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Research Article

A REVERSE HÖLDER INEQUALITY IN $L^{p(x)}(\Omega)$

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Abstract: *In this study, at first we provide a general overview of* $L^{p(x)}(\Omega)$ *spaces, also known as variable exponent Lebesgue spaces. They are a generalization of classical Lebesgue spaces* L^p *in the sense that constant exponent replaced by a measurable function. Then, based on classical Lebesgue space approach we prove a reverse of Hölder inequality in* $L^{p(x)}(\Omega)$. *Therefore, our proof in variable exponent Lebesgue space is very similar to that in classical Lebesgue space.*

Keywords: *Variable exponent Lebesgue space, measure, Radon–Nikodym derivative*

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1. Introduction

Variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ are certain cases of Orlicz–Musielak spaces, and at this point of view investigation of $L^{p(x)}(\Omega)$ date back to Hudzik [1] and Musielak [2]. But historically a paper by W**.** Orlicz can be considered as the originating paper in this field [3]. These important spaces are also known as generalized Lebesgue spaces. Since $L^{p(x)}(\Omega)$ space is a natural generalization of the classical $L^p(\Omega)$ space , therefore, the first question which comes to mind is: what types of properties $L^p(\Omega)$ space can be transferred to $L^{p(x)}(\Omega)$ space? Variable exponent Lebesgue spaces have found applications in many areas of mathematics**,** physics and differential equations. To name few of those application**s** areas: modeling electrorheological fluids, image restoration, the calculus of variations, the analysis of quasi-Newtonian fluids, partial differential equations, fluid flow in porous media, For various and concrete applications of these spaces we refer to [4-9]. For further, and more detailed properties of $L^{p(x)}$ spaces we refer to [10-12]. Next we introduce some notations**,** present some fundamental definitions and recall some basic results of $L^{p(x)}$ spaces. In this paper, a variable exponent function means a measurable bounded function such that $p(.):\Omega\to[1,\infty)$. p^+ and p^- notations stands for

$$p^+ = ess\sup\{p(x): x\in\Omega\},\quad p^- = ess\ \inf\{p(x): x\in\Omega\}.$$

We give modular functional $\rho_{p(x)}(\phi)=\rho(\phi): L^{p(x)}(\Omega)\to\mathbb{R}$ such that

$$\rho(\phi)=\int_{\Omega}|\phi(x)|^{p(x)}dx.$$

The space $L^{p(x)}(\Omega)$ is defined in the following way:

$$L^{p(x)}(\Omega)=\left\{\phi:\Omega\to\mathbb{R}\text{ measurable : }\int_{\Omega}|\phi(x)|^{p(x)}dx<\infty\text{ holds}\right\}.$$

Then $L^{p(x)}(\Omega)$ is a Banach space under the Luxemburg norm

$$\|\phi\|_{L^{p(x)}(\Omega)}=\|\phi\|_{p(x)}=\inf\left\{\omega>0:\int_{\Omega}\left|\frac{\phi(x)}{\omega}\right|^{p(x)}dx\leq 1\right\}. \quad (1)$$

If $p^{+}<\infty$, then in $L^{p(x)}(\Omega)$ space, the following inequality estimates a strong relationship between the modular functional and the norm

$$\min\left\{(\rho(\phi))^{\frac{1}{p^{+}}},(\rho(\phi))^{\frac{1}{p^{-}}}\right\}\leq\|\phi\|_{p(x)}\leq\max\left\{(\rho(\phi))^{\frac{1}{p^{+}}},(\rho(\phi))^{\frac{1}{p^{-}}}\right\}.$$

If $p(x)=p$ (constant) for all $x\in\Omega$, then space $L^{p(x)}(\Omega)$ agree with the classical Lebesgue space $L^{p}(\Omega)$ and these two norm values are equal. The topology of the function space $L^{p(x)}(\Omega)$ supplied with the norm (1) is equivalent to the topology of modular ρ convergence if and only if $p^{+}<\infty$. Notion of conjugate exponent from the classical case can be generalized to variable case by the similar formula

$$\frac{1}{p(x)}+\frac{1}{q(x)}=1$$

For any measurable function $\phi\in L^{p(.)}(\Omega)$ and $\varphi\in L^{q(.)}(\Omega)$ the Hölder like inequality

$$\int_{\Omega}\phi(x)\phi(x)dx\leq\beta\|\phi(x)\|_{p(x)}\|\varphi(x)\|_{q(x)}$$

holds.

We use the sign $|\Omega|$ to indicate the Lebesgue measure of a set $\Omega\subset\mathbb{R}^{n}$. Following shows us that when the exponent $p(x)$ is bounded then almost every $x\in\mathbb{R}^{n}$ is a Lebesgue point. This is shown in [13]. For $x\in\mathbb{R}^{n}$ and $t>0$, $B(x,t)$ stand for the open ball having center x and radius t. Let $p^{+}<\infty$. $\phi\in L^{p(x)}(\mathbb{R}^{n})$ then

$$\lim_{t\to 0}\frac{1}{|B(x,\ t)|}\int_{B(x,\ t)}|\phi(y)-\phi(x)|^{p(y)}dy=0$$

for almost every $x\in\mathbb{R}^n$.

