspaces* $N_p \subset N_{p,\mathbb{R}}$ *and* $N_{v^*} \subset N_{v^*,\mathbb{R}}$ *with* $dim N_p = dim N_{v^*} = j$, *the following inequalities are valid:*

$$Re\lambda_r(A) \leq \min_{\substack{(u,v)\gg 0 \\ u \in N_p, v \in N_{v^*}}} \frac{Re(Au,v)}{(u,v)} \leq Re\lambda_j(A), \tag{2.34}$$

and the following representation formulas hold:

$$Re\lambda_j(A) = \max_{\substack{dim N_p = j \\ dim N_{v^*} = j}} \min_{\substack{(u,v)\gg 0 \\ u \in N_p, v \in N_{v^*}}} \frac{Re(Au,v)}{(u,v)}. \tag{2.35}$$

Remark 2.11. *From (2.34), it follows*

$$\frac{Re(Au,v)}{(u,v)} \geq -\nu[-A] = \min_{j=1,\cdots,r} Re\lambda_j(A),\ (u,v) \gg 0,\ u \in N_{p,\mathbb{R}},\ v \in N_{v^*,\mathbb{R}}.$$

3. Generalized Rayleigh-Quotient Formulas for the Imaginary Parts of the Eigenvalues of a General Matrix

In this section, we want to state formulas for the representation of the imaginary parts of the eigenvalues of a general matrix $A \in \mathbb{C}^{n\times n}$ by Rayleigh quotients corresponding to those for the real parts. More precisely, max-, min-, min-max-, and max-min-representations are obtained corresponding to those in Section 2.
First, we want to state a relation corresponding to that of (2.19).

Lemma 3.1. *Let the conditions (C1′)-(C4′) be fulfilled. Then, with the denotations of Theorem 2.1 and (2.12),*

$$Im(A\,u,v) = \sum_{j=1}^{r} Im\lambda_j(A)\,(u,v_j^*)\,(p_j,v),\; u\in N_{p,\mathbb{R}},\; v\in N_{v^*,\mathbb{R}}. \tag{3.1}$$

Proof. Equation (3.1) follows directly from Lemma 2.2, Formula (2.18). □

Similarly to (2.28), we suppose that the eigenvalues $\lambda_1(A),\cdots,\lambda_r(A)$ of matrix A are arranged such that

$$Im\lambda_1(A) \ge Im\lambda_2(A) \ge \cdots \ge Im\lambda_r(A). \tag{3.2}$$

Then, we have the following series of theorems.

Theorem 3.2. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (3.2). Moreover, let the vector spaces $M_{p,k,\mathbb{R}}$ and $M_{v^*,k,\mathbb{R}}$ for $k=1,\cdots,r$ be defined by (2.24), (2.25) or (2.26),(2.27).*
Then,

$$Im\lambda_k(A) = \max_{\substack{(u,v)\gg 0\\ u\in M_{p,k,\mathbb{R}},v\in M_{v^*,k,\mathbb{R}}}} \frac{Im(Au,v)}{(u,v)},\; k=1,2,\cdots,r. \tag{3.3}$$

The maximum is attained for $u=p_k,\; v=v_k^*$.

Theorem 3.3. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (3.2).*
Then, for every $j=1,\cdots,r$ and every subspace $M_p \subset N_{p,\mathbb{R}}$ and $M_{v^*}\subset N_{v^*,\mathbb{R}}$ with $dim M_p = dim M_{v^*} = m = r+1-j$, the following inequalities are valid:

$$Im\lambda_j(A) \le \max_{\substack{(u,v)\gg 0\\ u\in M_p,v\in M_{v^*}}} \frac{Im(Au,v)}{(u,v)} \le Im\lambda_1(A), \tag{3.4}$$

and the following representation formulas hold:

$$Im\lambda_j(A) = \min_{\substack{dim M_p=m\\ dim M_{v^*}=m}} \max_{\substack{(u,v)\gg 0\\ u\in M_p,v\in M_{v^*}}} \frac{Im(Au,v)}{(u,v)}. \tag{3.5}$$

Remark 3.4. *From (3.4), it follows*

$$\frac{Im(Au,v)}{(u,v)} \le \max_{j=1,\cdots,r} Im\lambda_j(A),\; (u,v)\gg 0,\; u\in N_{p,\mathbb{R}},\; v\in N_{v^*,\mathbb{R}}.$$

Theorem 3.5. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (3.2). Moreover, let the vector spaces $N_{p,k,\mathbb{R}}$ and $N_{v^*,k,\mathbb{R}}$ for $k=1,\cdots,r$ be defined by (2.9) and (2.14).*
Then,

$$Im\lambda_k(A) = \min_{\substack{(u,v)\gg 0\\ u\in N_{p,k,\mathbb{R}},v\in N_{v^*,k,\mathbb{R}}}} \frac{Im(Au,v)}{(u,v)},\; k=1,2,\cdots,r. \tag{3.6}$$

The minimum is attained for $u=p_k,\; v=v_k^*$.

Theorem 3.6. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (3.2).*
Then, for every $j=1,\cdots,r$ and all subspaces $N_p\subset N_{p,\mathbb{R}}$ and $N_{v^*}\subset N_{v^*,\mathbb{R}}$ with $dim N_p = dim N_{v^*} = j$, the following inequalities are valid:

$$Im\lambda_r(A) \le \min_{\substack{(u,v)\gg 0\\ u\in N_p,v\in N_{v^*}}} \frac{Im(Au,v)}{(u,v)} \le Im\lambda_j(A), \tag{3.7}$$

and the following representation formulas hold:

$$Im\lambda_j(A) = \max_{\substack{dim N_p=j\\ dim N_{v^*}=j}} \min_{\substack{(u,v)\gg 0\\ u\in N_p,v\in N_{v^*}}} \frac{Im(Au,v)}{(u,v)}. \tag{3.8}$$

Remark 3.7. *From (3.7), it follows*

$$\frac{Im(Au,v)}{(u,v)} \ge \min_{j=1,\cdots,r} Im\lambda_j(A),\; (u,v)\gg 0,\; u\in N_{p,\mathbb{R}},\; v\in N_{v^*,\mathbb{R}}.$$

4. Generalized Rayleigh-Quotient Formula for the Moduli of the Eigenvalues of a General Matrix

Whereas in Sections 2 and 3 max-, min-, min-max-, and max-min-representations with generalized Rayleigh quotients for general matrices could be obtained, it seems that, for the moduli of eigenvalues, only a max-representation is possible. Some arguments why this is probably the case were already given in [10, Section 4].
Now, we want to state the max-representation. For this, we suppose that the eigenvalues $\lambda_1(A),\cdots,\lambda_r(A)$ of $A\in\mathbb{C}^{n\times n}$ are arranged such that

$$|\lambda_1(A)|\geq|\lambda_2(A)|\geq\cdots\geq|\lambda_r(A)|. \tag{4.1}$$

Herewith, one has the following theorem.

