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Kählerian Structure on the Product of Two Trans-Sasakian Manifolds

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Abstract

It's shown that for some changes of metrics and structural tensors, the product of two trans-Sasakian manifolds is a Kählerian manifold. This gives new positive answer and more generally to Blair-Oubiña's open question. (See [7] and [17]). Concrete examples are given.

Keywords: *Trans-Sasakian manifolds; Kählerian manifolds; product manifolds.*
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1. Introduction

On the product of two almost contact manifolds, A. Morimoto [11] defined a natural almost complex structure (see (4.2) in this paper) and proved that this almost complex structure is integrable if and only if the two factors are normal almost contact manifolds. Later, M. Capursi [8] investigated almost Hermitian geometry of the product of two almost contact metric manifolds with the product metric, with respect to the almost complex structure defined by Morimoto. He shows that this product is Hermitian, Kählerian, almost Kählerian or nearly Kählerian, if and only if, the two factors are normal, cosymplectic, almost cosymplectic or nearly cosymplectic respectively.

Extending ideas from Capursi and Morimoto, Blair and Oubiña [7] considered conformal and related changes of the product metric with respect to a family of almost complex structures (see relation (3.1)) containing the one of Morimoto. Under the Kähler condition on the product manifold, Blair and Oubina proved that if one factor is Sasakian, the other is not, but that locally the second factor is of the type studied by Kenmotsu. The resuls are more general and given in terms of trans-Sasakian, α-Sasakian and β-Kenmotsu structures, finally they asked the open question: What kind of change of the product metric will make both factors Sasakian?

In [18], Watanabe survey almost Hermitian, Kähler, almost quaternionic Hermitian and quaternionic Kähler structures, naturally constructed on products of manifolds with almost contact metric and Sasakian structures and open intervals, as an application of these constructions. Next, he investigated almost Hermitian structures, naturally defined on the product manifolds of two almost contact metric and Sasakian manifolds, and asked some problems related to these topics.

In the same direction, Özdemir and al. [14], gave some properties that each factor should satisfy to make the almost Hermitian structure on the product manifold in a certain class of almost Hermitian manifolds.

Recently, in [2], we introduced the notion of generalized doubly D-homothetic bi-warping. we gave an application to some questions of the characterization of certain geometric structures. Our work has supported the point of view of the Calabi-Eckmann manifolds that the almost Hermitian structures defined on the product of two Sasakian manifolds are never Kählerian.

Here, again we based on the open question of Blair-Oubiña (see [7],[18]), but with a new techniques which requires a change in the two directions, metrics and structural tensors of the two Trans-Sasakian manifolds,

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which gave a positive response to the question see theorem (5.1)(main theorem).

This paper is organized in the following way:
Section 2, is devoted to the background of the structures which will be used in the sequel. In Section 3, we introduce a new deformation of almost contact metric structure and we give some geometric properties. Section 3 is devoted to the construction of a class of interesting examples in dimension 3. In the last section, we focus on our main goal where we construct Kählerian manifold using the product of two Trans-Sasakian manifolds with a concrete example.

2. Review Of Needed Notions

An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold. For an almost Hermitian manifold (M, J, g) we thus have

$$J^2 = -1, \qquad g(JX, JY) = g(X, Y), \tag{2.1}$$

where X and Y denote arbitrary vector fields on M.
An almost complex structure J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_j vanishes, with

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y].$$

For an almost Hermitian manifold (M, J, g), we define the fundamental Kähler form Ω as

$$\Omega(X, Y) = g(X, JY).$$

(M, J, g) is then called almost Kähler if Ω is closed i.e. $\mathrm{d}\Omega = 0$ where d denotes the exterior derivative.
An almost Kähler manifold with integrable J is called a Kähler manifold, and thus is characterized by the conditions: $\mathrm{d}\Omega = 0$ and $N_J = 0$.
One can prove that both these conditions combined are equivalent with the single condition

$$\nabla J = 0.$$

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1, 1)$ tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi \quad and \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector fields X, Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.
Such a manifold is said to be a contact metric manifold if $\mathrm{d}\eta = \phi$, where $\phi(X, Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M.
On the other hand, the almost contact metric structure of M is said to be normal if