For two functions ϕ and φ defined in $\mathbb{R}^n$, the convolution of ϕ and φ, denoted by $\phi*\varphi$, given by the formula:

$$\phi*\varphi(x)=\int_{\mathbb{R}^n}\phi(x-y)\varphi(y)\,dy$$

A useful inequality for convolution is Young's inequality. The Young's inequality is not true with full generality in $L^{p(\cdot)}$:

$$\|\phi*\varphi\|_{p(x)}\le\alpha\|\phi\|_{p(x)}\|\varphi\|_1$$

the inequality is valid if and only if $p(x)$ is constant.

2. Methods

Since the result that we wanted to prove was proved in general measure space rather than Lebesgue measure in classical Lebesgue spaces , we also state and prove our result in general measure space. Thus, by means of classical the $L^p(\Omega)$ approach we prove the following theorem.

Lemma 2.1. Let (X,M,ν) be a σ- finite measure space such that $\nu(X)=\infty$. Then there exists a measurable function $\psi\notin L^1(X,M,\nu)$ and $\psi\in L^{q(x)}(X,M,\nu)$ for all measurable variable exponent $q(x)$ satisfies $q^->1$ and $q^+<\infty$ conditions.

Proof. There exists disjoint sets $A_1,\ A_2,\ A_3,\ldots$ in M such that $1\le\nu(A_k)<\infty$ for each k and $X=\bigcup_{k=1}^{\infty}A_k$. Define $\psi(x)=\dfrac{1}{k.\nu(A_k)}$ on each A_k.

Now we have

$$\int_X|\psi|d\nu=\sum_{k=1}^{\infty}\int_{A_k}|\psi|d\nu=\sum_{k=1}^{\infty}\frac{1}{k}=\infty$$

This means that $\psi\notin L^1(X,M,\mu)$.

For $1<q^-\le q(x)\le q^+<\infty$, also we have

$$\int_X |\psi|^{q(.)} d\nu = \sum_{k=1}^{\infty} \int_{A_k} |\psi|^{q(.)} d\nu$$
$$= \sum_{k=1}^{\infty} \int_{A_k} \frac{1}{k^{q(.)}.\left[\nu(A_k)\right]^{q(.)}} d\nu$$
$$\le \sum_{k=1}^{\infty} \int_{A_k} \frac{1}{k^{q^-}.\nu(A_k)} d\nu$$
$$\le \sum_{k=1}^{\infty} \frac{1}{k^{q^-}} < \infty$$

This means that $\psi \in L^{q(.)}(X, M, \nu)$.

3. Results

Theorem 3.1. Let (X, M, μ) be a σ - finite measure space such that $\mu(X) = \infty$. Assume a measurable variable exponent $p(x)$ satisfies $p^- > 1$, $p^+ < \infty$ conditions and $|\phi|$ is finite μ - a.e. on X. If $\phi\varphi \in L^1(X, M, \mu)$ for each $\varphi \in L^{q(x)}(X, M, \mu)$ then $\phi \in L^{p(x)}(X, M, \mu)$.

Proof. By the method of contradiction, let us assume the opposite, namely that $\phi \notin L^{p(.)}(X, M, \mu)$. Let us, now, obtain a new measure on (X, M) as follows

$$\nu(A) = \int_A |\phi|^{p(.)} d\mu \quad \text{for } A \in M.$$

Then ν is also a σ - finite measure due to (X, M, μ), σ - finite and $|\phi|^{p(.)}$ finite a.e., μ -on X. Also, by Radon–Nikodym derivative we have $dv = |f|^{p(.)} d\mu$. It is important to be aware $\nu(X) = \int_X |\phi|^{p(.)} d\mu = \infty$, since we assume $\phi \notin L^{p(.)}(X, M, \mu)$. By Lemma 2.1. there is a measurable function ψ satisfying $\psi \notin L^1(X, M, \nu)$ and $\psi \in L^{q(.)}(X, M, \nu)$. Let us consider a function φ on X as follows, $\varphi = \psi |\phi|^{p(.)-1}$.

$$\int_X |\varphi|^{q(.)} d\mu = \int_X |\psi|^{q(.)} |\phi|^{p(.)} d\mu = \int_X |\psi|^{q(.)} dv < \infty$$

This gives us $\varphi \in L^q(X, M, \mu)$. We have also

$$\int_X |\phi\varphi| d\mu = \int_X |\psi| |\phi|^{p(.)} d\mu = \int_X |\psi| d\nu = \infty$$

This gives us $\varphi\phi \notin L^1(X, M, \mu)$. Hence our assumption led to a contradiction, since we have assumed $\varphi\phi \in L^1(X, M, \mu)$, and thus ϕ must be an element of $L^{p(.)}(X, M, \mu)$ space.

4. Discussion

By applying the classical methods of constant case, we obtained a reverse of Hölder inequality in $L^{p(x)}(\Omega)$ space. However, the case $p^{-}=1$ still remain open in this context.

The compliance to Research and Publication Ethics: This work was carried out by obeying research and ethics rules.

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