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Some Special Matrices with Harmonic Numbers

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Abstract

In this paper, we define a particular $n\times n$ matrix $H=[H_{k_{i,j}}]_{i,j=1}^{n}$ and its Hadamard exponential matrix $e^{\circ H}=[e^{H_{k_{i,j}}}]$, where $k_{i,j}=min(i,j)$ and H_n is the n^{th} harmonic number. Then we investigate the determinants and inverses of these matrices. Moreover, we presented the Euclidean norm and two upper bounds and lower bounds for the spectral norm of these matrices. Finally, we present an illustrative example.

Keywords: *Harmonic number, Spectral norm, Hadamard inverse, determinant.*
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1. Introduction

Matrices and matrix operations play an important role in almost every branch of mathematics, natural and social sciences and engineering. In a general statement, a matrix norm is a map from all $m\times n$ matrices space to $\mathbb{R}$ which satisfies certain properties. The norm of a matrix is a non-negative real number which is a measure of the magnitude of the matrix. It is a measure of how large its elements are. It is a way of determining the "size" of a matrix that is not necessarily related to how many rows or columns the matrix has. There are several different ways of defining a matrix norm but they all share the same certain properties. Let $A=(a_{ij})$ be an $n\times n$ matrix, then the maximum column length norm, denoted by $c_1(.)$, and the maximum row length norm, denoted by $r_1(.)$, are defined as following:

$$c_1(A)=max_j\sqrt{\sum_i |a_{ij}|^2} \quad , \quad r_1(A)=max_i\sqrt{\sum_j |a_{ij}|^2}. \tag{1.1}$$

The ℓ_p norm of A is defined by

$$\|A\|_p=(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^p)^{\frac{1}{p}}. \tag{1.2}$$

For $p=2$, this norm is called Frobenius or Euclidean norm and denoted by $\|A\|_{\mathbb{E}}$.
The spectral norm of A is defined by

$$\|A\|_2=\sqrt{\max_{1\le i\le n}\lambda_i}, \tag{1.3}$$

where λ_i is the eigenvalue of matrix AA^H, here A^H is conjugate transpose of matrix A.
For $m\times n$ matrices $A=(a_{ij})$ and $B=(b_{ij})$, the Hadamard product is defined by $A\circ B=(a_{ij}b_{ij})$ and if $A=B\circ C$, the Hadamard product of B and C satisfies as follow:

$$\|A\|_2\le r_1(B)c_1(C). \tag{1.4}$$

There is a relation between frobenius and spectral norm, that is

$$\frac{1}{\sqrt{n}}\|A\|_{\mathbb{E}}\le\|A\|_2\le\|A\|_{\mathbb{E}}. \tag{1.5}$$

The Hadamard exponential and Hadamard inverse of the matrix $A=(a_{ij})_{m\times n}$ is defined by $e^{\circ A}=(e^{a_{ij}})$ and $A^{\circ(-1)}=(\frac{1}{a_{ij}})$, respectively [2].

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Let us consider a square matrix M as below:

$$M = \begin{bmatrix} K & b \\ b^T & c \end{bmatrix}$$

where K be an $n \times n$ nonsigular matrix and b is an $n \times 1$ matrix, also c is a real number. The inverse of M is

$$M^{-1} = \begin{bmatrix} K^{-1} + \frac{1}{l}K^{-1}bb^TK^{-1} & -\frac{1}{l}K^{-1}b \\ -\frac{1}{l}b^TK^{-1} & c \end{bmatrix}, \tag{1.6}$$

where $l = c - b^TK^{-1}b$, [13].
The n^{th} harmonic number, denoted by H_n, is defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k},$$

where $H_0 = 0$. It has a very long history, the famous Pythagoras of Samos was the first to encounter the harmonic series. In his famous work, J. Stirling (1730) found the asymptotic formula for factorial $n!$ and then L. Euler (1740) used harmonic numbers and introduced the generalized harmonic numbers. The n^{th} harmonic number H_n also can be expressed by Stirling numbers,

$$H_n = \frac{S(n+1,2)}{n!}$$

where $S(n,2)$ is the Stirling number of first kind.
From [3], [4] and [17], we have some interesting properties of harmonic numbers:

$$\sum_{k=1}^{n-1} H_k = nH_n - n, \tag{1.7}$$

$$\sum_{k=1}^{n} H_k^2 = (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n + 2, \tag{1.8}$$

$$\sum_{k=1}^{n} kH_k^2 = \frac{(n-1)(n+2)}{2}H_{n+1}^2 - \frac{n^2-3n-7}{2}H_{n+1} + \frac{n^2-9n-10}{4}. \tag{1.9}$$

For more information about harmonic numbers and some generalizations of it one can see [3]-[5], [17] and [20].

