On Bicomplex Jacobsthal-Lucas Numbers

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Article Info

Abstract

In this study we introduced a sequence of bicomplex numbers whose coefficients are chosen from the sequence of Jacobsthal-Lucas numbers. We also present some identities about the known some fundamental identities such as the Cassini's, Catalan's and Vajda's identities.

1. Introduction and preliminaries

Quaternionic numbers, defined by Hamilton in 1843, led to the existence of an algebraic structure with all the properties of real and complex numbers, except the property of change of multiplication. There are also many studies on quaternions' coefficients(see, [1],[2],[3]). The recognition and identification of bicomplex numbers was made by James Cockle[4]. Cockle defined a bicomplex number as $b = z_1 + \mathbf{j}z_2$ using the new unit $\mathbf{j}$, which Hamilton described, inspired by the definition of quaternions. In 1892, Segre gave different interpretations of this algebra by studying again on bicomplex numbers algebra[5]. In [6], Price, while dealing with the field properties of bicomplex numbers, some authors [7], [8], [9],[10] made some important studies on bicomplex holomorphic functions. In the last years, some studies have been done on bicomplex numbers. In particular, one can refer to the related sources for some studies made using different representations of these numbers [11], [9]. Furthermore, there are some studies on bicomplex numbers and their algebraic, geometric, topological and dynamic properties. For some of these studies, the reader can look at [12], [13], [4],[14], [15]. Recently, the studies conducted by selecting the coefficients of bicomplex numbers from different sequences have attracted attention. Because it is easier to work with these new sequences created using the properties of the selected a sequence, it is also easier to find application areas in other fields. One of these studies, the coefficients by selecting from the Fibonacci sequence and using idempotent notation, by made Halici[11]. Jacobsthal and Jacobsthal-Lucas sequences have a rich history, especially in view of its relationship to the Fibonacci numbers these are studied by some authors. Cerin, in [13], examined the products of Jacobsthal numbers and gave sum of their squares. A. F. Horadam studied the Jacobsthal representation numbers and Fibonacci quaternions [16],[17]. Szynal-Liana and Włoch worked on Jacobsthal quaternions[18]. Diana, in [19], considered Fibonacci octonions and generalized Fibonacci-Lucas octonions. Jacobsthal and Jacobsthal-Lucas sequences, respectively, are defined by the aid of the following recursive relations[20]: For $n \geq 2$

$$J_n = J_{n-1} + 2J_{n-2}; \quad J_0 = 0, \quad J_1 = 1$$

and

$$j_n = j_{n-1} + 2j_{n-2}; \quad j_0 = 2, \quad j_1 = 1.$$

Moreover, these sequences can be also given by the following formulas: For $n \geq 0$

$$J_n = \frac{1}{3}\left\{2^n - (-1)^n\right\}, \quad j_n = 2^n + (-1)^n.$$

We listed some important identities used some of them:

$$\sum_{i=1}^{\infty} j_i x^{i-1} = (1+4x)(1-x-2x^2)^{-1}.$$

$$j_{n+1} + j_n = 3.2^n, \ n \geq 0.$$

$$j_{n+1} - j_n = 2^n - 2(-1)^n, \ n \geq 0.$$

$$j_{n+r} + j_{n-r} = 2^{n-r}(2^{2r} + 1) + 2(-1)^{n-r}, \ n \geq r.$$

$$j_{n+r} - j_{n-r} = 2^{n-r}(2^{2r} - 1), \ n \geq r.$$

$$j_n = J_{n+1} + 2J_{n-1}.$$

$$j_n J_n = J_{2n}, \ n \geq 0.$$

$$J_m j_n + J_n j_m = 2J_{m+n}; \quad J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}.$$

The set of bicomplex numbers is as follows:

$$BC = \left\{ z_1 + z_2 \mathbf{j} \mid z_1, \ z_2 \in C, \ \mathbf{j}^2 = -1 \right\}.$$

Here, $\mathbf{i}, \mathbf{j}$ are different and commuting imaginary units, $\mathbf{ij} = \mathbf{ji}, \mathbf{ii} = \mathbf{jj} = -1$. Hyperbolic unit $\mathbf{k}$ arises from the multiplication of the two imaginary units $\mathbf{i}$ and $\mathbf{j}$; $\mathbf{ij} = \mathbf{k}$. In the set BC, the addition operation is component-wise and the multiplication operation is done taking into account the multiplication of the base elements.

