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Araştırma Makalesi / Research Article

On a Closed Subspace of $L^{p(.)}(\Omega)$

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Abstract

In this study, we first give a description of $L^{p(.)}(\Omega)$ spaces. These spaces are an important generalization of classical Lebesgue spaces. We mention their various applications in engineering and physics fields. Thereafter, as it is naturally, one of the main task in $L^{p(.)}(\Omega)$ spaces is to generalize known properties classical Lebesgue spaces $L^p(\Omega)$ to $L^{p(.)}(\Omega)$ spaces. Provided that measure of the set Ω is finite, we extend a theorem which about a closed subspace of $L^{p(.)}(\Omega)$ space, from constant exponent to variable exponent. Our proof method based on embedding between $L^{p(.)}(\Omega)$-$L^p(\Omega)$ spaces and the proof of constant case. The essence of the method is to take advantage of properties of Hilbert space $L^2(\Omega)$, and also based on the use of the closed graph theorem and finite measure of the set Ω.

Keywords: Variable Exponent, Lebesgue Space, Closed Subspace.

$L^{p(.)}(\Omega)$nin Kapalı Bir Alt Uzayı Üzerine

Öz

Bu çalışmada önce $L^{p(.)}(\Omega)$ uzaylarını tanıtıyoruz. Bu uzaylar klasik Lebesgue uzaylarının önemli bir genelleştirmesidir. Bunların mühendislik ve fizikte bulunan çeşitli uygulamalarına değiniyoruz. Sonra, doğal olarak beklenildiği gibi, $L^{p(.)}(\Omega)$uzaylarındaki en önemli işlerden biri $L^p(\Omega)$ klasik Lebesgue uzaylarının bilinen özelliklerini $L^{p(.)}(\Omega)$uzaylarına genelleştirmektir. Ω kümesinin ölçümü sonlu olmak koşulluyla, $L^{p(.)}(\Omega)$nın bir kapalı alt uzayı ile ilgili bir teoremi sabit üslüden değişken üslüye genişletiyoruz. İspatımızın yöntemi $L^{p(.)}(\Omega)$-$L^p(\Omega)$ uzayları arasındaki gömülmeye ve sabit durumun ispatına dayanmaktadır. Yöntemin esası $L^2(\Omega)$ Hilbert uzayının özelliklerinin avantajlarından yararlanmak ayrıca kapalı grafik teoremi ve Ω kümesinin sonlu ölçümlü olmasına dayanmaktadır.

Anahtar kelimeler: Değişken Üs, Lebesgue Uzayı, Kapalı Alt Uzay.

1. Introduction

The variable exponent Lebesgue function spaces $L^{p(.)}(\Omega)$ are a quite important and very useful generalization of the classical Lebesgue function spaces $L^p(\Omega)$. $L^{p(.)}(\Omega)$ spaces are obtained by

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substituting the variable exponent $p(.)$ for constant exponent p. The variable Lebesgue spaces are not only important for purely theoretical reasons but have broad applications, e.g. partial differential equations, the calculus of variations [1-4], in the context of engineering and physical: the modeling of electrorheological fluids [5], (electrorheological fluids are substances whose flow properties vary when subjected to an electric field), hydromechanics of quasi-Newtonian fluids [6], analysis of fluid flow in porous media [7], magnetostatics problems [8], and image reconstruction [9]. The basics on $L^{p(.)}(\Omega)$ spaces may be seen in the crucial articles by Kováčik and Rákosník [10], and Fan and D. Zhao [11]. Also, for a monograph treatment and a general discussion of $L^{p(.)}(\Omega)$ spaces we refer the reader to [2,3].

Let $\Omega \subset \mathrm{R}^n$ be a measurable set. An variable exponent function $p(.)$ is defined to be a bounded measurable function satisfying $p(.):\Omega \to [1,\infty)$. Let $E \subset \Omega$ some notations about $p(.)$ defined as follows

$$p_E^+ = ess\sup_{x\in E} p(x), \quad p_E^- = ess\inf_{x\in E} p(x), \quad p^+ = p_\Omega^+, \quad p^- = p_\Omega^-.$$

We will always impose $p^+ < \infty$ restriction to exponents without even mentioning it, in other words, we always assume that $p^+ < \infty$. We use the notation $|E|$ for the measure of the a Lebesgue measurable set $E \subset \Omega$. We will usually need to distinguish between variable and constant exponents, therefore, variable exponent function will be shown by $p(.)$

Definition of variable exponent Lebesgue spaces $L^{p(.)}(\Omega)$ and that of $L^p(\Omega)$ spaces look alike. $L^{p(.)}(\Omega)$ is the space of all measurable functions $u:\Omega \to \mathrm{R}$ whose modular functional satisfy

$$\rho_{L^{p(.)}(\Omega)}(u) = \int_\Omega |u(x)|^{p(x)} dx < \infty$$

condition. We give a norm (called the Luxemburg norm) on $L^{p(.)}(\Omega)$ function space by setting

$$\|u\|_{L^{p(.)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{L^{p(.)}(\Omega)}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

