Journal of Multidisciplinary Modeling and Optimization 3(2) (2020),70-79.

Bezier Curve Based Smoothing Penalty Function for Constrained Optimization

Ahmet SAHINER[1,a], Nurullah YILMAZ[1,b], Gulden KAPUSUZ[1,c] and Gamze OZKARDAS[1,d]

[1]Deparment of Mathematics,
Suleyman Demirel University, Isparta, TURKEY
[a] ahmetsahiner@sdu.edu.tr
[b] nurullahyilmaz@sdu.edu.tr
[c] guldenkapusuz92@gmail.com
[d] gamzeozkrds@gmail.com

Received: 01.02.2021, Accepted: 07.03.2021, Published: 25.03.2021

Abstract — *In this study, we consider nonlinear inequality constrained optimization problems. We introduce l_1 exact penalty function approach with a new smoothing function based on Bezier curve. Then, we propose a new algorithm by using the differentiation based methods to solve for solving l_1 exact penalty functions. Finally, we apply our algorithm to test problems to demonstrate the effectiveness of the algorithm.*

Keywords: l_1 penalty function, Smoothing, Non-smooth optimization.
Mathematics Subject Classification: 90C30, 65D10, 90C26.

1 Introduction

In this study, we deal with the constrained optimization problem as follows

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f(x) \\ &s.t.\ \ g_i(x) \leq 0, \quad i = 1, 2, ..., m. \end{aligned} \tag{1}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I = \{1, 2, ..., m\}$ are continuously differentiable functions. The set of feasible solution is defined as $G_0 := \{x \in \mathbb{R}^n | g_i(x) \leq 0, i = 1, 2, ...m\})$ and it is assumed that G_0 is not empty.

The penalty function is used in order to transform a constrained problem to an unconstrained one. The following problem is one of the well-known penalty form of problem 1:

$$\min_{x\in\mathbb{R}^n} F_2(x, \rho) = f(x) + \rho \sum_{i=1}^{m} \left(g_i^+(x)\right)^2, \tag{2}$$

Cite as: *A. Sahiner, N. Yilmaz, G. Kapusuz and G. Ozkardas, Bezier curve based smoothing penalty function for constrained optimization, Journal of Multidisciplinary Modeling and Optimization 3(2) (2020),70-79.*

where $\rho > 0$ is a penalty parameter and $g_i^+(x) = \max\{0, g_i(x)\}$, $i \in I$. Clearly, $F_2(x,\rho)$ is continuously differentiable exact penalty function. According to Zangwill [1], an exact penalty function has been defined by

$$\min_{x \in \mathbb{R}^n} F_1(x,\rho) = f(x) + \rho \sum_{i=1}^{m} g_i^+(x). \tag{3}$$

The obvious difficulty in minimization of F_1 is the non-differentiability of F_1 which originates from the presence of "$\max$" operator (when the power of $\max$ is equal 1). The exact l_1 penalty function has been studied by many interesting studies [2, 3]. The penalty approach is used many areas such as academic problems: image processing problems [4], min-max problems [5], PDE constrained control optimization problems [6] and also many engineering problems [7].

One of the most popular way of solving these kind of non-smooth problems is smoothing techniques. The idea of behind the smoothing techniques is based on the approximation to the non-smooth objective function by smooth functions. The degree of approximation is controlled by parameters. The first studies are on smoothing techniques [8, 9, 10, 11, 12]. In order to improve the smoothing approaches, different types of valuable techniques and algorithms are developed [13, 14, 16, 15, 17]. Smoothing techniques are widely used for solving exact penalty functions. The first study is given in [18] and many new studies has been arisen with different smoothing techniques [19, 20, 21, 22, 24, 25, 26]. The smoothing exact penalty functions has been an active research area in recent years [27, 28, 29]

In this paper, we first present a new smoothing function based on Bezier curve. Then, we apply smoothing approach with exact penalty functions and construct the smoothing l_1 exact penalty functions. Finally, we develop a new algorithm by using the differentiation based methods and the implementation of our algorithm to test problems is demonstrated.

2 Preliminaries

Throughout the paper, x^k is denoted as local minimizer and x^* is denoted as the global minimizer. $\mathbb{R}_+$ denote the non-negative real numbers and $\|\cdot\|$ denote the Euclidean norm.