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) + 2\mathrm{d}\eta(X, Y)\xi = 0, \tag{2.3}$$

for any X and Y vector fields on M, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

An almost contact metric structures (φ, ξ, η, g) on M is said to be:

$$\begin{cases} (a) : Sasaki \;\Leftrightarrow\; \phi = \mathrm{d}\eta \; and \; (\varphi, \xi, \eta) \; is \; normal, \\ (b) : Cosymplectic \;\Leftrightarrow\; \mathrm{d}\phi = \mathrm{d}\eta = 0 \; and \; (\varphi, \xi, \eta) \; is \; normal, \\ (c) : Kenmotsu \;\Leftrightarrow\; \mathrm{d}\eta = 0, \; \mathrm{d}\phi = 2\phi \wedge \eta \;\; and \; (\varphi, \xi, \eta) \; is \; normal. \end{cases} \tag{2.4}$$

These manifolds can be characterized through their Levi-Civita connection, by requiring

$$\begin{cases} (1) : Sasaki \;\Leftrightarrow\; (\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \\ (2) : Cosymplectic \;\Leftrightarrow\; \nabla\varphi = 0, \\ (3) : Kenmotsu \;\Leftrightarrow\; (\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X. \end{cases} \tag{2.5}$$

In [15], the author proves that (φ, ξ, η, g) is trans-Sasakian structure if and only if it is normal and

$$\mathrm{d}\eta = \alpha\phi, \qquad \mathrm{d}\phi = 2\beta\eta \wedge \phi, \tag{2.6}$$

where $\alpha = \frac{1}{2n}\delta\phi(\xi)$, $\beta = \frac{1}{2n}div\xi$ and δ is the codifferential of g.
It is well known that the trans-Sasakian condition may be expressed as an almost contact metric structure satisfying

$$(\nabla_X\varphi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) + \beta\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big). \tag{2.7}$$

From this formula one easily obtains

$$\nabla_X\xi = -\alpha\varphi X - \beta\varphi^2 X, \tag{2.8}$$

$$(\nabla_X\eta)Y = \alpha g(X, \varphi Y) + \beta g(\varphi X, \varphi Y). \tag{2.9}$$

It is clear that a trans-Sasakian manifold of type $(1, 0)$ is a Sasakian manifold and a trans-Sasakian manifold of type $(0, 1)$ is a Kenmotsu manifold. A trans-Sasakian manifold of type $(0, 0)$ is called a cosymplectic manifold.
From [9] we know that for a 3-dimensional trans-Sasakian manifold we have

$$2\alpha\beta + \xi(\alpha) = 0. \tag{2.10}$$

The relation between trans-Sasakian, α-Sasakian and, β-Kenmotsu structures was discussed by Marrero [10].

Proposition 2.1. *(Marrero [10]) A trans-Sasakian manifold of dimension ≥ 5 is either α-Sasakian, β-Kenmotsu or cosymplectic.*

Proposition 2.2. *(Marrero [10]) Let M be a 3-dimensional Sasakian manifold with structure tensors (φ, ξ, η, g), $f > 0$ a non-constant function on M and $\overline{g} = fg + (1-f)\eta\otimes\eta$. Then $(\varphi, \xi, \eta, \overline{g})$ is a trans-Sasakian structure of type $\big(\frac{1}{f}, \frac{1}{2}\xi(\ln f)\big)$.*

For more background on these manifolds, we recommend the references [4, 5, 19].