Theorem 4.1. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (4.1). Moreover, let the vector spaces $M_{p,k,\mathbb{R}}$ and $M_{v^*,k,\mathbb{R}}$ for $k=1,\cdots,r$ be defined by (2.24), (2.25) or (2.26),(2.27).*
Then,

$$|\lambda_k(A)|=\max_{\substack{(u,v)\gg 0\\ u\in M_{p,k,\mathbb{R}},v\in M_{v^*,k,\mathbb{R}}}}\frac{|(Au,v)|}{(u,v)},\ k=1,2,\cdots,r. \tag{4.2}$$

The maximum is attained for $u=p_k,\ v=v_k^*$.

5. Generalized Rayleigh-Quotient Formulas for a General Matrix with Real Eigenvalues

In Section 4, we have observed that, for the moduli of the eigenvalues of a general matrix, one obtains only a max-representation with generalized Rayleigh quotients. However, for $A\in\mathbb{C}^{n\times n}$ with

$$\sigma(A)\subset\mathbb{R},$$

one gets generalized Rayleigh-quotient formulas for the eigenvalues themselves. And it goes without saying that these imply Rayleigh-quotient representations for the moduli if all eigenvalues are nonnegative such as $\lambda_1(A^*A),\cdots,\lambda_r(A^*A)$ (where $r=n$).
So, let $A\in\mathbb{C}^{n\times n}$ with spectrum $\sigma(A)\subset\mathbb{R}$. Further, let the eigenvalues be arranged according to

$$\lambda_1(A)\geq\lambda_2(A)\geq\cdots\geq\lambda_r(A). \tag{5.1}$$

Then, we obtain the following series of corollaries following from Theorems 2.7 - 2.10, as the case may be.

Corollary 5.1. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let $\sigma(A)\subset\mathbb{R}$, and let the eigenvalues of A be arranged according to (5.1). Moreover, let the vector spaces $M_{p,k,\mathbb{R}}$ and $M_{v^*,k,\mathbb{R}}$ for $k=1,\cdots,r$ be defined by (2.24), (2.25) or (2.26),(2.27).*
Then,

$$\lambda_k(A)=\max_{\substack{(u,v)\gg 0\\ u\in M_{p,k,\mathbb{R}},v\in M_{v^*,k,\mathbb{R}}}}\frac{(Au,v)}{(u,v)},\ k=1,2,\cdots,r. \tag{5.2}$$

The maximum is attained for $u=p_k,\ v=v_k^*$.

Corollary 5.2. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let $\sigma(A)\subset\mathbb{R}$, and let the eigenvalues of A be arranged according to (5.1).*
Then, for every $j=1,\cdots,r$ and every subspace $M_p\subset N_{p,\mathbb{R}}$ and $M_{v^*}\subset N_{v^*,\mathbb{R}}$ with $\dim M_p=\dim M_{v^*}=m=r+1-j$, the following inequalities are valid:

$$\lambda_j(A)\leq\max_{\substack{(u,v)\gg 0\\ u\in M_p,v\in M_{v^*}}}\frac{(Au,v)}{(u,v)}\leq\lambda_1(A), \tag{5.3}$$

and the following representation formulas hold:

$$\lambda_j(A)=\min_{\substack{\dim M_p=m\\ \dim M_{v^*}=m}}\max_{\substack{(u,v)\gg 0\\ u\in M_p,v\in M_{v^*}}}\frac{(Au,v)}{(u,v)}. \tag{5.4}$$

Remark 5.3. *From (5.3), it follows*

$$\frac{(Au,v)}{(u,v)}\leq\max_{j=1,\cdots,r}\lambda_j(A),\ (u,v)\gg 0,\ u\in N_{p,\mathbb{R}},\ v\in N_{v^*,\mathbb{R}}.$$

Corollary 5.4. *Let the conditions ($C1'$)-($C4'$) be fulfilled. Further, let $\sigma(A) \subset \mathbb{R}$, and let the eigenvalues of A be arranged according to (5.1). Moreover, let the vector spaces $N_{p,k,\mathbb{R}}$ and $N_{v^*,k,\mathbb{R}}$ for $k = 1, \cdots, r$ be defined by (2.9) and (2.14).*
Then,

$$\lambda_k(A) = \min_{\substack{(u,v)\gg 0 \\ u\in N_{p,k,\mathbb{R}}, v\in N_{v^*,k,\mathbb{R}}}} \frac{(Au,v)}{(u,v)}, \; k = 1,2,\cdots,r. \tag{5.5}$$

The minimum is attained for $u = p_k$, $v = u_k^*$.

Corollary 5.5. *Let the conditions ($C1'$)-($C4'$) be fulfilled. Further, let $\sigma(A) \subset \mathbb{R}$, and let the eigenvalues of A be arranged according to (5.1).*
Then, for every $j = 1, \cdots, r$ and all subspaces $N_p \subset N_{p,\mathbb{R}}$ and $N_{v^*} \subset N_{v^*,\mathbb{R}}$ with $\dim N_p = \dim N_{v^*} = j$, the following inequalities are valid:

$$\lambda_r(A) \le \min_{\substack{(u,v)\gg 0 \\ u\in N_p, v\in N_{v^*}}} \frac{(Au,v)}{(u,v)} \le \lambda_j(A), \tag{5.6}$$

and the following representation formulas hold:

$$\lambda_j(A) = \max_{\substack{\dim N_p = j \\ \dim N_{v^*} = j}} \; \min_{\substack{(u,v)\gg 0 \\ u\in N_p, v\in N_{v^*}}} \frac{(Au,v)}{(u,v)}. \tag{5.7}$$

Remark 5.6. *From (5.6), it follows*

$$\frac{(Au,v)}{(u,v)} \ge \min_{j=1,\cdots,r} \lambda_j(A), \; (u,v) \gg 0, \; u \in N_{p,\mathbb{R}}, \; v \in N_{v^*,\mathbb{R}}.$$

6. Application

In this section, an application of the obtained results is presented. More precisely, a new formula for $\rho(A)$ is stated; its derivation is similar to that of [8, (79)]. First, known formulas for this quantity are recapitulated.

Known formulas for the spectral radius of $A \in \mathbb{C}^{n\times n}$

One formula is given by

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}, \tag{6.1}$$

see [4, Chapter I, p.27], where in (6.1) the spectral radius $\rho(A)$ is independent of the used operator norm $\|\cdot\|$.
Another representation is

$$\rho(A) = \max_{j=1,\cdots,n} |\lambda_j(A)|, \tag{6.2}$$

cf. [4, Chapter I,(5.12), p.38].