The smoothing function of non-smooth functions is defined by the following definition:

Definition 1. *[30] Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\varepsilon > 0$. The function $\tilde{f} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is called a smoothing function of $f(x)$, if $\tilde{f}(\cdot,\varepsilon)$ is continuously differentiable in $\mathbb{R}^n$ for any fixed ε, and for any $x \in \mathbb{R}^n$,*

$$\lim_{z \to x, \varepsilon \to 0} \tilde{f}(z,\varepsilon) = f(x).$$

The Bezier curve is successfully used for smoothing of the $\min$ operator in [31] to obtain filled function for global optimization. We plan to construct a new smoothing function for penalty problem by the help of Bezier curve. A Bezier curve is defined as follows:

Definition 2. *[32] A Bezier curve of degree n is a parametric curve with control points $P_0, P_1, ..., P_n$, and it is expressed in terms of Bernstein polynomials given by*

$$B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i$$

where the binomial coefficients are

$$\left(\begin{array}{c} n \\ i \end{array}\right) = \begin{cases} \frac{n!}{i!(n-i)!} & if \quad 0 \leq i \leq n \\ 0 & else \end{cases}$$

Therefore, a Bezier curve of degree n is explicitly defined by

$$\beta(t) = \sum_{i=0}^{n} B_i^n(t) P_i, \qquad t \in [0,1].$$

In general, finding exact solution is quite hard task for the complicated constrained optimization problems. Therefore, the approximate solution is useful for these types of problems.The $\varepsilon-$feasible solution for inequality constrained optimization problems is defined as follows:

Definition 3. *[20] Assume $\varepsilon > 0$, a point x_ε is called $\varepsilon-$feasible solution of problem* (1), *if*

$$g_i(x) \leq \varepsilon, \quad i = 1, 2, \ldots, m.$$

3 A New Smoothing Approach Based on Bezier Curve for Exact Penalty Functions

Let us define the $h : \mathbb{R} \to \mathbb{R}$ such that $h(t) = \max\{t, 0\}$. It easy to see that, the function $h(t)$ is re-written as

$$h(t) = t\chi_A(t), \tag{4}$$

where $A = \{t \in \mathbb{R} : t > 0\}$ and $\chi_A : \mathbb{R} \to \mathbb{R}$ is indicator function of a set A defined as

$$\chi_A(t) = \begin{cases} 1 & , t \in A, \\ 0 & , t \notin A. \end{cases}$$

Considering the Eqn. (4), if anyone smooth out the function $\chi_A(t)$, then smoothing function of $h(t)$ is obtained. Therefore, we plan to construct a new smoothing function by the help of Bezier curves. The smoothing function is obtained as follows:

$$\tilde{h}(t, \varepsilon) = t\tilde{\chi_A}(t, \varepsilon),$$

where $\tilde{\chi}_A(x, \varepsilon)$ is the smoothing function of indicator function $\chi_A(t)$ and

$$\tilde{\chi}_A(t, \varepsilon) = \begin{cases} 0 & , t \leq -\varepsilon/2 \\ \dfrac{(t + 0.5\varepsilon)^2}{\varepsilon^3} (3\varepsilon - 2(t + 0.5\varepsilon)) & , -\varepsilon/2 \leq t < \varepsilon/2 \\ 1 & , t \geq \varepsilon/2 \end{cases}$$

It can easily verify that the function $\tilde{h}(t, \varepsilon)$ is continuously differentiable on $\mathbb{R}$.

Lemma 1. *Assume that $\varepsilon > 0$ then*

$$0 \leq h(t) - \tilde{h}(t, \varepsilon) \leq \frac{\varepsilon}{4} \tag{5}$$

for any $t \in \mathbb{R}$.

Proof. Since $\chi_A(t) = \tilde{\chi}_A(t,\varepsilon)$ when $t \notin [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$, it is enough to show that the inequality (5) holds for any $t \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. For $\varepsilon > 0$ we have

$$\begin{aligned} 0 \le h(t) - \tilde{h}(t,\varepsilon) &= t\chi_A(t) - t\tilde{\chi_A}(t,\varepsilon) \\ &\le \frac{\varepsilon}{4}. \end{aligned}$$

It completes the proof. □

It can be concluded from the Lemma 1 that $\tilde{h}(t,\varepsilon) \to h(t)$ as $\varepsilon \to 0$.