In literature, many authors have investigated some properties of matrices such as inverse, determinants and norms, with entries some well-known special numbers such as Fibonacci and Lucas numbers and harmonic numbers. In [1] the author studied Hadamard exponential Hankel matrix and get some bounds for them. The authors, in [3], investigated spectral norm, determinant and inversion of particular matrices of the form $[H_i]_{i,j=1}^n$ and $[H_{i+j}]_{i,j}^n$ where H_k is the k^{th} harmonic number. Norms of circulant matrices and $r-$Circulant matrices with Hyper harmonic numbers. In [6], the authors determined bounds for the spectral and ℓ_p norm of Cauchy-Hankel matrices. The authors, in [7], established a lower bound and upper bound for the ℓ_p norms of the Khatri-Rao product of Cauchy-Hankel matrix of the form $H_n = [\frac{1}{\frac{1}{2}+(i+j)}]$. The authors, in [13], obtained some bounds for the spectral norm of particular matrix of the form $A = [a^{min(i,j)}]_{i,j=0}^{n-1}$ where a is a real positive number. In [16], the authors investigated some norms of Toeplitz matrices with Fibonacci and Lucas numbers. For more information one can see [1] and [7]-[21].

2. Main Results

In this section, we consider a particular $n \times n$ matrix $H = [H_{k_{ij}}]_{i,j=1}^n$ and its Hadamard exponential matrix $e^{\circ H} = [e^{H_k}]$, where $k_{i,j} = min(i,j)$ and H_n is the n^{th} harmonic number. In other words, these matrices are represented as below:

$$H = [H_{min(i,j)}]_{i,j=1}^n = \begin{bmatrix} H_1 & H_1 & H_1 & \cdots & H_1 \\ H_1 & H_2 & H_2 & \cdots & H_2 \\ H_1 & H_2 & H_3 & \cdots & H_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ H_1 & H_2 & H_3 & \cdots & H_n \end{bmatrix} \tag{2.1}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1+\frac{1}{2} & 1+\frac{1}{2} & \cdots & 1+\frac{1}{2} \\ 1 & 1+\frac{1}{2} & 1+\frac{1}{2}+\frac{1}{3} & \cdots & 1+\frac{1}{2}+\frac{1}{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1+\frac{1}{2} & 1+\frac{1}{2}+\frac{1}{3} & \cdots & 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \end{bmatrix}, \tag{2.2}$$

and

$$e^{\circ H} = [e^{H_{min(i,j)+1}}]_{i,j=1}^{n} = \begin{bmatrix} e^{H_1} & e^{H_1} & e^{H_1} & \cdots & e^{H_1} \\ e^{H_1} & e^{H_2} & e^{H_2} & \cdots & e^{H_2} \\ e^{H_1} & e^{H_2} & e^{H_3} & \cdots & e^{H_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{H_1} & e^{H_2} & e^{H_3} & \cdots & e^{H_n} \end{bmatrix}, \tag{2.3}$$

where H_k is the k^{th} harmonic number. The determinant, inverse and principal minors of these matrices are obtained. Then, we find some upper and lower bounds for the spectral norm of these matrices. Moreover, some other spectacular properties of these matrices are represented.

Theorem 2.1. *Let H be a matrix as in the matrix (2.2), then*

$$det(H) = H_1 \prod_{i=2}^{n} (H_i - H_{i-1}) = \frac{1}{n!}.$$

Proof. By using elementary row operations on the matrix (2.2), we have

$$det(H) = det \begin{bmatrix} H_1 & H_1 & H_1 & \cdots & H_1 \\ 0 & H_2 - H_1 & H_2 - H_1 & \cdots & H_2 - H_1 \\ 0 & 0 & H_3 - H_2 & \cdots & H_3 - H_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & H_n - H_{n-1} \end{bmatrix}.$$

So, we get

$$det(H) = H_2 \prod_{i=2}^{n} (H_{i+1} - H_i) = \prod_{i=2}^{n} \frac{1}{i} = \frac{1}{n!}.$$