This norm makes $L^{p(.)}(\Omega)$ space a Banach space [10]. One labour-saving property of these spaces is that [10]

$$\rho_{L^{p(.)}(\Omega)}(u_n) \to 0 \quad \Leftrightarrow \|u_n\|_{L^{p(.)}(\Omega)} \to 0.$$

If $u \equiv 0$, then it is not hard to see the value of norm is 0, but we can not replace $\lambda = 0$. Therefore, apart from special case $u \equiv 0$, the infimum in the definion of Luxemburg norm can always be obtained. In other words, given Ω and $p(.)$, then

$$\rho_{L^{p(.)}(\Omega)}\left(\frac{u}{\|u\|_{L^{p(.)}(\Omega)}}\right) = 1$$

for all $u \in L^{p(.)}(\Omega)$ except $u \equiv 0$ [2].

Also, when $p(x) = p$, a positive constant, the space $L^{p(.)}(\Omega)$ coincides with the classical Lebesgue space $L^{p}(\Omega)$ and the norm $\|u\|_{L^{p(.)}(\Omega)}$ reduces to the standard $\|u\|_{L^{p}(\Omega)} = \left(\int_{\Omega}|u(x)|^{p} dx\right)^{\frac{1}{p}}$ norm in $L^{p}(\Omega)$ space. We recall some more essential properties of variable Lebesgue spaces. We will use some of them even without any mention since they are basic tools of $L^{p(.)}(\Omega)$ spaces. Similar to constant case, for given $p(.)$, conjugate variable exponent $p'(.)$ can defined by setting

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,\ x \in \Omega.$$

Given $p(.)$, $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$. Then the product uv is an element of $L^{1}(\Omega)$ and Hölder type inequality

$$\int_{\Omega} uvdx \leq 2\|u\|_{L^{p(.)}(\Omega)}\|v\|_{L^{p'(.)}(\Omega)}$$

holds [10].

In [10,11], given $p(.)$ and Ω, then the following advantageous relations between norm and modular hold

i) $\|u\|_{L^{p(.)}(\Omega)} < 1(=1;>1) \Leftrightarrow \rho_{L^{p(.)}(\Omega)}(u) < 1(=1;>1)$

ii) $\|u\|_{L^{p(.)}(\Omega)} > 1$ then $\left(\|u\|_{L^{p(.)}(\Omega)}\right)^{p^{-}} \leq \rho_{L^{p(.)}(\Omega)}(u) \leq \left(\|u\|_{L^{p(.)}(\Omega)}\right)^{p^{+}}$

iii) $\|u\|_{L^{p(.)}(\Omega)} < 1$ then $\left(\|u\|_{L^{p(.)}(\Omega)}\right)^{p^{+}} \leq \rho_{L^{p(.)}(\Omega)}(u) \leq \left(\|u\|_{L^{p(.)}(\Omega)}\right)^{p^{-}}$

Lemma 1.1. [2,10] Let Ω be a measurable set satisfying $|\Omega| < \infty$ and given exponents $p(.)$, $q(.)$ such that $p(x) \leq q(x)$. Then the inclusion $L^{q(x)}(\Omega) \subset L^{p(x)}(\Omega)$ holds. We also have the following inequality

$$\|u\|_{L^{p(.)}(\Omega)} \leq (1+|\Omega|)\|u\|_{L^{q(.)}(\Omega)}. \tag{1}$$

We also need to recall the well-known result from functional analysis as known closed graph theorem: Let B_1 and B_2 be any two Banach spaces. If $H : B_1 \to B_2$ is any closed linear map, then H is bounded.

Lemma 2.1. Let $\phi_1, \phi_2, ..., \phi_k$ be pairwise orthogonal vectors in an inner product space. Then

$$\left\|\sum_{j=1}^{k}\phi_j\right\|^2=\sum_{j=1}^{k}\left\|\phi_j\right\|^2$$

holds [12].

2. Main Results

The constant exponent case of the following theorem was proved in [13], (see also [12]). In fact, our proof of the following theorem based on latter.

Theorem 2.1. Let $\Omega\subset \mathrm{R}^n$ be a measurable set such that $|\Omega|<\infty$. If V is a closed subspace of variable Lebesgue space $L^{p(.)}(\Omega)$ that satisfying $V\subset L^\infty(\Omega)$. Then the space V is finite dimensional.