New formula for the spectral radius of $A \in \mathbb{C}^{n\times n}$

Let the conditions ($C1'$)-($C4'$) be fulfilled. Then, from Theorem 4.1, as Application, we deduce the new formula

$$\rho(A) = \max_{\substack{(u,v)\gg 0 \\ u\in N_{p,\mathbb{R}}, v\in N_{v^*,\mathbb{R}}}} \frac{|(Au,v)|}{(u,v)}. \tag{6.3}$$

7. New Generalized Numerical Ranges

In this section, a series of known numerical ranges are recapitulated and new numerical ranges of a general matrix are defined.

Known numerical range of $A \in \mathbb{C}^{n\times n}$ with respect to the full space $\mathbb{C}^n$

According to [12, Section 5.4,(5)], the *numerical range* of $A \in \mathbb{C}^{n\times n}$ with respect to the full space $\mathbb{C}^n$ is defined by

$$W_{\mathbb{C}^n}(A) = \left\{ z \in \mathbb{C} \,|\, z = \frac{(Au,u)}{(u,u)}, 0 \ne u \in \mathbb{C}^n \right\}, \tag{7.1}$$

which is a convex subset of $\mathbb{C}$. Employing this definition to A^*A instead of A, we obtain

$$W_{\mathbb{C}^n}(A^*A) = \left\{ x \in \mathbb{R}_0^+ \,|\, x = \frac{(A^*Au,u)}{(u,u)} = \frac{(Au,Au)}{(u,u)}, 0 \ne u \in \mathbb{C}^n \right\}, \tag{7.2}$$

which is a convex subset of $\mathbb{R}_0^+$. One has

$$W_{\mathbb{C}^n}(A^*A) = \left[\min_{j=1,\cdots,n} \lambda_j(A^*A), \max_{j=1,\cdots,n} \lambda_j(A^*A)\right] = [\frac{1}{\|A^{-1}\|_2^2}, \|A\|_2^2] \tag{7.3}$$

where $\frac{1}{\|A^{-1}\|_2^2}$ has to be interpreted as zero if A is singular.

The following four definitions of generalized numerical ranges are new.

Generalized numerical range of $A \in \mathbb{C}^{n\times n}$ with respect to the subspaces $N_{p,\mathbb{R}}$ and $N_{v^*,\mathbb{R}}$

Let the conditions $(C1')$-$(C4')$ be fulfilled. Then, we define the *generalized numerical range* of A with respect to the subspaces $N_{p,\mathbb{R}}$ and $N_{v^*,\mathbb{R}}$ by

$$W_{N_{p,\mathbb{R}},N_{v^*,\mathbb{R}},gen.}(A) = \left\{ z \in \mathbb{C} \,|\, z = \frac{(Au,v)}{(u,v)},\, (u,v) \gg 0,\, u \in N_{p,\mathbb{R}},\; v \in N_{v^*,\mathbb{R}} \right\}. \tag{7.4}$$

Real part of the generalized numerical range of $A \in \mathbb{C}^{n\times n}$ with respect to the subspaces $N_{p,\mathbb{R}}$ and $N_{v^*,\mathbb{R}}$

Let the conditions $(C1')$-$(C4')$ be fulfilled. Then, we define the *real part of the generalized numerical range* of A with respect to the subspaces $N_{p,\mathbb{R}}$ and $N_{u^*,\mathbb{R}}$ by

$$Re[W_{N_{p,\mathbb{R}},N_{v^*,\mathbb{R}},gen.}(A)] = \left\{ x \in \mathbb{R} \,|\, x = \frac{Re(Au,v)}{(u,v)},\, (u,v) \gg 0,\, u \in N_{p,\mathbb{R}},\; v \in N_{v^*,\mathbb{R}} \right\}. \tag{7.5}$$

Imaginary part of the generalized numerical range of $A \in \mathbb{C}^{n\times n}$ with respect to the subspaces $N_{p,\mathbb{R}}$ and $N_{v^*,\mathbb{R}}$

Let the conditions $(C1')$-$(C4')$ be fulfilled. Then, we define the *imaginary part of the generalized numerical range* of A with respect to the subspaces $N_{p,\mathbb{R}}$ and $N_{v^*,\mathbb{R}}$ by

$$Im[W_{N_{p,\mathbb{R}},N_{v^*,\mathbb{R}},gen.}(A)] = \left\{ x \in \mathbb{R} \,|\, x = \frac{Im(Au,v)}{(u,v)},\, (u,v) \gg 0,\, u \in N_{p,\mathbb{R}},\; v \in N_{v^*,\mathbb{R}} \right\}. \tag{7.6}$$

Modulus of the generalized numerical range of $A \in \mathbb{C}^{n\times n}$ with respect to the subspaces $N_{p,\mathbb{R}}$ and $N_{v^*,\mathbb{R}}$

Let the conditions $(C1')$-$(C4')$ be fulfilled. Then, we define the *modulus of the generalized numerical range* of A with respect to the subspaces $N_{p,\mathbb{R}}$ and $N_{v^*,\mathbb{R}}$ by

$$|[W_{N_{p,\mathbb{R}},N_{v^*,\mathbb{R}},gen.}(A)]| = \left\{ x \in \mathbb{R}_0^+ \,|\, x = \frac{|(Au,v)|}{(u,v)},\, (u,v) \gg 0,\, u \in N_{p,\mathbb{R}},\; v \in N_{v^*,\mathbb{R}} \right\}. \tag{7.7}$$

8. Numerical example

In this section, we check some of the formulas of Section 2 on an example from the theory of linear dynamical systems.

8.1. A two-mass vibration model

We take up the multi-mass vibration model of [5], shown in *Fig.8.1*

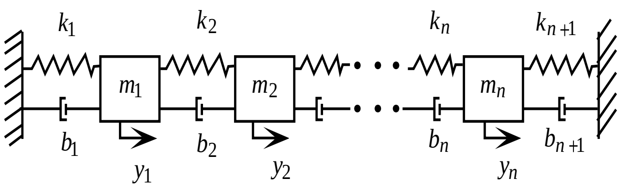

Fig.8.1: Multi-mass vibration model

and study the case $\mathbf{n} = \mathbf{2}$ as in [11]. For the sake of completeness, we give again the details. The associated initial value problem is given by

$$M\,\ddot{y} + B\,\dot{y} + K\,y = 0,\; y(0) = y_0,\, \dot{y}(0) = \dot{y}_0,$$

where $y = [y_1, y_2]^T$ and

$$M = \left[\begin{array}{c|c} m_1 & 0 \\ \hline 0 & m_2 \end{array}\right],$$

$$B = \left[\begin{array}{c|c} b_1 + b_2 & -b_2 \\ \hline -b_2 & b_2 + b_3 \end{array}\right],$$

$$K = \left[\begin{array}{c|c} k_1+k_2 & -k_2 \\ \hline -k_2 & k_2+k_3 \end{array}\right],$$

with the *mass, damping,* and *stiffness matrices* M, B, and K, as the case may be, and the *displacement vector* y. In *state-space description*, this problem takes the form

$$\dot{x} = Ax,\, t \geq 0,\, x(0) = x_0, \tag{8.1}$$

where $x = [y^T, z^T]^T$, $z = \dot{y}$, and where the *system matrix* A is given by

$$A = \left[\begin{array}{c|c} 0 & E \\ \hline -M^{-1}K & -M^{-1}B \end{array}\right].$$