□

Theorem 2.2. *Let H be a matrix as in the matrix (2.1), then H is invertible and the inverse of H is a symmetric tridiagonal matrix of the form*

$$H^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -2 & 5 & -3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -3 & 7 & -4 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -4 & 9 & -5 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -(n-1) & 2n-1 & -n \\ 0 & 0 & 0 & \cdots & 0 & 0 & -n & n \end{bmatrix}.$$

Proof. By Theorem 2.1, it is known that H is nonsingular, so it is invertible.
Lets prove the inverse by principle mathematical induction (PMI), on n. It verifies for $n = 2$, i.e.:

$$H = \begin{bmatrix} 1 & 1 \\ 1 & 1 + \frac{1}{2} \end{bmatrix}$$

then we have

$$H^{-1} = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}.$$

Assume that the result provides for n, that is, there exist $H = [H_{k_{ij}}]_{n \times n}$, and $H^{-1} = [H_{k_{ij}}]_{n \times n}^{-1}$. Thus, by taking $b = (H_1, H_2, \cdots, H_n)^T$, $b^T = (H_1, H_2, \cdots, H_n)$ and $c = H_{n+1}$ along with equation (1.6), the proof is completed for $n+1$. So the result is true for each n. □

Theorem 2.3. *Let H be a matrix as the matrix (2.1) then the Euclidean norm of H is*

$$\|H\|_{\mathbb{E}} = \sqrt{(n+1)\left[(n+1)H_{n+1}^2 - (3n+4)H_{n+1} + \frac{7n+6}{2}\right]}.$$

Proof. By definition of the Euclidean norm, we have

$$\|H\|_{\mathbb{E}}^2 = \left[\left(\sum_{i=1}^{n}\sum_{j=1}^{n} |H_{K_{ij}}|^2\right)^{\frac{1}{2}}\right]^2.$$

Thus,

$$\|H\|_{\mathbb{E}}^2 = \sum_{k=1}^{n} (2n - 2k + 1) H_k^2 = (2n+1)\sum_{k=1}^{n} H_k^2 - 2\sum_{k=1}^{n} k H_k^2.$$

$$= (2n+1)\sum_{k=1}^{n} H_k^2 - 2\left(\sum_{k=1}^{n} k H_{k+1}^2 + \sum_{k=1}^{n} H_k^2 - n H_{n+1}^2\right)$$

$$= (2n-1)\sum_{k=1}^{n} H_k^2 - 2\sum_{k=1}^{n} kH_{k+1}^2 + 2nH_{n+1}^2.$$

According to (1.8) and (1.9), we obtain

$$\|H\|_{\mathbb{E}}^2 = (2n-1)\left[(n+1)H_{n+1}^2 - (2n+3)H_{n+1} + (2n+2)\right] + 2nH_{n+1}^2$$
$$-2\left[\frac{(n+2)(n-1)}{2}H_{n+1}^2 - \frac{n^2-3n-7}{2}H_{n+1} + \frac{n^2-9n-10}{4}\right].$$

By some computations, we have

$$\|H\|_{\mathbb{E}}^2 = (n+1)^2H_{n+1}^2 - (3n^2+7n+4)H_{n+1} + \frac{7n^2+13n+6}{2}$$
$$= (n+1)^2H_{n+1}^2 - (3n+4)(n+1)H_{n+1} + \frac{(7n+6)(n+1)}{2}.$$

Thus, by taking the square root of the equation:

$$\|H\|_{\mathbb{E}} = \sqrt{(n+1)\left[(n+1)H_{n+1}^2 - (3n+4)H_{n+1} + \frac{7n+6}{2}\right]}.$$

Thus the proof is completed. □

Corollary 2.4. *Let H be a matrix as in (2.1) Then we have the following upper and lower bounds for the spectral norm of H.*

$$\frac{1}{\sqrt{n}}\sqrt{(n+1)\left[(n+1)H_{n+1}^2 - (3n+4)H_{n+1} + \frac{7n+6}{2}\right]} \leq \|H\|_2$$
$$\leq \sqrt{(n+1)\left[(n+1)H_{n+1}^2 - (3n+4)H_{n+1} + \frac{7n+6}{2}\right]}.$$

Proof. The proof can be seen easily by exploiting the inequality (1.5) and Corollary 2.4. □

Theorem 2.5. *Let H be a matrix as in (2.1) then we have the following upper bound for the spectral norm of H.*

$$\|H\|_2 \leq \sqrt{((n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n+2)(nH_n^2 - (2n+1)H_n + 2n+1)}.$$