Proof. Let define map $I:V\to L^\infty(\Omega)$ by $I(u)=u$. Given a sequence $\{u_n\}$ in V such that $u_n\to u$ in V and $I(u_n)=u_n\to\varphi$ in $L^\infty(\Omega)$. Since $\left\|u_n(x)-\varphi(x)\right\|_{L^\infty(\Omega)}\to 0$ and modular convergence and norm convergence are equivalent, so there is a positive integer N such that for all $n\geq N$ we have the following inequality

$$\begin{aligned}\int_\Omega\left|u_n(x)-\varphi(x)\right|^{p(x)}dx&\leq\int_\Omega\left(\left\|u_n(x)-\varphi(x)\right\|_{L^\infty(\Omega)}\right)^{p^-}dx\\&=\left(\left\|u_n(x)-\varphi(x)\right\|_{L^\infty(\Omega)}\right)^{p^-}|\Omega|.\end{aligned}$$

Then we have $\int_\Omega\left|u_n(x)-\varphi(x)\right|^{p(x)}dx\to 0$ as $n\to\infty$. Hence from the uniqueness of limit $u=\varphi$ a.e. on Ω. This indicates that the graph of map I is closed. Therefore, by the closed graph theorem there is a constant $a>0$ such that for all $u\in V$ we have

$$\|u\|_{L^\infty(\Omega)}\leq a\|u\|_{L^{p(.)}(\Omega)}. \tag{2}$$

From (1) and (2) we have for all $u\in V$

$$\|u\|_{L^\infty(\Omega)}\leq a\|u\|_{L^{p(.)}(\Omega)}\leq a\left(1+|\Omega|\right)\|u\|_{L^{p^+}(\Omega)}.$$

Since p^+ is a constant, from argument that used in the proof of clasicall case [see 12, Theorem13.34] there exists a constant $b>0$ such that

$$\|u\|_{L^{p^+}(\Omega)}\leq b\|u\|_{L^2(\Omega)}. \tag{3}$$

Hence we have

$$\|u\|_{L^\infty(\Omega)}\leq ba\left(1+|\Omega|\right)\|u\|_{L^2(\Omega)}. \tag{4}$$

(The rest proof of the theorem is same as classical case but for sake of completeness we write it here.) Now we have replaced V inside $L^2(\Omega)$. By considering various advantages features of $L^2(\Omega)$ we can prove that V must be finite dimensional. Let $\{v_1, v_2, \dots, v_n\}$ be a set of linearly independent functions in V ; without loss of generality, we can accept that these functions are orthonormal in $L^2(\Omega)$ space. Hence

$$\int_\Omega v_i(x)v_j(x)\,dx = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

We will show that $n \leq \left(ba\left(1+|\Omega|\right)\right)^2 |\Omega|$. For each rational numbers $r = (r_1, r_2, \dots, r_n)$ such that $\sum_{i=1}^{n}(r_i)^2 \leq 1$. Consider a function

$$H_r = \sum_{i=1}^{n} r_i v_i.$$

Since V is a vector space, we have $H_r \in V$. By lemma 2.1. we have

$$\|H_r\|_{L^2(\Omega)} = \left\|\sum_{i=1}^{n} r_i v_i\right\|_{L^2(\Omega)} = \left(\sum_{i=1}^{n}(r_i)^2\right)^{\frac{1}{2}} \leq 1.$$

By (4) we have

$$\|H_r\|_{L^\infty(\Omega)} \leq ba\left(1+|\Omega|\right)\|H_r\|_{L^2(\Omega)} \leq ba\left(1+|\Omega|\right).$$

Hence there is a set of $|A_r| = 0$ such that

$$\left|\sum_{i=1}^{n} r_i v_i(x)\right| \leq ba\left(1+|\Omega|\right)$$

for all $x \in \Omega \setminus A_r$. Let A represent the union of the countable collection of all sets E_r taken over all rational numbers $r = (r_1, r_2, \dots, r_n)$ such that $\sum_{i=1}^{n}(r_i)^2 \leq 1$. Then we obtain

$$\left|\sum_{i=1}^{n} r_i v_i(x)\right| \leq ba\left(1+|\Omega|\right) \tag{5}$$

for all $x \in \Omega \setminus A$ and any choice of rational numbers $r = (r_1, r_2, \dots, r_n)$ such that $\sum_{i=1}^{n}(r_i)^2 \leq 1$. By continuity, inequality (5) is valid for every real numbers $r = (r_1, r_2, \dots, r_n)$ such that $\sum_{i=1}^{n}(r_i)^2 \leq 1$. But, for any x satisfying this relation, we must have

$$\sum_{i=1}^{n} \left| v_i(x) \right|^2 \leq \left(ba\left(1+|\Omega|\right) \right)^2 . \tag{6}$$

Moreover, this inequality is valid a.e. on Ω. From (6) we have

$$\int_\Omega \sum_{i=1}^{n} \left| v_i(x) \right|^2 dx \leq \int_\Omega \left(ba\left(1+|\Omega|\right) \right)^2 dx.$$

Then, by taking integration, we obtain $n \leq \left(ba\left(1+|\Omega|\right) \right)^2 |\Omega|$. This complete the proof.

Author's Contributions

This paper entirely belongs to me.

Statement of Conflicts of Interest

No potential conflict of interest was reported by the author.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

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