3. Deformation of trans-Sasakian manifolds

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. Here, we will introduce a new deformation of almost contact metric structures where we deform the metric g and the structural tensor φ simultaneously.
For all X and Y vector fields on M, we consider a change of structure tensors of the form

$$\tilde{\varphi}X = \varphi X + \theta(\varphi X)\xi, \quad \tilde{\xi} = \xi, \quad \tilde{\eta} = \eta - \theta, \quad \tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = fg(\varphi X, \varphi Y), \tag{3.1}$$

where θ is a 1-form orthogonal to η and f a positive function on M.
Note that the metric $\tilde{g}$ can be written as

$$\begin{aligned} \tilde{g}(X,Y) &= fg(\varphi X, \varphi Y) - \tilde{\eta}(X)\tilde{\eta}(Y) \\ &= fg(X,Y) + (1-f)\eta(X)\eta(Y) - \eta(X)\theta(Y) - \theta(X)\eta(Y) + \theta(X)\theta(Y). \end{aligned}$$

Proposition 3.1. *The structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an almost contact metric structure.*

Proof. The proof follows by a routine calculation, just using (2.2). □

In particular, if $\theta = 0$ then we get

$$\tilde{g} = fg + (1-f)\eta\otimes\eta,$$

and this deformation was studied by Marrero [10].

Remark 3.1. In this new deformation we required the orthogonality between θ and η. But, If we canceled this condition and we took $\theta = (1-h)\eta$ and $\tilde{\xi} = \frac{1}{h}\xi$ with h a non zero function on M, we get

$$\tilde{g} = fg + (h^2 - f)\eta\otimes\eta.$$

This deformation appeared in [1] and [13]. In addition, if $f = 1$ then we have D-isometric deformation [3], but for $h = f$ we get the deformation of Blair [6] and for $h = f = a$ where a is a positive constant we obtain D-homothetic deformation [17].

The above remark indicates that this deformation is more general than other deformations in the literature.

We denote by $\tilde{N}^{(1)}$ the tensor field of type $(1,2)$ on M defined for all X and Y vector fields on M by

$$\tilde{N}^{(1)}(X,Y) = [\tilde{\varphi},\tilde{\varphi}](X,Y) + 2\mathrm{d}\tilde{\eta}(X,Y)\xi,$$

where

$$[\tilde{\varphi},\tilde{\varphi}](X,Y) = \tilde{\varphi}^2[X,Y] + [\tilde{\varphi}X,\tilde{\varphi}Y] - \tilde{\varphi}[\tilde{\varphi}X,Y] - \tilde{\varphi}[X,\tilde{\varphi}Y].$$

By long direct calculation, using formulas (3.1) one can get

$$\begin{aligned}\tilde{N}^{(1)}(X,Y) &= N^{(1)}(X,Y) + \theta\big(N^{(1)}(X,Y)\big)\xi \\ &- \theta(\varphi X)\Big(N^{(3)}(Y) + \theta\big(N^{(3)}(Y)\big)\xi\Big) - \theta(\varphi Y)\Big(N^{(3)}(X) + \theta\big(N^{(3)}(X)\big)\xi\Big) \\ &+ 2\mathrm{d}\theta\big(\tilde{\varphi}X,\tilde{\varphi}Y\big)\xi - 2\mathrm{d}\theta(X,Y)\xi, \end{aligned} \tag{3.2}$$

with $N^{(3)}$ is a tensor fields on M given by

$$N^{(3)}(X) = \big(L_\xi\varphi)(X) = \varphi[X,\xi] - [\varphi X,\xi],$$

where L_ξ denotes the Lie derivative with respect to the vector field ξ.