By the help of the smoothing and penalty formulation we can construct the following problem

$$\min_{x\in\mathbb{R}^n} \tilde{F}_1\left(x,\varepsilon,\rho\right) \tag{6}$$

instead of the problem given in (3). Here the function $\tilde{F}_1(x,\rho,\varepsilon)$ is defined as

$$\tilde{F}_1(x,\varepsilon,\rho) := f(x) + \rho \sum_{i=1}^{m} \tilde{h}(g_i(x),\varepsilon).$$

Now, we are ready to give the following theoretical results.

Theorem 3.1. *Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$ then,*

$$0 \le F_1(x,\rho) - \tilde{F}_1(x,\varepsilon,\rho) \le \frac{m}{4}\rho\varepsilon. \tag{7}$$

Proof. From Lemma 1 we obtain

$$\begin{aligned} F_1(x,\rho) - \tilde{F}_1(x,\varepsilon,\rho) &= \rho\sum_{i=1}^{m} h(g_i(x)) - \rho\sum_{i=1}^{m}\tilde{h}(g_i(x),\varepsilon) \\ &= \rho\sum_{i=1}^{m}\Big(h(g_i(x)) - \tilde{h}(g_i(x),\varepsilon)\Big) \\ &\le \frac{m}{4}\rho\varepsilon. \end{aligned}$$

□

Theorem 3.2. *Suppose that $\{\varepsilon_j\} \to 0$ and x^j is a solution of* (6) *for any $\rho > 0$. Assume that $\overline{x}$ is an accumulation point of $\{x^j\}$. Then $\overline{x}$ is an optimal solution for* (3).

Proof. The proof is obtained from the Theorem 3.1. □

Theorem 3.3. *Let x^* be an optimal solution for the problem* (3) *and $\overline{x}$ be an optimal solution for the problem* (6). *Then we have the following:*

$$0 \le F_1(x^*,\rho) - \tilde{F}_1(\overline{x},\varepsilon,\rho) \le \frac{m\rho\varepsilon}{4}. \tag{8}$$

Proof. From the Theorem 3.1 we have the following:

$$\begin{aligned} F_1(x^*,\rho)-\tilde{F}_1(x^*,\rho,\varepsilon) &\leq F_1(x^*,\rho)-\tilde{F}_1(\bar{x},\varepsilon,\rho) \\ &\leq F_1(\bar{x},\rho)-\tilde{F}_1(\bar{x},\varepsilon,\rho) \\ &\leq \frac{m\rho\varepsilon}{4}. \end{aligned}$$

□

Theorem 3.4. *Let x^* be an optimal solution for* (3)*, $\overline{x}$ be an optimal solution for* (6) *and let x^* be a feasible solution for (P) and $\overline{x}$ be an $\varepsilon-$ feasible solution for (P), then we have*

$$0 \leq f(x^*)-f(\bar{x}) \leq \frac{m\rho\varepsilon}{2}. \tag{9}$$

Proof. From the Theorem 3.3, we have

$$\begin{aligned} F_1(x^*,\rho)-\tilde{F}_1(\bar{x},\varepsilon,\rho) &= f(x^*)+\rho\sum_{i=1}^{m}h(g_i(x^*))-\left(f(\bar{x})+\rho\sum_{i=1}^{m}\tilde{h}\left(g_i(\bar{x}),\varepsilon\right)\right) \\ &\leq \frac{m\rho\varepsilon}{4} \end{aligned}$$

and since $\sum_{i=1}^{m} h(g_i(x^*))=0$, we obtain

$$\rho\sum_{i=1}^{m}\tilde{h}\left(g_i(\bar{x}),\varepsilon\right) \leq f(\bar{x})-f(x^*) \leq \rho\sum_{i=1}^{m}\tilde{h}\left(g_i(\bar{x}),\varepsilon\right)+\frac{m\rho\varepsilon}{4}.$$

Since $\bar{x}$ is $\varepsilon-$feasible then we have

$$\rho\sum_{i=1}^{m}\tilde{h}\left(g_i(\bar{x}),\varepsilon\right) \leq \frac{m\rho\varepsilon}{4}.$$

Therefore, we obtain

$$0 \leq f(x^*)-f(\bar{x}) \leq \frac{m\rho\varepsilon}{2}.$$

□

4 Algorithm and Numerical Examples

In this section, we first propose an algorithm to solve (6) as follows:

Penalty Function Algorithm (PFA)

Step 1 Choose the initial point x^0. Determine $\varepsilon_0>0$, $\rho_0>0$, $0<\delta<1$, and $M>1$, let $k=0$ and go to Step 2.