Like in [11], as numerical values for the quantities not yet specified, we choose $b_1 = 1/4$, $k_2 = 2^3 = 8$. On the whole, this delivers the following data:

$$m_1 = m_2 = 1;\ b_1 = 1/4, b_2 = 0, b_3 = 1/4;\ k_1 = 1/64 = 1/2^4, k_2 = 8, k_3 = 1/64 = 1/2^4,$$

which leads to

$$\begin{aligned} M &= \left[\begin{array}{c|c} m_1 & 0 \\ \hline 0 & m_2 \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right], \\ B &= \left[\begin{array}{c|c} b_1+b_2 & -b_2 \\ \hline -b_2 & b_2+b_3 \end{array}\right] = \left[\begin{array}{c|c} 0.25 & 0 \\ \hline 0 & 0.25 \end{array}\right], \\ K &= \left[\begin{array}{c|c} k_1+k_2 & -k_2 \\ \hline -k_2 & k_2+k_3 \end{array}\right] = \left[\begin{array}{c|c} 1/64+8 & -1/2 \\ \hline -1/2 & 8+1/64 \end{array}\right] = \left[\begin{array}{c|c} 8.015625 & -0.5 \\ \hline -0.5 & 8.015625 \end{array}\right]. \end{aligned}$$

Further, we choose

$$t_0 = 0$$

as well as

$$y_0 = [-1, 1]^T$$

and

$$\dot{y}_0 = [-1, -1]^T,$$

but y_0 and $\dot{y}_0$ are not needed here.

8.2. Computation of important quantities

Using the Matlab routine *jordan*, one obtains

$$\begin{aligned} \lambda_1(A) &= -0.1250 + 4.0000i, \\ \lambda_2(A) &= -0.1250 - 4.0000i, \\ \lambda_3(A) &= -0.1250, \\ \lambda_4(A) &= \lambda_3(A). \end{aligned} \tag{8.2}$$

The pertinent eigenvectors and principal vectors are

$$\left[p_1^{(1)}, p_1^{(2)}, p_1^{(3)}, p_2^{(3)}\right] = [p_1, p_2, p_3, p_4.]$$

They are unnormed. The algebraic multiplicities are thus $m_1 = m_2 = 1$ and $m_3 = 2$. So, here, $r = 3$.
For the adjoint matrix A^*, we obtain

$$\begin{aligned} \lambda_1(A^*) &= -0.1250 - 4.0000i, \\ \lambda_2(A^*) &= -0.1250 + 4.0000i, \\ \lambda_3(A^*) &= -0.1250, \\ \lambda_4(A^*) &= \lambda_3(A^*). \end{aligned} \tag{8.3}$$

The associated eigenvectors and principal vectors are

$$\left[u_1^{(1)^*}, u_1^{(2)^*}, u_1^{(3)^*}, u_2^{(3)^*}\right] = [u_1^*, u_2^*, u_3^*, u_4^*.]$$

They are also unnormed.
In [11], we biorthogonalized these vectors based on Theorem 1 such that the relations

$$(p_k^{(i)}, u_l^{(i)^*}) = \begin{cases} 1, & l = m_i - k + 1 \\ 0, & l \neq m_i - k + 1 \end{cases} \tag{8.4}$$

and

$$(p_k^{(i)}, u_l^{(j)^*}) = 0,\ i \neq j. \tag{8.5}$$

So, with

$$v_l^{(i)^*} = u_{m_i-l+1}^{(i)}{}^*, \tag{8.6}$$

one has then the biorthogonality relations

$$(p_k^{(i)}, v_l^{(j)^*}) = \delta_{kl}\delta_{ij}. \tag{8.7}$$

The details of the biorthogonalization can be found in [11]. Define

$$\begin{aligned} v_1^* &= v_1^{(1)^*} = u_1^{(1)^*} = u_1^*, \\ v_2^* &= v_1^{(2)^*} = u_1^{(2)^*} = u_2^*, \\ v_3^* &= v_1^{(3)^*} = u_2^{(3)^*} = u_4^*, \\ v_4^* &= v_2^{(3)^*} = u_1^{(3)^*} = u_3^*. \end{aligned}$$

Herewith,

$$(p_i, v_j^*) = \delta_{ij},\ i,j = 1,\cdots,4 \tag{8.8}$$

where

$$p_1 = \begin{bmatrix} 0.364602 \\ -0.364602 \\ -0.045575 + 1.458408i \\ 0.045575 - 1.458408i \end{bmatrix}, \qquad v_1^* = \begin{bmatrix} 0.685679 + 0.021427i \\ -0.685679 - 0.021427i \\ 0 + 0.171420i \\ 0 - 0.171420i \end{bmatrix},$$

$$p_2 = \begin{bmatrix} 0.364602 \\ -0.364602 \\ -0.045575 - 1.458408i \\ 0.045575 + 1.458408i \end{bmatrix}, \qquad v_2^* = \begin{bmatrix} 0.685679 - 0.021427i \\ -0.685679 + 0.021427i \\ 0 - 0.171420i \\ 0 + 0.171420i \end{bmatrix},$$

$$p_3 = \begin{bmatrix} 0.707107 \\ 0.707107 \\ -0.088388 \\ -0.088388 \end{bmatrix}, \qquad v_3^* = \begin{bmatrix} 0.707107 \\ 0.707107 \\ 0 \\ 0 \end{bmatrix},$$

$$p_4 = \begin{bmatrix} 0 \\ 0 \\ 0.712610 \\ 0.712610 \end{bmatrix}, \qquad v_4^* = \begin{bmatrix} 0.087706 \\ 0.087706 \\ 0.701646 \\ 0.701646 \end{bmatrix}.$$

As in [11], we add the followings remarks.