Proposition 3.2. *Let (φ,ξ,η,g) be a normal almost contact metric structure on M. The almost contact metric structure $(\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is normal if and only if*

$$\mathrm{d}\theta\big(\varphi X,\varphi Y\big) = \mathrm{d}\theta(X,Y).$$

Proof. Firstly, we have

$$N^{(1)}(X,Y) = 0 \Rightarrow N^{(1)}(\varphi X,\xi) = [\xi,\varphi X] - \varphi[\xi,X] = N^{(3)}(X) = 0.$$

So, If (φ,ξ,η,g) is normal then from (3.2), we obtain

$$\tilde{N}^{(1)}(X,Y) = 2\mathrm{d}\theta\big(\tilde{\varphi}X,\tilde{\varphi}Y\big)\xi - 2\mathrm{d}\theta(X,Y)\xi. \tag{3.3}$$

Suppose that

$$\mathrm{d}\theta\big(\varphi X,\varphi Y\big) = \mathrm{d}\theta(X,Y).$$

For $Y=\xi$ we get for all X vector field on M,

$$\mathrm{d}\theta(X,\xi) = 0. \tag{3.4}$$

Applying (3.4) and (2.2) in (3.3) we obtain $\tilde{N}^{(1)}(X,Y) = 0$.
For the inverse, suppose that $\tilde{N}^{(1)} = 0$ and taking $Y=\xi$ we obtain for all X vector field on M,

$$\mathrm{d}\theta(X,\xi) = 0. \tag{3.5}$$

Applying (3.5) in (3.3) we get

$$\mathrm{d}\theta\big(\varphi X,\varphi Y\big) = \mathrm{d}\theta(X,Y).$$

□

Through the rest of this paper, we consider $(M^{2n+1},\varphi,\xi,\eta,g)$ is a trans-Sasakian manifold of type (α,β) i.e., we have

$$\mathrm{d}\eta = \alpha\phi, \qquad \mathrm{d}\phi = 2\beta\eta\wedge\phi.$$

Theorem 3.1. *Let (M,φ,ξ,η,g) be a Trans-Sasakian manifold of type (α,β). If*

$$\mathrm{d}\theta = \mathrm{d}\eta \qquad \textit{and} \qquad \mathrm{d}(\ln f) = -2\beta\eta$$

then, $(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ define in (3.1) is a cosymplectic manifold.

Proof. According to the formulas (2.4), $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is cosymplectic manifold if and only if

$$\mathrm{d}\tilde{\eta} = \mathrm{d}\tilde{\phi} = 0 \quad and \quad (\tilde{\varphi}, \tilde{\xi}, \tilde{\eta})\ is\ normal.$$

Firstly, we have $\tilde{\eta} = \eta - \theta$ which implies $\mathrm{d}\tilde{\eta} = 0$. Secondly, for all X and Y vector fields on M, the fundamental 2-form $\tilde{\phi}$ of $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is

$$\tilde{\phi}(X, Y) = \tilde{g}(X, \tilde{\varphi}Y),$$

we can check that is very simply as follows:

$$\tilde{\phi} = f\phi. \tag{3.6}$$

Since $\mathrm{d}(\ln f) = -2\beta\eta$ and $\mathrm{d}\phi = 2\beta\eta \wedge \phi$, then

$$\begin{aligned} \mathrm{d}\tilde{\phi} &= \mathrm{d}f \wedge \phi + f\mathrm{d}\phi \\ &= f\big(\mathrm{d}(\ln f) + 2\beta\eta\big) \wedge \phi \\ &= 0. \end{aligned}$$

Finally, we have

$$\mathrm{d}\theta = \mathrm{d}\eta = \alpha\phi \Rightarrow \mathrm{d}\theta\big(\varphi X, \varphi Y\big) = \mathrm{d}\theta(X, Y),$$

according to the proposition 3.2 the structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$ is normal . This completes the proof of the theorem. □

Corollary 3.1. *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional Trans-Sasakian manifold of type (α, β). If*

$$\mathrm{d}\theta = \mathrm{d}\eta \qquad and \qquad \xi(\ln f) = -2\beta$$

then, $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ define in (3.1) is a cosymplectic manifold.