Step 2 Use x^k as an initial point to solve (6). Let x^{k+1} be the solution.

Step 3 If x^{k+1} is $\varepsilon-$feasible for (1), then stop and x^{k+1} is the optimal solution. If not, determine $\rho_{k+1}=M\rho_k$, $\varepsilon_{k+1}=\delta\varepsilon_k$ and $k=k+1$, then go to Step 2.

In order to guaranteed that the algorithm is worked straightly, we have to prove the following theorem.

Theorem 4.1. *Assume that the set*

$$\underset{x\in\mathbb{R}^n}{argmin}\ \tilde{F}_1(x,\varepsilon,\rho) \tag{10}$$

is not empty for $\rho \in [\rho_0, \infty)$ *and* $\varepsilon \in (0, \varepsilon_0]$. *Further assume that* x^k *is generated by* PFA *when* $\delta M < 1$. *If* $\{x^k\}$ *has a limit point, then the limit point of* x^k *is the solution for the problem* (1).

Proof. Assume $\overline{x}$ is a limit point of $\{x^k\}$. Then there exists set $K \subset \mathbb{N}$, such that $x^k \to \overline{x}$ for $k \in K$. We have to show that $\overline{x}$ is the optimal solution for (1). Thus, it is sufficient to show (i) $\overline{x} \in G_0$ and (ii) $f(\overline{x}) \leq \inf_{x\in G_0} f(x)$.

i. Let us consider the contrary that $\overline{x} \notin G_0$, i.e. for sufficiently large $k \in K$, there exist $\tau_0 > 0$ and $i_0 \in \{1, 2, \ldots, m\}$ such that

$$g_{i_0}(x^k) \geq \tau_0 > 0.$$

Since x^j is the global minimum according $k-$th values of the parameters ρ_k, ε_k, for any $x \in G_0$ we have

$$\begin{aligned} F_1(x^k,\varepsilon_k,\rho_k) &= f(x^k) + \rho_k(\tau_0 + \frac{\varepsilon_k}{2}) + \frac{(m-1)}{2}\rho_k\varepsilon_k \\ &= f(x^k) + \rho_k\tau_0 + \frac{m}{2}\rho_k\varepsilon_k \\ &\leq f(x) + \frac{m}{2}\rho_k\varepsilon_k. \end{aligned}$$

If $k \to \infty$ then, $\rho \to \infty$, $\rho_k\varepsilon_k \to 0$ and $\rho_k\tau_0 \to \infty$. Thus, $f(x)$ takes infinite values on G_0 and it contradicts with the boundedness of f on G_0.

ii. By considering the Step 2 in *PFA* and for any $x \in G_0$,

$$\tilde{F}_1(x^k,\varepsilon_k,\rho_k) \leq \tilde{F}_1(x,\varepsilon_k,\rho_k) = f(x) + \frac{1}{4}m\rho_k\varepsilon_k$$

When $k \to \infty$, we have $f(\overline{x}) \leq f(x)$.

□

Now we are ready to apply *PFA* to numerical examples. The *PFA* is programmed in Matlab R2016A. For these tables we use some symbols in order to abbreviate the expressions. The symbols are described as follows:

$Iter$	:	The total number of iterations.
Obj	:	The value of solution minimum point x^*.
$C.val$	:	The maximum value of error value for constraints.
$Time$	:	The total time in seconds.

We consider the 4 different test problems which are given in details [26].

Problem 1. *Let us consider the Example in [19]*

$$\begin{aligned} \min f(x) &= x_1^2 + x_2^2 - \cos(17x_1) - \cos(17x_2) + 3 \\ \textit{s.t.} \quad & g_1(x) = (x_1 - 2)^2 + x_2^2 - 1.6^2 \leq 0, \\ & g_2(x) = x_1^2 + (x_2 - 3)^2 - 2.7^2 \leq 0, \\ & 0 \leq x_1 \leq 2, \ \ 0 \leq x_2 \leq 2. \end{aligned}$$

The global minimum is obtained at a point $x^* = (0.7254, 0.3993)$ *with the corresponding value* 1.8376.