Remark 8.1. *The vector $p_2^{(3)}$ is a principal vector of stage 2 for matrix A. But, since it is normed such that $(p_2^{(3)}, v_2^{(3)^*}) = 1$ instead of $\|p_2^{(3)}\|_2 = 1$, we have* **not** *$Ap_2^{(3)} = \lambda_3 p_2^{(3)} + p_1^{(3)}$, but instead, $Ap_2^{(3)} = \lambda_3 p_2^{(3)} + \gamma_1^{(3)} p_1^{(3)}$ with a factor $\gamma_1^{(3)} \neq 0$, $\gamma_1^{(3)} \neq 1$. Similarly, $u_2^{(3)^*}$ is principal vector of stage 2 for A^*. But, due to the biorthogonalization process, we have* **not** *$A^* u_2^{(3)^*} = \overline{\lambda_3} u_2^{(3)^*} + u_1^{(3)^*}$, but instead, $A^* u_2^{(3)^*} = \overline{\lambda_3} u_2^{(3)^*} + \delta_1^{(3)} u_1^{(3)^*}$ with a factor $\delta_1^{(3)} \neq 0$, $\delta_1^{(3)} \neq 1$. We leave it to the reader to check this numerically on our example.*

Remark 8.2. *Due to the foregoing remark, Formula (2.17) looks somewhat different. But, Formula (2.18) remains valid which is the important point since the subsequent findings are based on Formula (2.18), not on Formula (2.17).*

8.3. Numerical check of Theorems 2.6 and 2.9

From Theorem 2.6, Formula (2.30) and Theorem 2.9, Formula (2.33), we conclude

$$\min_{j=1,2,3} Re\lambda_j(A) \leq \frac{Re(Au,v)}{(u,v)} \leq \max_{j=1,2,3} Re\lambda_j(A),$$

$(u,v) \gg 0$, $u \in N_{p,\mathbb{R}}$, $v \in N_{v^*,\mathbb{R}}$ by setting $k=1$, there. This can also be written as

$$\frac{Re(Au,v)}{(u,v)} \in \left[\min_{j=1,2,3} Re\lambda_j(A), \max_{j=1,2,3} Re\lambda_j(A) \right],$$

$(u,v) \gg 0,\ u \in N_{p,\mathbb{R}},\ v \in N_{v^*,\mathbb{R}}$. We check this for a series of vectors. One has

$$\left[\min_{j=1,2,3} Re\lambda_j(A),\ \max_{j=1,2,3} Re\lambda_j(A)\right] = [-0.1250, -0.1250];$$

in other words

$$\frac{Re(Au,v)}{(u,v)} = -0.1250,\ (u,v) \gg 0,\ u \in N_{p,\mathbb{R}},\ v \in N_{v^*,\mathbb{R}}.$$

Let

$$\begin{array}{rcl} u_1 & = & -5p_1 + 3p_3, \\ v_1 & = & -4v_1^* + 2v_3^*. \end{array}$$

Then, $u_1 \in N_{p,\mathbb{R}}$ and $v_1 \in N_{v^*,\mathbb{R}}$ as well as $(u_1,v_1) \gg 0$, and one obtains

$$u_1 = \begin{bmatrix} 0.298311 \\ 3.944330 \\ -0.037289 - 7.292039i \\ -0.493041 + 7.292039i \end{bmatrix}, \qquad v_1 = \begin{bmatrix} -1.328504 - 0.085710i \\ 4.156931 + 0.085710i \\ 0 - 0.685679i \\ 0 + 0.685679i \end{bmatrix},$$

$$\begin{array}{rcl} (Au_1,v_1) & = & -3.250000000000000 + 80.000000000000000i, \\ (u_1,v_1) & = & 26.000000000000004, \end{array}$$

and thus

$$\frac{Re(Au_1,v_1)}{(u_1,v_1)} \doteq -0.125000000000000.$$

Let

$$\begin{array}{rcl} u_2 & = & 3p_2, \\ v_2 & = & -4v_1^* + 2v_2^*. \end{array}$$

Then, $u_2 \in N_{p,\mathbb{R}}$ and $v_2 \in N_{v^*,\mathbb{R}}$ as well as $(u_2,v_2) \gg 0$, and one obtains

$$u_2 = \begin{bmatrix} 1.093806 \\ -1.093806 \\ -0.136726 - 4.375223i \\ 0.136726 + 4.375223i \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -1.371359 - 0.128565i \\ 1.371359 + 0.128565i \\ 0 - 1.028519i \\ 0 + 1.028519i \end{bmatrix},$$

$$\begin{array}{rcl} (Au_2,v_2) & = & -0.750000000000000 - 24.000000000000000i, \\ (u_2,v_2) & = & 6, \end{array}$$

and thus

$$\frac{Re(Au_2,v_2)}{(u_2,v_2)} \doteq -0.125000000000000.$$

Let

$$\begin{array}{rcl} u_3 & = & -5p_1 + 3p_2 - 4p_3, \\ v_3 & = & -4v_1^* + 2v_2^* - 2v_3^*. \end{array}$$

Then, $u_3 \in N_{p,\mathbb{R}}$ and $v_3 \in N_{v^*,\mathbb{R}}$ as well as $(u_3,v_3) \gg 0$, and one obtains

$$u_3 = \begin{bmatrix} -3.557631 \\ -2.099223 \\ 0.444704 - 11.667262i \\ 0.262403 + 11.667262i \end{bmatrix}, \qquad v_3 = \begin{bmatrix} -2.785572 - 0.128565i \\ -0.042855 + 0.128565i \\ 0 - 1.028519i \\ 0 + 1.028519i \end{bmatrix},$$

$$\begin{array}{rcl} (Au_3,v_3) & = & -4.250000000000000 + 56.000000000000000i, \\ (u_3,v_3) & = & 34, \end{array}$$

and thus

$$\frac{Re(Au_3,v_3)}{(u_3,v_3)} \doteq -0.125000000000000.$$

Let

$$\begin{array}{rcl} u_4 & = & -5p_1 + 3p_2 + 6p_3 - 4p_4, \\ v_4 & = & -2v_1^* + 4v_2^* + 2v_3^* - 3v_4^*. \end{array}$$

Then, $(u_4, v_4) \gg 0$, and one obtains

$$u_4 = \begin{bmatrix} 3.513437 \\ 4.971845 \\ -3.289618 - 11.667262i \\ -3.471919 + 11.667262i \end{bmatrix}, \qquad v_4 = \begin{bmatrix} 2.522455 - 0.128565i \\ -0.220262 + 0.128565i \\ -2.104939 - 1.028519i \\ -2.104939 + 1.028519i \end{bmatrix},$$

$$\begin{aligned} (Au_4, v_4) &= -13.812257748298542 - 7.999999999999989i, \\ (u_4, v_4) &= 46, \end{aligned}$$

and thus

$$\begin{aligned} \frac{Re(Au_4, v_4)}{(u_4, v_4)} &= -0.709454874462791 \\ &\neq -0.1250 \end{aligned}$$

which is not surprising since $u_4 \notin N_{p,\mathbb{R}} = [p_1, p_2, p_3]$ and $v_4 \notin N_{v^*,\mathbb{R}} = [v_1^*, v_2^*, v_3^*]$. Recall at this point that $r = 3$ and $v_j^* = u_{m_j}^{(j)^*}$, $j = 1,2,3$ are the principal vectors of maximum stage associated with the eigenvalues $\lambda_j(A^*), j = 1,2,3$, as the case may be. With $m_1 = m_2 = 1$ and $m_3 = 2$, therefore $v_1^* = u_1^{(1)^*}$ and $v_2^* = u_1^{(2)^*}$ are eigenvectors and $v_3^* = u_2^{(3)^*}$ is a principal vector of stage 2 whereas $v_4^* = u_1^{(3)^*}$ is an eigenvector and thus not a principal vector of maximum stage.
Let

$$\begin{aligned} u_5 &= [1,2,3,4]^T \in \mathbb{R}^4, \\ v_5 &= [4,3,2,1]^T \in \mathbb{R}^4. \end{aligned}$$