Proof. By definition of Hadamard product for matrix H, we have

$$H = \mathfrak{A} \circ \mathfrak{B},$$

where

$$\mathfrak{A} = \begin{bmatrix} H_1 & 1 & 1 & \cdots & 1 \\ H_1 & H_2 & 1 & \cdots & 1 \\ H_1 & H_2 & H_3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \\ H_1 & H_2 & H_3 & \cdots & H_n \end{bmatrix}$$

and

$$\mathfrak{B} = \begin{bmatrix} 1 & H_1 & H_1 & \cdots & H_1 \\ 1 & 1 & H_2 & \cdots & H_2 \\ 1 & 1 & 1 & \cdots & H_3 \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

By definition of maximum row length norm and maximum column length norm, we have

$$r_1(\mathfrak{A}) = max_i\sqrt{\sum_j |a_{ij}|^2}$$
$$= \sqrt{\sum_{i=1}^{n} H_i^2} = \sqrt{(n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n+2},$$
$$c_1(\mathfrak{B}) = max_j\sqrt{\sum_i |b_{ij}|^2}$$
$$= \sqrt{\sum_{i=1}^{n-1} H_i^2 + 1} = \sqrt{nH_n^2 - (2n+1)H_n + 2n+1}.$$

Consequently, by the Hadamard product (1.4), we get

$$\|H\|_2 \leq \sqrt{((n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n+2)(nH_n^2 - (2n+1)H_n + 2n+1)}.$$

□

Theorem 2.6. *Let H be a matrix as the matrix (2.1), then H is a positive definite matrix and all eigenvalues of H are positive.*

Proof. By Theorem 2.1, we know that for each $n \geq 1$, $det(H)$ is positive. So all leading principal minors of H are positive. Thus, by [18], H is positive definite matrix. Consequently, all eigenvalues of H are positive. $\square$

Example 2.7. *Let H be a matrix as in (2.1). Determinants and eigenvalues of H for some values of n are represented in Table 1 and Table 2.*

Table 1: Determinants

n	det(H) (for $n > 2$ is rounded off to four decimal places)
2	0.5
3	0.1667
4	0.0417
5	0.0083
6	0.0014
7	0.0001984

Table 2: Eigenvalues

n	Eigenvalues of H (is rounded off to four decimal places)
2	0.2192, 2.2808
3	0.1292, 0.3333, 3.0708
4	0.0913, 0.1763, 0.4545, 5.6945
5	0.0701, 0.1192, 0.2221, 0.5826, 7.7060
6	0.0567, 0.0890, 0.1450, 0.2685, 0.7164, 9.8745
7	0.0474, 0.0705, 0.1062, 0.1701, 0.3159, 0.8549, 12.1779

Theorem 2.8. *Let H be a matrix as in (2.1) and Δ_n denotes the leading principal minor of H, in exact $\Delta_1 = 1, \Delta_2 = \frac{1}{2!}, \Delta_3 = \frac{1}{3!}, \cdots, \Delta_n = \frac{1}{n!}$, then we have*
i. $\Delta_{n-2} = \frac{n-1}{n}\Delta_{n-1}^2$,
ii. $\Delta_1\Delta_2\Delta_3\cdots\Delta_n = \prod_{k=1}^{n}\frac{1}{k!}$.

Proof. It follows from Theorem 2.1 by direct calculation. $\square$

Theorem 2.9. *Let H be a matrix as in (2.1) then determinant of Hadamard inverse of H is*

$$det(H^{\circ(-1)}) = \prod_{k=2}^{n}\frac{-1}{H_{k-1}(kH_{k-1}+1)}.$$

Proof. Let H be a matrix as in (2.1). Then, by definition of Hadamard inverse, we have

$$H^{\circ(-1)} = \begin{bmatrix} \frac{1}{H_1} & \frac{1}{H_1} & \frac{1}{H_1} & \cdots & \frac{1}{H_1} \\ \frac{1}{H_1} & \frac{1}{H_2} & \frac{1}{H_2} & \cdots & \frac{1}{H_2} \\ \frac{1}{H_1} & \frac{1}{H_2} & \frac{1}{H_3} & \cdots & \frac{1}{H_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{H_1} & \frac{1}{H_2} & \frac{1}{H_3} & \cdots & \frac{1}{H_n} \end{bmatrix}.$$