Proof. Let $\{e_0 = \xi, e_1, e_2\}$ be the frame of vector fields and $\{\theta^0 = \eta, \theta^1, \theta^2\}$ be the dual frame of differential 1-forms on M. Then,

$$\phi = 2\theta^2 \wedge \theta^1,$$

and

$$\mathrm{d}(\ln f) = \xi(\ln f)\eta + \theta^1(\ln f)e_1 + \theta^2(\ln f)e_2.$$

Thus

$$\mathrm{d}(\ln f) \wedge \phi = \xi(\ln f)\eta \wedge \phi.$$

Then, from the theorem 3.1, we obtain

$$\xi(\ln f) = -2\beta.$$

□

4. A class of examples

For this construction, we rely on our example in [2]. We denote the Cartesian coordinates in a 3-dimensional Euclidean space $\mathbb{R}^3$ by (x, y, z) and define a symmetric tensor field g by

$$g = \begin{pmatrix} \rho^2 + \tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix},$$

where ρ and τ are functions on $\mathbb{R}^3$ such that $\rho \neq 0$ everywhere.
Further, we define an almost contact metric (φ, ξ, η) on $\mathbb{R}^3$ by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\tau & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \eta = (-\tau, 0, 1).$$

The fundamental 1-form η and the 2-forme ϕ have the forms,

$$\eta = dz - \tau dx \qquad and \qquad \phi = -2\rho^2 dx \wedge dy,$$

and hence

$$\begin{aligned} \mathrm{d}\eta &= \tau_2\, dx \wedge dy + \tau_3\, dx \wedge dz, \\ \mathrm{d}\phi &= -4\rho_3\rho\, dx \wedge dy \wedge dz, \end{aligned}$$

where $\rho_i = \frac{\partial \rho}{\partial x_i}$ and $\tau_i = \frac{\partial \tau}{\partial x_i}$.

Knowing that the components of the Nijenhuis tensor N_φ (2.3) can be written as,

$$N^i_{kj} = \varphi^l_k(\partial_l \varphi^i_j - \partial_j \varphi^i_l) - \varphi^l_j(\partial_l \varphi^i_k - \partial_k \varphi^i_l) + \eta_k(\partial_j \xi^i) - \eta_j(\partial_k \xi^i),$$

where the indices i, j, k and l run over the range $1, 2, 3$, then by a direct computation we can verify that

$$N^i_{kj} = 0, \qquad \forall i, j, k.$$

implying that the structure (φ, ξ, η, g) is normal. From (2.6), the structure (φ, ξ, η, g) is a Trans-Sasakian when

$$\tau_2 = -2\alpha\rho^2, \qquad \rho_3 = \beta\rho \qquad and \qquad \tau_3 = 0.$$

Knowing that θ is a 1-form orthogonal to η, i.e. $\theta(\xi) = 0$ then θ has the following form

$$\theta = a\, dx + b\, dy,$$

where a and b are two functions on $\mathbb{R}^3$. According to Corollary 3.1, the structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is cosymplectic if

$$\mathrm{d}\theta = \mathrm{d}\eta \qquad and \qquad \xi(\ln f) = -2\beta,$$

which give

$$a_2 - b_1 = -2\alpha\rho^2 \qquad and \qquad \frac{f_3}{f} = -2\beta.$$

Under these informations and using formulas (3.1), one can get

$$\tilde{g} = \begin{pmatrix} f\rho^2 + (a+\tau)^2 & b(a+\tau) & -(a+\tau) \\ b(a+\tau) & f\rho^2 + b^2 & -b \\ -(a+\tau) & -b & 1 \end{pmatrix}$$

and

$$\tilde{\varphi} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ b & -(a+\tau) & 0 \end{pmatrix}, \quad \tilde{\xi} = \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \tilde{\eta} = \big(-(a+\tau), -b, 1 \big).$$