Here, one obtains

$$\begin{aligned} (Au_5, v_5) &= 29.437500000000000, \\ (u_5, v_5) &= 20, \end{aligned}$$

and thus

$$\frac{Re(Au_5, v_5)}{(u_5, v_5)} = 1.471875000000000 \neq -0.1250$$

which is neither surprising since $(u_5, v_5) \not\gg 0$ due to

$$\alpha^{(5)} := (\alpha_k^{(5)})_{k=1,\cdots,4} = ((u_5, v_k^*))_{k=1,\cdots,4} = \begin{bmatrix} -0.685679 + 0.192847i \\ -0.685679 - 0.192847i \\ 2.121320 \\ 5.174642 \end{bmatrix}$$

and

$$\beta^{(5)} := (\beta_k^{(5)})_{k=1,\cdots,4} = ((p_k, v_5))_{k=1,\cdots,4} = \begin{bmatrix} 0.319027 + 1.458408i \\ 0.319027 - 1.458408i \\ 4.684582 \\ 2.137829 \end{bmatrix}.$$

8.4. Computational aspects

In this subsection, we say something about the used computer equipment and the computation times.
(i) As to the *computer equipment*, the following hardware was available: an Intel Core2 Duo Processor at 3166 GHz, a 500 GB mass storage facility, and two 2048 MB high-speed memories. As software package for the computations, we used MATLAB, Version 7.11.
(ii) The *computation time t* of an operation was determined by the command sequence *t1=clock; operation; t=etime(clock,t1)*. It is put out in seconds, rounded to four decimal places. For the computation of the eigenvalues of matrix A in Subsection 8.2, we used the command [XA,DA]=eig(A); the pertinent computation time was less than 0.0001 s.

9. Numerical Example 2

In this section, we proceed in a similar way as in Section 8. Here, we present an example of a real non-diagonalizable matrix A with real eigenvalues.

9.1. The matrix A and its eigenvalues and principal vectors

We take the matrix A from [1, Example 5.2, p.82]. So, let

$$A=\begin{bmatrix} 4 & 1 & 1 \\ 2 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}.$$

In [1], the eigenvalues are given as

$$\begin{aligned} \lambda_1 &= 6, \\ \lambda_2 &= 3, \\ \lambda_3 &= \lambda_2, \end{aligned}$$

where the numbering is such that $\lambda_1 \geq \lambda_2$. According to [1], the associated right eigenvectors are given as

$$p_1 = p_1^{(1)} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix},\ p_2 = p_1^{(2)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix};$$

they are unnormed. Here, $m_1 = 1$ and $m_2 = 2$. So, here, $r = 2$. A corresponding principal vector $p_3 = p_2^{(2)}$ is not given in [1]. Further, there, the vectors u_1^* and u_2^* of $A^* = A^T$ are given as

$$u_1^* = u_1^{(1)^*} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ u_2^* = u_1^{(2)^*} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix};$$

they are also unnormed. Again, the principal vector $u_3^* = u_2^{(2)^*}$ is not given in [1].
From

$$Ap_2^{(2)} = \lambda_2(A)p_2^{(2)} + p_1^{(2)},$$

one can determine a principal vector $p_2^{(2)}$, and from

$$A^*u_2^{(2)^*} = \lambda_2(A^*)u_2^{(2)^*} + u_1^{(2)^*},$$

a principal vector $u_2^{(2)^*}$. By hand calculation, we obtain

$$p_2^{(2)} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ u_2^{(2)^*} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

if we choose the third component of $p_2^{(2)}$ and the first component of $u_2^{(2)^*}$ as zero. At this point, we remind that these principal vectors of stage 2 are only determined up to an associated eigenvector.

9.2. Auxiliary computational results

Using the Matlab routine *eig.m*, we obtain

$$\begin{aligned} \lambda_1 &= 6, \\ \lambda_2 &= 3, \\ \lambda_3 &= \lambda_2. \end{aligned}$$

The pertinent computed biorthonormal right eigenvectors and principal vectors

$$\left[p_1^{(1)}, p_1^{(2)}, p_2^{(2)}\right] = [p_1, p_2, p_3]$$

are unnormed. The algebraic multiplicities are thus $m_1 = 1$ and $m_2 = 2$.
For the adjoint matrix A^*, we obtain

$$\begin{aligned} \lambda_1(A^*) &= \lambda_1(A), \\ \lambda_2(A^*) &= \lambda_2(A), \\ \lambda_3(A^*) &= \lambda_2(A). \end{aligned} \tag{9.1}$$

The associated eigenvectors and principal vectors are

$$\left[u_1^{(1)^*}, u_1^{(2)^*}, u_2^{(2)^*}\right] = [u_1^*, u_2^*, u_3^*]$$

are also unnormed.

As in [11], we biorthogonalized these vectors based on Theorem 2.1 such that the relations (8.4) - (8.7) hold. The details of the biorthogonalization can be found in [11].
Define

$$\begin{aligned} v_1^* &= v_1^{(1)^*} = u_1^{(1)^*} = u_1^*, \\ v_2^* &= v_1^{(2)^*} = u_2^{(2)^*} = u_3^*, \\ v_3^* &= v_1^{(3)^*} = u_1^{(2)^*} = u_2^*. \end{aligned}$$

Herewith,

$$(p_i, v_j^*) = \delta_{ij},\ i,j = 1,\cdots,3 \tag{9.2}$$

where

$$p_1 = \begin{bmatrix} -0.577350269189626 \\ -0.769800358919501 \\ -0.384900179459750 \end{bmatrix}, \qquad v_1^* = \begin{bmatrix} -0.577350269189626 \\ -0.577350269189626 \\ -0.577350269189626 \end{bmatrix},$$

$$p_2 = \begin{bmatrix} 0 \\ -1.427248064296125 \\ 1.427248064296125 \end{bmatrix}, \qquad v_2^* = \begin{bmatrix} -0.622799155329218 \\ 0.077849894416152 \\ 0.778498944161523 \end{bmatrix},$$

$$p_3 = \begin{bmatrix} -0.816496580927726 \\ 1.632993161855452 \\ -0.816496580927726 \end{bmatrix}, \qquad v_3^* = \begin{bmatrix} -0.816496580927726 \\ 0.408248290463863 \\ 0.408248290463863 \end{bmatrix}.$$

These results are based on the eigenvectors and principal vectors computed by using the Matlab routine *jordan*. We leave it to the reader to compute these vectors by starting with the unnormed vectors stated in Section 9.1. The result is somewhat different. This outcome is not surprising since the principal vectors are determined only up to eigenvectors for the treated matrix A.
We conclude this section by mentioning that, here, similar remarks hold to those at the end of Section 8.2.