Now, let's define a new bicomplex sequence whose coefficients are selected from Jacobsthal Lucas sequence. If we denote nth bicomplex Jacobsthal number by jBQ_n, then we can write it as follows: For $n \geq 0$

$$jBQ_n = j_n + \mathbf{i} j_{n+1} + \mathbf{j} j_{n+2} + \mathbf{ij} j_{n+3},$$

where j_n is nth Jacobsthal-Lucas number. Let us write the set of such numbers as follows:

$$BC_j = \left\{ j_{C_n} + j_{C_{n+2}} \mathbf{j} \mid j_{C_n} = j_n + \mathbf{i} j_{n+1}, j_{C_{n+2}} = j_{n+2} + \mathbf{i} j_{n+3}, \mathbf{j}^2 = -1 \right\}.$$

Then one can write

$$BC_j = \{ jBQ_0, jBQ_1, jBQ_2, \ldots, jBQ_n, \ldots \}.$$

Notice that the nth element of this sequence satisfy the following recurrence relation

$$jBQ_n = jBQ_{n-1} + 2jBQ_{n-2}.$$

For any two elements jBQ_n, jBQ_m the algebraic operations are as follows:

$$jBQ_n + jBQ_m = (j_n + j_m) + (j_{n+1} + j_{m+1})\mathbf{i} + (j_{n+2} + j_{m+2})\mathbf{j} + (j_{n+3} + j_{m+3})\mathbf{ij},$$

$$jBQ_n - jBQ_m = (j_n - j_m) + (j_{n+1} - j_{m+1})\mathbf{i} + (j_{n+2} - j_{m+2})\mathbf{j} + (j_{n+3} - j_{m+3})\mathbf{ij},$$

$$jBQ_n jBQ_m = sc(jBQ_n jBQ_m) + vec(jBQ_n jBQ_m),$$

where the scalar and vectorial parts are follows.

$$sc(jBQ_n jBQ_m) = (j_n j_m - j_{n+1} j_{m+1} - j_{n+2} j_{m+2} + j_{n+3} j_{m+3})$$

$$vec(jBQ_n jBQ_m) = (j_n j_{m+1} + j_{n+1} j_m - j_{n+2} j_{m+3} - j_{n+3} j_{m+2})\mathbf{i}$$

$$+ (j_n j_{m+2} - j_{n+1} j_{m+3} + j_{n+2} j_m - j_{n+3} j_{m+1})\mathbf{j}$$

$$+ (j_n j_{m+3} + j_{n+1} j_{m+2} + j_{n+2} j_{m+1} + j_{(n+3)} j_m)\mathbf{ij}$$

respectively.

2. Main identities

In this section we provide some fundamental identities such as the Cassini's, Catalan's, Vajda's identities. The elements of the sequence $\{jBQ_n\}_{n \geq 0}$ can be seen as the coefficients of the power series of the corresponding generating function.

Now, let us give the generating function for the bicomplex Jacobsthal-Lucas numbers.

Theorem 2.1. *The generating function for the numbers jBQ_n is*

$$G(t) = \frac{jBQ_0(1-t) + jBQ_1 t}{1 - t - 2t^2}.$$

Proof. Suppose $G(t)$ is a generating function for jBQ_n

$$G(t) = \sum_{i=0}^{\infty} jBQ_i t^i.$$

Multiply this function by t and t^2, respectively, to take advantage of the recursive relation. Thus, we write