Problem 2. *Let us consider the example in [21],*

$$\begin{aligned} \min f(x) &= 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3 \\ \textit{s.t.} \quad & x_1^2 + x_2^2 + x_3^2 - 25 = 0, \\ & (x_1 - 5)^2 + x_2^2 + x_3^2 - 25 = 0 \\ & (x_1 - 5)^2 + (x_2 - 5)^2 + (x_3 - 5)^2 - 25 \leq 0 \end{aligned}$$

The global minimum is obtained at a points $x^* = (2.5000, 4.2196, 0.9721)$ *and the value of the point is* 944.2157.

Problem 3. *The Rosen-Suzuki problem in [19]*

$$\begin{aligned} \min f(x) &= x_1^2 + x_2^2 + 2x^3 + x_4^2 - 5x_1 - 21x_3 + 7x_4 \\ \textit{s.t.} \quad & 2x_1^2 + x_2^2 + x_3^2 + 2x_1 + x_2 + x_4 - 5 \leq 0, \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8 \leq 0, \\ & x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10 \leq 0. \end{aligned}$$

In the paper [19], the obtained global value is obtained as -44.23040.

Problem 4. *Let us consider the Example in [21, 18]*

$$\begin{aligned} \min f(x) &= 10x_2 + 2x_3 + x_4 + 3x_3 + 4x_6 \\ \textit{s.t.} \quad & x_1 + x_2 - 10 = 0, \\ & -x_1 + x_3 + x_4 + x_5 = 0, \\ & -x_2 - x_3 + x_5 + x_6 = 0, \\ & 10x_1 - 2x_3 + 3x_4 - 2x_5 - 16 \leq 0, \\ & x_1 + 4x_3 + x_5 - 10 \leq 0, \\ & 0 \leq x_1 \leq 12, \ 0 \leq x_2 \leq 18, \\ & 0 \leq x_3 \leq 5, \ 0 \leq x_4 \leq 12, \\ & 0 \leq x_5 \leq 1, 0 \leq x_6 \leq 16, \end{aligned}$$

In the paper [21], the obtained global minimum value is obtained as 117.000004.

The *PFA* is applied to test problems and the detailed result is presented in Table 1. In Table 1, the total number of function iterations, the value of the objective function at the optimal point, the maximum error values of constraints at the optimal point and the total spending time obtained from our algorithm and competing algorithms have been reported. The numerical results show that our algorithm is present better results among the all algorithms.

Table 1: The numerical results

	PFA				Algorithm I				Algorithm II			
Problem No	Iter	Obj	C.val	Time	Iter	Obj	C.val	Time	Iter	Obj	C.val	Time
1	3	1.8376	−0.0000	0.446089	3	1.8376	−0.0000	0.458735	3	1.8376	−0.0000	0.482673
2	2	944.2156	0.0000	0.345145	4	944.2157	0.0000	0.486354	3	944.2157	0.0000	0.448798
3	3	−44.2338	−0.0000	0.444549	3	−44.2338	−0.0000	0.519692	4	−44.2322	−0.0000	0.552898
4	4	117.0100	0.0000	0.474952	3	117.0182	0.0000	0.795644	3	117.0071	0.0000	0.884352

5 Conclusion

In this study, we propose new smoothing technique based on Bezier Curve for l_1 exact penalty function. We design a new algorithm to solve smoothing penalty expression of the problem (1). We perform some numerical experiments on test problems and obtain satisfactorily results.

Our new smoothing technique needs to tune just one parameter. Thus, it is easy to set the best parameter value in the process of the algorithm. It can be conclude that our approach provide good approximations to this kind of penalty functions. The algorithm is user friendly and effective. It has fast convergence properties in comparing with the other penalty algorithms. Moreover, the numerical results consolidate the efficiency of the algorithm.

Acknowledgments

The authors declare that this work is completely private effort and it has not been supported by any governmental, financial, educational or otherwise establishment.

Conflict of Interest Declaration

The authors declare that there is no conflict of interest statement.

Ethics Committee Approval and Informed Consent

The authors declare that there is no ethics committee approval and/or informed consent statement.

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Ahmet Sahiner, ORCID: https://orcid.org/0000-0002-4945-2476
Nurullah YILMAZ, ORCID: https://orcid.org/0000-0001-6429-7518
Gulden Kapusuz, ORCID: https://orcid.org/0000-0002-6316-1501
Gamze Ozkardas, ORCID: https://orcid.org/0000-0003-4262-0120