By using elementary row operations, we get

$$det(H^{\circ(-1)}) = det\begin{bmatrix} \frac{1}{H_1} & \frac{1}{H_1} & \frac{1}{H_1} & \cdots & \frac{1}{H_1} \\ 0 & \frac{1}{H_2}-\frac{1}{H_1} & \frac{1}{H_2}-\frac{1}{H_1} & \cdots & \frac{1}{H_2}-\frac{1}{H_1} \\ 0 & 0 & \frac{1}{H_3}-\frac{1}{H_2} & \cdots & \frac{1}{H_3}-\frac{1}{H_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{H_n}-\frac{1}{H_{n-1}} \end{bmatrix}.$$

So we get

$$\begin{aligned} det(H^{\circ(-1)}) &= \frac{1}{H_1}\prod_{k=2}^{n}\left(\frac{1}{H_k}-\frac{1}{H_{k-1}}\right) = \prod_{k=2}^{n}\left(\frac{H_{k-1}-H_k}{H_kH_{k-1}}\right) \\ &= \prod_{k=2}^{n}\left(\frac{\frac{-1}{k}}{H_kH_{k-1}}\right) = \prod_{k=2}^{n}\frac{-1}{H_{k-1}(kH_{k-1}+1)}. \end{aligned}$$

$\square$

Theorem 2.10. *Let $e^{\circ H}$ be a matrix as in (2.3), then we have*

$$det(e^{\circ H}) = e\prod_{k=2}^{n}(e^{H_k}-e^{H_{k-1}}) = e\prod_{k=2}^{n}e^{H_{k-1}}(\sqrt[k]{e}-1).$$

Proof. The proof of the theorem can be done by following the similar steps of Theorem 2.1. ◻

Theorem 2.11. *Let $e^{\circ H}$ be a matrix as in (2.3), then $e^{\circ H}$ is invertible and the inverse of $e^{\circ H}$ is $(e^{\circ H})^{-1}=$*

$$\begin{bmatrix} \frac{1}{e^{H_1}-e^{\frac{1}{2}}} & \frac{1}{e^{H_1}-e^{H_2}} & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{e^{H_1}-e^{H_2}} & \mu_2 & \frac{1}{e^{H_2}-e^{H_3}} & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{e^{H_2}-e^{H_3}} & \ddots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \ddots & \frac{1}{e^{H_{n-2}}-e^{H_{n-1}}} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{e^{H_{n-2}}-e^{H_{n-1}}} & \mu_{n-1} & \frac{1}{e^{H_{n-1}}-e^{H_n}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{e^{H_{n-1}}-e^{H_n}} & \frac{-1}{e^{H_{n-1}}-e^{H_n}} \end{bmatrix},$$

where

$$\mu_2 = (1-\frac{e^{H_1}}{e^{H_1}-e^{H_2}}-\frac{e^{H_2}}{e^{H_2}-e^{H_3}})\frac{1}{e^{H_2}},$$

and

$$\mu_{n-1} = (1-\frac{e^{H_{n-2}}}{e^{H_{n-2}}-e^{H_{n-1}}}-\frac{e^{H_{n-1}}}{e^{H_{n-1}}-e^{H_n}})\frac{1}{e^{H_{n-1}}},$$

Proof. We can prove this theorem by similar method which is used in Theorem 2.2. ◻

Theorem 2.12. *Let $e^{\circ H}$ be a matrix as the matrix (2.1), then $e^{\circ H}$ is a positive definite matrix and all eigenvalues of $e^{\circ H}$ are positive.*

Proof. From Theorem 2.10, we know that for each $n \geq 1$, the determinant of $e^{\circ H}$ is positive. So all leading principal minors of $e^{\circ H}$ are positive. Thus by [18], $e^{\circ H}$ is positive definite matrix. Consequently, all eigenvalues of $e^{\circ H}$ are positive. ◻

Theorem 2.13. *Assume that $e^{\circ H}$ is a matrix which is given in (2.3). Then,*

$$\|e^{\circ H}\|_{\mathbb{E}} = \sqrt{(2n+1)\sum_{k=1}^{n} e^{2H_k} - 2\sum_{k=1}^{n} k e^{2H_k}}.$$

Proof. The Euclidean norm of $e^{\circ H}$ can be written as

$$\left\|e^{\circ H}\right\|_{\mathbb{E}}^2 = \sum_{k=1}^{n}(2n-2k+1)e^{2H_k} = (2n+1)\sum_{k=1}^{n} e^{2H_k} - 2\sum_{k=1}^{n} k e^{2H_k}.$$

Thus, the proof is clear.