9.3. Numerical check of Corollaries 5.2 and 5.5

From Corollary 5.2, Formula (5.3) and Corollary 5.5, Formula (5.7), we conclude

$$\min_{j=1,2} \lambda_j(A) \le \frac{(Au,v)}{(u,v)} \le \max_{j=1,2} \lambda_j(A),$$

$(u,v) \gg 0,\ u \in N_{p,\mathbb{R}} = [p_1, p_2]_{\mathbb{R}},\ v \in N_{v^*,\mathbb{R}} = [v_1^*, v_2^*]_{\mathbb{R}}$ by setting $k = 1$, there. This can also be written as

$$\frac{(Au,v)}{(u,v)} \in \left[\min_{j=1,2} \lambda_j(A), \max_{j=1,2} \lambda_j(A)\right],$$

$(u,v) \gg 0,\ u \in N_{p,\mathbb{R}},\ v \in N_{v^*,\mathbb{R}}$. We check this for a series of vectors. One has

$$\left[\min_{j=1,2} \lambda_j(A), \max_{j=1,2} \lambda_j(A)\right] = [3;6].$$

Let

$$\begin{aligned} u_1 &= -5p_1 + 3p_2, \\ v_1 &= -4v_1^* + 2v_2^*. \end{aligned}$$

Then, $u_1 \in N_{p,\mathbb{R}}$ and $v_1 \in N_{v^*,\mathbb{R}}$ as well as $(u_1, v_1) \gg 0$, and one obtains

$$u_1 = \begin{bmatrix} 2.886751345948128 \\ -0.432742398290872 \\ 6.206245090187128 \end{bmatrix}, \qquad v_1 = \begin{bmatrix} 1.063802766100067 \\ 2.465100865590808 \\ 3.866398965081549 \end{bmatrix},$$

$$\begin{aligned} (Au_1, v_1) &= 138.0000000000000, \\ (u_1, v_1) &= 25.999999999999996, \end{aligned}$$

and thus

$$\frac{(Au_1, v_1)}{(u_1, v_1)} \doteq 5.307692307692308 \in [3;6].$$

Let

$$\begin{aligned} u_2 &= 3p_2, \\ v_2 &= -4v_1^* + 2v_2^*. \end{aligned}$$

Then, $u_2 \in N_{p,\mathbb{R}}$ and $v_2 \in N_{v^*,\mathbb{R}}$ as well as $(u_2, v_2) \gg 0$, and one obtains

$$u_2 = \begin{bmatrix} 0 \\ -4.281744192888376 \\ 4.281744192888376 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1.063802766100067 \\ 2.465100865590808 \\ 3.866398965081549 \end{bmatrix},$$

$$\begin{aligned} (Au_2, v_2) &= 17.999999999999996, \\ (u_2, v_2) &= 5.999999999999998, \end{aligned}$$

and thus

$$\frac{(Au_2, v_2)}{(u_2, v_2)} \doteq 3.000000000000000 \in [3; 6].$$

Let

$$\begin{aligned} u_3 &= -2p_1 + 2p_2, \\ v_3 &= -2v_1^* + 2v_2^*. \end{aligned}$$

Then, $u_3 \in N_{p,\mathbb{R}}$ and $v_3 \in N_{v^*,\mathbb{R}}$ as well as $(u_3, v_3) \gg 0$, and one obtains

$$u_3 = \begin{bmatrix} 1.154700538379251 \\ -1.314895410753249 \\ 3.624296487511752 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} -0.090897772279185 \\ 1.310400327211556 \\ 2.711698426702298 \end{bmatrix},$$

$$\begin{aligned} (Au_3, v_3) &= 36, \\ (u_3, v_3) &= 7.999999999999999, \end{aligned}$$

and thus

$$\frac{(Au_3, v_3)}{(u_3, v_3)} \doteq 4.500000000000001 \in [3; 6].$$

Let

$$\begin{aligned} u_4 &= -5p_1 + 3p_2 + 6p_3, \\ v_4 &= -2v_1^* + 4v_2^* + 2v_3^*. \end{aligned}$$

Then, $(u_4, v_4) \gg 0$, and one obtains

$$u_4 = \begin{bmatrix} -2.012228139618228 \\ 9.365216572841840 \\ 1.307265604620772 \end{bmatrix}, \qquad v_4 = \begin{bmatrix} -2.969489244793074 \\ 2.282596696971587 \\ 5.085192895953069 \end{bmatrix},$$

$$\begin{aligned} (Au_4, v_4) &= 145.7298612851365, \\ (u_4, v_4) &= 34, \end{aligned}$$

and thus

$$\frac{(Au, v_4)}{(u_4, v_4)} = 4.286172390739309 \in [3; 6]$$

even though $u_4 \notin N_{p,\mathbb{R}} = [p_1, p_2]_{\mathbb{R}}$ and $v_4 \notin N_{v^*,\mathbb{R}} = [v_1^*, v_2^*]_{\mathbb{R}}$.
Let

$$\begin{aligned} u_5 &= [1, 2, 3]^T \in \mathbb{R}^3, \\ v_5 &= [3, 2, 1]^T \in \mathbb{R}^3. \end{aligned}$$

Here, one obtains

$$\begin{aligned} (Au_5, v_5) &= 67, \\ (u_5, v_5) &= 10, \end{aligned}$$

and thus

$$\frac{(Au_5, v_5)}{(u_5, v_5)} = 6.700000000000000 \notin [3; 6]$$

which is not surprising since $(u_5, v_5) \not\gg 0$ due to

$$\alpha^{(5)} := (\alpha_k^{(5)})_{k=1,2,3} = ((u_5, v_k^*))_{k=1,2,3} = \begin{bmatrix} -3.464101615137755 \\ 1.868397465987655 \\ 1.224744871391589 \end{bmatrix}$$

and

$$\beta^{(5)} := (\beta_k^{(5)})_{k=1,2,3} = ((p_k, v_5))_{k=1,2,3} = \begin{bmatrix} -3.656551704867629 \\ -1.427248064296125 \\ 0.000000000000000 \end{bmatrix}.$$