with $\tilde{\phi} = -2dx \wedge dy$. Since $\mathrm{d}\tilde{\phi} = \mathrm{d}\tilde{\eta} = 0$ and $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$ is normal then, $(\mathbb{R}^3, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a cosymplectic manifold.
Finally, we propose some concrete cases:
1) : $\rho = \mathrm{e}^z, \quad \tau = -2y, \quad f = \alpha = \mathrm{e}^{-2z}, \quad \beta = 1, \quad \theta = ydx - xdy.$
2) : $\rho = z, \quad \tau = -2y, \quad f = \alpha = \frac{1}{z^2}, \quad \beta = \frac{1}{z}, \quad \theta = -2xdy.$
3) : $\rho = x, \quad \tau = -2x^2 y, \quad f = 1, \quad \alpha = 1, \quad \beta = 0, \quad \theta = -x^2 dy.$
4) : $\rho = \mathrm{e}^z, \quad \tau = \mathrm{e}^x, \quad f = \mathrm{e}^{-2z}, \quad \alpha = 0, \quad \beta = 1, \quad \theta = ydx - xdy.$
We can construct further examples on $\mathbb{R}^3$ by the similar way.

5. Kählerian structure on the product of two Trans-Sasakian structures

It's natural to research for some conditions to construct a Kählerian manifold on the product of two Sasakian manifolds or more generally two Trans-Sasakian manifolds.

Is very well known that the product metric of two cosymplectic manifolds is Kählerian (see [8]). According to the previous section, if we consider $(M_1,\varphi_1,\xi_1,\eta_1,g_1)$ and $(M_2,\varphi_2,\xi_2,\eta_2,g_2)$ two Trans-Sasakian manifolds then, $(M_1,\tilde\varphi_1,\tilde\xi_1,\tilde\eta_1,\tilde g_1)$ and $(M_2,\tilde\varphi_2,\tilde\xi_2,\tilde\eta_2,\tilde g_2)$ are two cosymplectic manifolds where

$$\tilde\varphi_1 X=\varphi_1 X+\theta_1(\varphi_1 X)\xi_1,\qquad \tilde\xi_1=\xi_1,\qquad \tilde\eta_1=\eta_1-\theta_1,$$
$$\tilde g_1(\tilde\varphi_1 X,\tilde\varphi_1 Y)=f_1 g_1(\varphi_1 X,\varphi_1 Y)$$

and

$$\tilde\varphi_2 X=\varphi_2 X+\theta_2(\varphi_2 X)\xi_2,\qquad \tilde\xi_2=\xi_2,\qquad \tilde\eta_2=\eta_2-\theta_2,$$
$$\tilde g_2(\tilde\varphi_2 X,\tilde\varphi_2 Y)=f_2 g_2(\varphi_2 X,\varphi_2 Y),$$

with θ_1 and θ_2 are two 1-forms on M_1 and M_2 respectively such that

$$\mathrm{d}\theta_1=\mathrm{d}\eta_1\qquad and\qquad \mathrm{d}\theta_2=\mathrm{d}\eta_2.$$

Define an almost complex structure J on $M_1\times M_2$ by

$$J(X_1,X_2)=\big(\tilde\varphi_1 X_1-\tilde\eta_2(X_2)\tilde\xi_1\ ,\ \tilde\varphi_2 X_2+\tilde\eta_1(X_1)\tilde\xi_2\big),\tag{5.1}$$

and a Riemannian metric g on $M_1\times M_2$ defined by

$$g\big((X_1,X_2),(Y_1,Y_2)\big)=\tilde g_1(X_1,Y_1)+\tilde g_2(X_2,Y_2).\tag{5.2}$$

Using formulas (2.1), one can prove that $(M=M_1\times M_2,g,J)$ is an almost Hermitian manifold.