$$tG(t) = jBQ_0 t + jBQ_1 t^2 + \ldots + jBQ_n t^{n+1} + \ldots.$$

$$t^2 G(t) = jBQ_0 t^2 + jBQ_1 t^3 + \ldots + jBQ_n t^{n+2} + \ldots.$$

Using the characteristic equation and making needed arrangements, we get

$$G(t)(1 - t - 2t^2) = jBQ_0 + t(jBQ_1 - jBQ_0),$$

$$G(t) = \sum_{i=0}^{\infty} jBQ_i t^i$$

which is desired result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 2.2. *For integers n, r such that $n \geq r$, we have*

$$i) \quad jBQ_n + jBQ_{n+1} = 3\underline{\alpha}\alpha^n.$$

$$ii) \quad jBQ_{n+r} - jBQ_{n-r} = \underline{\alpha}(2^{n+r} - 2^{n-r}).$$

Proof. Using the Binet formula the accuracy of the desired equations can be easily seen. $\qquad\qquad\qquad\qquad\qquad\square$

Theorem 2.3. *For the bicomplex Jacobsathal-Lucas sequence jBQ_n, the following equality is then provided:*

$$\sum_{s=1}^{n} jBQ_s = \frac{1}{2}(jBQ_{n+2} - jBQ_2).$$

Proof. In order to prove the claim we will use the following equalities:

$$\{j_n\}_{n \geq 0} = \{2, 1, 5, 7, 17, 31, 65, 127, 257, 511, \ldots\}$$

and

$$\sum_{i=1}^{n} j_i = \frac{j_{n+2} - 5}{2}.$$

So, we have

$$\sum_{s=1}^{n} jBQ_s = \sum_{s=1}^{n} j_s + \mathbf{i} \sum_{s=1}^{n} j_{s+1} + \mathbf{j} \sum_{s=1}^{n} j_{s+2} + \mathbf{ij} \sum_{s=1}^{n} j_{s+3}.$$

This implies that

$$\sum_{s=1}^{n} jBQ_s = \frac{1}{2} \{(j_{n+2} + \mathbf{i}j_{n+3} + \mathbf{j}j_{n+4} + \mathbf{ij}j_{n+5}) - (5 + 7\mathbf{i} + 17\mathbf{j} + 31\mathbf{ij})\}.$$

This proves our result. That is,

$$\sum_{s=1}^{n} jBQ_s = \frac{1}{2}(jBQ_{n+2} - jBQ_2).$$

The following formula gives any element of the sequence BC_j and this formula is known as the Binet formula. $\qquad\qquad\square$

Theorem 2.4. *(Binet's Formula) For every positive integer n, the following equality holds.*

$$jBQ_n = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n$$

where

$$\underline{\alpha} = (1 + 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{ij}) \text{ and } \underline{\beta} = (1 - \mathbf{i} + \mathbf{j} - \mathbf{ij}).$$

Proof. The general term of the sequence $\{jBQ_n\}_{n \geq 0}$ is $jBQ_n = A\alpha^n + B\beta^n$. Using the roots α, β of the characteristic equation associated with the recurrence $jBQ_n = jBQ_{n-1} + 2jBQ_{n-2}$ we get the values A and B as follows:

$$A = \frac{jBQ_1 - jBQ_0\beta}{3}, \quad B = \frac{-jBQ_1 + jBQ_0\alpha}{3}.$$

Writing these values in the equation $jBQ_n = A\alpha^n + B\beta^n$, we have

$$jBQ_n = \frac{jBQ_1 - jBQ_0\beta}{3}\alpha^n + \frac{-jBQ_1 + jBQ_0\alpha}{3}\beta^n,$$

$$jBQ_n = (1 + 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{ij})\alpha^n + (1 - \mathbf{i} + \mathbf{j} - \mathbf{ij})\beta^n$$

which is desired. $\qquad\square$

Theorem 2.5. *For the integers n, m such that $n \geq m$, Catalan's identity is follows:*

$$jBQ_{n+m} jBQ_{n-m} - jBQ_n^2 = \underline{\alpha}\underline{\beta}(\alpha\beta)^{n-m}\left\{(\alpha\beta)^{2m} - 2(\alpha\beta)^m + 1\right\}.$$