10. Conclusion

It has been shown that there exist generalized Rayleigh-quotient representations of the real and imaginary parts of the eigenvalues of general matrices that parallel those for the eigenvalues of diagonalizable matrices. As in that case, for the moduli, only a max-representation could be stated. The special case of general matrices with real eigenvalues has also been considered. The main difference to the case of diagonalizable matrices is that the space $N_{u^*,\mathbb{R}} = [u_1^*,\cdots,u_n^*]_{\mathbb{R}}$ is replaced by the space $N_{v^*,\mathbb{R}} = [v_1^*,\cdots,u_r^*]_{\mathbb{R}}$ where $v_j^* = v_1^{(j)^*} = u_{m_j}^{(j)^*}$ are the principal vectors of largest stage m_j pertinent to the eigenvalues $\lambda_j(A^*) = \overline{\lambda_j(A)}$ for $j = 1,\cdots,r$. As application, a new formula for the spectral radius $\rho(A)$ for general matrices is obtained. On a numerical example from the theory of linear dynamical systems with non-diagonalizable system matrix A (Example 1), we check that $\dfrac{Re(Au,v)}{(u,v)} \in [\min\limits_{j=1,\cdots,r} Re\lambda_j(A), \max\limits_{j=1,\cdots,r} Re\lambda_j(A)]$, $(u,v) \gg 0$, $u \in N_{p,\mathbb{R}}$, $v \in N_{v^*,\mathbb{R}}$. On a further example (Example 2), also for a non-diagonalizable matrix A, this time with real eigenvalues, we check numerically that $\dfrac{(Au,v)}{(u,v)} \in [\min\limits_{j=1,\cdots,r} \lambda_j(A), \max\limits_{j=1,\cdots,r} \lambda_j(A)]$, $(u,v) \gg 0$, $u \in N_{p,\mathbb{R}}$, $v \in N_{v^*,\mathbb{R}}$. We mention that, in the case of diagonalizable matrices, the results of [10] are obtained back since then $r = n$ and $m_j = 1$ for $j = 1,\cdots,r$ so that then $N_{v^*,\mathbb{R}} = N_{u^*,\mathbb{R}}$ and $N_{\sigma(A)} = \mathbb{C}^n$. The paper is of interest on its own in the areas of Linear Algebra and Numerical Analysis. Beyond this, it could be of value to mathematicians and engineers, in general.

References

[1] R.T. Gregory, D.L. Karney, *A Collection of Matrices for Testing Computational Algorithms*, Wiley-Interscience, New York, London Sydney Toronto, 1969.
[2] R.A. Horn, Ch.R. Johnson, *Matrix Analysis*, University Press, Cambridge, 22nd printing, 2009.
[3] R.A. Horn, Ch.R. Johnson, *Topics in Matrix Analysis*, University Press, Cambridge, 10th printing, 2008.
[4] T. Kato, *Perturbation Theory for Linear Operators*, Springer, New York, 1966.
[5] L. Kohaupt, *Construction of a biorthogonal system of principal vectors of the matrices A and A^* with applications to the initial value problem* $\dot{x} = Ax$, $x(t_0) = x_0$, J. Comput. Math. Optim., **3**(3) (2007),163-192.
[6] L. Kohaupt, *Biorthogonalization of the principal vectors for the matrices A and A^* with application to the computation of the explicit representation of the solution $x(t)$ of* $\dot{x} = Ax$, $x(t_0) = x_0$, Appl. Math. Sci., **2**(20) (2008), 961-974.
[7] L. Kohaupt, *Solution of the vibration problem* $M\ddot{y} + B\dot{y} + Ky = 0$, $y(t_0) = y_0$, $\dot{y}(t_0) = \dot{y}_0$ without *the hypothesis* $BM^{-1}K = KM^{-1}B$ *or* $B = \alpha M + \beta K$, Appl. Math. Sci., **2**(41) (2008)1989-2024.
[8] L. Kohaupt, *Generalized Rayleigh-quotient representation of the eigenvalues of self-adjoint matrices*, J. Algebra Appl. Math., **14**(1) (2016)1-26.
[9] L. Kohaupt, *Rayleigh-quotient representation of the real parts, imaginary parts, and moduli of the eigenvalues of diagonalizable matrices*, J. Math. Sci. Model., **2**(2) (2019), 82-98.
[10] L. Kohaupt, *Generalized Rayleigh-quotient representation of the real parts, imaginary parts, and moduli of the eigenvalues of diagonalizable matrices*, J. Adv. Math., **14**(2) (2018), 7702-7728.
[11] L. Kohaupt, *Rayleigh-quotient representation of the real parts, imaginary parts, and moduli of the eigenvalues of general matrices*, J. Math. Sci. Model., **3**(2) (2020), 55-75.
[12] F. Stummel, K. Hainer, *Introduction to Numerical Analysis*, Scottish Academic Press, Edinburgh, 1980.
[13] L.H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965.
[14] R. Zurmühl, S. Falk, *Matrizen und ihre Anwendungen, Teil 1: Grundlagen (Matrices and Their Applications, Part 1: Foundations)*, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.

Appendix

As in [10], with a minor additional hypothesis, the generalized min-, min-max-, and max-min-representaions for the moduli of eigenvalues can be proven. In this Appendix, we show this, but restrict ourselves to the min-max-representation. The minor additional hypothesis is $p_j \in M_p$ and $v_j^* \in M_{v^*}$. A further advantage of this additional hypothesis is that the proofs simplify. But, we omit the proof since it is similar to that in the case when matrix A is diagonalizable in [10].
We have the following theorem.

Theorem 10.1. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (4.1). Then, for every $j = 1,\cdots,n$ and every subspace $M_p \subset N_{p,\mathbb{R}}$ and $M_{v^*} \subset N_{v^*,\mathbb{R}}$ with $dim M_p = dim M_{v^*} = m = n+1-j$ where* **additionally** $\mathbf{p_j} \in \mathbf{M_p}$ *and* $\mathbf{v_j^*} \in \mathbf{M_{v^*}}$, *the following inequalities are valid:*

$$|\lambda_j(A)| \leq \max_{\substack{(u,v)\gg 0 \\ u\in M_p, v\in M_{v^*}}} \frac{|(Au,v)|}{(u,v)} \leq |\lambda_1(A)|,$$

and the following representation formulas hold:

$$|\lambda_j(A)| = \min_{\substack{dim M_p = m,\, p_j\in M_p \\ dim M_{v^*} = m,\, v_j^*\in M_{v^*}}} \max_{\substack{(u,v)\gg 0 \\ u\in M_p, v\in M_{v^*}}} \frac{|(Au,v)|}{(u,v)}.$$

Remark 10.2. *We mention that, with the above additional hypotheses, the proofs of Theorems 2.6 - 2.10, Theorems 3.2 - 3.5, Theorem 4.1, and Corollaries 5.1 - 5.5 get also simpler.*