Therefore, summing up the arguments above, we have the following main theorem:

Theorem 5.1. *Let $(M_1,\varphi_1,\xi_1,\eta_1,g_1)$ and $(M_2,\varphi_2,\xi_2,\eta_2,g_2)$ be two Trans-Sasakian manifolds. Then $M=M_1\times M_2$ equiped with the alomst Hermitian structure (J,g) defined by (5.1) and (5.2) is a Kählerian manifold.*

Example 5.1. For this construction, we will use the two structures (1) and (2) from the previous class of examples. The first Trans-Sasakian structure is given by

$$g_1=\begin{pmatrix} e^{2z}+4y^2 & 0 & 2y\\ 0 & e^{2z} & 0\\ 2y & 0 & 1\end{pmatrix},\ \varphi_1=\begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 2y & 0\end{pmatrix},\ \xi_1=\begin{pmatrix}0\\0\\1\end{pmatrix},\quad \eta_1=(2y,0,1).$$

The structure cosymplectic extracted by the deformation is:

$$\tilde g_1=\begin{pmatrix} 1+y^2 & xy & y\\ xy & 1+x^2 & x\\ y & x & 1\end{pmatrix},\ \tilde\varphi_1=\begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ -x & y & 0\end{pmatrix},\ \tilde\xi_1=\begin{pmatrix}0\\0\\1\end{pmatrix},\quad \tilde\eta_1=(y,x,1).$$

For the second Trans-Sasakian manifold $(\mathbb{R}^3,\varphi_2,\xi_2,\eta_2,g_2)$ we denote the Cartesian coordinates in $\mathbb{R}^3$ by (u,v,w) and define the Sasakian structure $(\varphi_2,\xi_2,\eta_2,g_2)$ by

$$g_2=\begin{pmatrix} w^2+4v^2 & 0 & 2v\\ 0 & w^2 & 0\\ 2v & 0 & 1\end{pmatrix},\quad \varphi_2=\begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 2v & 0\end{pmatrix},$$
$$\xi_2=\begin{pmatrix}0\\0\\1\end{pmatrix},\qquad \eta_2=(2v,0,1).$$

The second cosymplectic structure extracted is:

$$\tilde g_2=\begin{pmatrix} 1+4v^2 & 4uv & 2v\\ 4uv & 1+4u^2 & 2u\\ 2v & 2u & 1\end{pmatrix},\quad \tilde\varphi_2=\begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ -2u & 2v & 0\end{pmatrix},$$

$$\tilde{\xi}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \tilde{\eta}_2 = (2v, 2u, 1).$$

Now, we use the product of the two Trans-Sasakian manifolds $(\mathbb{R}^3, \varphi_1, \xi_1, \eta_1, g_1)$ and $(\mathbb{R}^3, \varphi_2, \xi_2, \eta_2, g_2)$ and using (5.1) and (5.2) with a straightforward computation we can get the associated matrices of g and J on $\mathbb{R}^6$:

$$g = \begin{pmatrix} 1+y^2 & xy & y & 0 & 0 & 0 \\ xy & 1+x^2 & x & 0 & 0 & 0 \\ y & x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & w^2+4v^2 & 0 & 2v \\ 0 & 0 & 0 & 0 & w^2 & 0 \\ 0 & 0 & 0 & 2v & 0 & 1 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -x & y & 0 & -2v & -2u & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ y & x & 1 & -2u & 2v & 0 \end{pmatrix}.$$

Knowing that the components of ∇J (2.3) can be written as,

$$(\nabla_{\partial_i} J)\partial_j = \sum_{m=1}^{6} \partial_i J^j_m \partial_m + \sum_{a,m=1}^{6} J^j_a \Gamma^m_{ia} \partial_m - \sum_{k,m=1}^{6} J^k_m \Gamma^k_{ij} \partial_m,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, then by a direct computation we can verify that

$$(\nabla_{\partial_i} J)\partial_j = 0, \qquad \forall i, j, k.$$

implying that $(\mathbb{R}^6, J, g)$ is a Kählerian manifold.

Remark 5.1. The case where $(M_1, \varphi_1, \xi_1, \eta_1, g_1)$ and $(M_2, \varphi_2, \xi_2, \eta_2, g_2)$ are two Sasakian manifolds, ie for $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 0$, is a positive answer for Blair-Oubi$\tilde{n}a$'s open question (see [7], [18]).

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