Proof. From the Binet formula, we write

$$jBQ_{n+m} jBQ_{n-m} - jBQ_n^2 = (\underline{\alpha}\alpha^{n+m} + \underline{\beta}\beta^{n+m})(\underline{\alpha}\alpha^{n-m} + \underline{\beta}\beta^{n-m}) - (\underline{\alpha}\alpha^n + \underline{\beta}\beta^n)^2.$$

If we use the fact $\underline{\alpha}\underline{\beta} = \underline{\beta}\underline{\alpha}$, then we get

$$jBQ_{n+m} jBQ_{n-m} - jBQ_n^2 = \underline{\alpha}\underline{\beta}(\alpha^{n+m}\beta^{n-m} + \alpha^{n-m}\beta^{n+m} - 2\alpha^n\beta^n).$$

$$jBQ_{n+m} jBQ_{n-m} - jBQ_n^2 = \underline{\alpha}\underline{\beta}(\alpha^n\beta^n)\left\{\frac{\alpha^m}{\beta^m - 1} + \frac{\beta^m}{\alpha^m - 1}\right\}.$$

Hence, after the some calculations we obtain the following equality which is desired result:

$$jBQ_{n+m} jBQ_{n-m} - jBQ_n^2 = \underline{\alpha}\underline{\beta}(\alpha\beta)^{n-m}\left\{(\alpha\beta)^{2m} - 2(\alpha\beta)^m + 1\right\}.$$

$\square$

Theorem 2.6. *For $n \geq 1$, Cassini's identity is follows:*

$$jBQ_{n-1} jBQ_{n+1} - jBQ_n^2 = 9\underline{\alpha}\underline{\beta}(\alpha\beta)^{n-1}.$$

Proof. One can see that $\underline{\alpha}\underline{\beta} = \underline{\beta}\underline{\alpha}$. Also, by considering the multiplication rules of the base elements we write

$$jBQ_{n-1} jBQ_{n+1} - (jBQ_n)^2 = \underline{\alpha}\underline{\beta}(\alpha^{n-1}\beta^{n+1} + \alpha^{n+1}\beta^{n-1} - 2\alpha^n\beta^n).$$

By making the necessary adjustments and calculations, we obtain

$$jBQ_{n-1} jBQ_{n+1} - jBQ_n^2 = 9\underline{\alpha}\underline{\beta}(\alpha\beta)^{n-1}.$$

Thus, we completed the proof. $\square$

Notice that Theorem 2.6 is an immediate consequence of Theorem 2.5. In this case, $m = 1$, Catalan's formula induces the Cassini's formula:

$$jBQ_{n-1} jBQ_{n+1} - jBQ_n^2 = 9\underline{\alpha}\underline{\beta}(\alpha\beta)^{n-1}.$$

Theorem 2.7. *(d'Ocagne Identity). For the elements jBQ_n, we have*

$$jBQ_m jBQ_{n+1} - jBQ_n jBQ_{m+1} = 3\underline{\alpha}\underline{\beta}(\alpha^n\beta^m - \alpha^m\beta^n).$$

Proof. From the Binet's formula, the equality

$$jBQ_m jBQ_{n+1} - jBQ_n jBQ_{m+1}$$

is equal to this:

$$\underline{\alpha}\underline{\beta}\left\{\alpha^m\beta^n(\beta - \alpha) + \alpha^n\beta^m(\alpha - \beta)\right\}.$$

Hence, we get

$$jBQ_m jBQ_{n+1} - jBQ_n jBQ_{m+1} = 3\underline{\alpha}\underline{\beta}(\alpha^n\beta^m - \alpha^m\beta^n).$$

$\square$

3. Conclusion

In this study, we introduced the bicomplex number sequence whose coefficients are chosen from the Jacobsthal-Lucas sequence. We gave algebraic properties of the elements of this sequence and obtained their Binet formula. We also obtained the well known some generalized identities related with these numbers such as Cassini and Catalan identities. It should be note that due their applications in different areas, the study of these numbers can give other properties and applications.

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