◻

Corollary 2.14. *Suppose that $e^{\circ H}$ is a matrix as in the matrix form (2.3). Then,*

$$\frac{1}{\sqrt{n}}\sqrt{(2n+1)\sum_{k=1}^{n} e^{2H_k} - 2\sum_{k=1}^{n} k e^{2H_k}} \leq \|e^{\circ H}\|_2 \leq \sqrt{(2n+1)\sum_{k=1}^{n} e^{2H_k} - 2\sum_{k=1}^{n} k e^{2H_k}}.$$

Proof. The proof can be seen easily by using theorem above and the inequality (1.5). ◻

Theorem 2.15. *Let $e^{\circ H}$ be a matrix as in (2.3), then we have the following upper bound for the spectral norm of $e^{\circ H}$.*

$$\|e^{\circ H}\|_2 \leq \sqrt{(e^{2H_n}+n-1)(e^{2H_{n+1}}+n-1)}.$$

Proof. By definition of Hadamard product for $e^{\circ H}$, we have

$$e^{\circ H} = \mathfrak{F} \circ \mathfrak{G}$$

where

$$\mathfrak{F} = \begin{bmatrix} e^{H_1} & 1 & 1 & \cdots & 1 \\ e^{H_1} & e^{H_2} & 1 & \cdots & 1 \\ e^{H_1} & e^{H_2} & e^{H_3} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{H_1} & e^{H_2} & e^{H_3} & \cdots & e^{H_n} \end{bmatrix}$$

and

$$\mathfrak{G} = \begin{bmatrix} 1 & e^{H_1} & e^{H_1} & \cdots & e^{H_1} \\ 1 & 1 & e^{H_2} & \cdots & e^{H_2} \\ 1 & 1 & 1 & \cdots & e^{H_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

By definition of maximum row length norm and maximum column length norm we have

$$r_1(\mathfrak{F}) = max_i \sqrt{\sum_j |a_{ij}|^2} = \sqrt{\sum_{k=1}^{n} e^{2H_k}}.$$

Also we have

$$c_1(\mathfrak{G}) = max_j \sqrt{\sum_i |b_{ij}|^2} = \sqrt{\sum_{k=1}^{n-1} e^{2H_k} + 1}.$$

Thus, according to (1.4), we obtain

$$\|e^{\circ H}\|_2 \leq \sqrt{(\sum_{k=1}^{n} e^{2H_k})(\sum_{k=1}^{n-1} e^{2H_k} + 1)}.$$

□

3. Numerical Examples

In this section, we give an illustrative example that we calculate all results for the harmonic 5×5 matrix whose entries are the harmonic numbers.

Example 3.1. *Let H be a matrix as in matrix form (2.1) for $n = 5$. Then, the matrix H is*

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{11}{6} & \frac{11}{6} & \frac{11}{6} \\ 1 & \frac{3}{2} & \frac{11}{6} & \frac{25}{12} & \frac{25}{12} \\ 1 & \frac{3}{2} & \frac{11}{6} & \frac{25}{12} & \frac{137}{60} \end{bmatrix},$$

which can be found from the MATLAB code in Table 2. From the Theorem 2.1, the determinant of H can be calculated as

$$\det(H) = \frac{1}{5!} = \frac{1}{120}.$$

The inverse of the matrix H can be written as

$$H^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 & 0 \\ -2 & 5 & -3 & 0 & 0 \\ 0 & -3 & 7 & -4 & 0 \\ 0 & 0 & -4 & 9 & -5 \\ 0 & 0 & 0 & -5 & 5 \end{bmatrix}.$$

By the light of Theorem 2.3, the Euclidean norm of the matrix H can be calculated as

$$||H||_E \approx 7.7324$$

For the spectral norm of H, the following inequality can be evaluted from Corollary 2.4:

$$1.5465 \leq ||H||_2 \leq 7.7324.$$

Let us define the following two matrix

$$\mathfrak{A} = \begin{bmatrix} H_1 & 1 & 1 & 1 & 1 \\ H_1 & H_2 & 1 & 1 & 1 \\ H_1 & H_2 & H_3 & 1 & 1 \\ H_1 & H_2 & H_3 & H_4 & 1 \\ H_1 & H_2 & H_3 & H_4 & H_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{3}{2} & 1 & 1 & 1 \\ 1 & \frac{3}{2} & \frac{11}{6} & 1 & 1 \\ 1 & \frac{3}{2} & \frac{11}{6} & \frac{25}{12} & 1 \\ 1 & \frac{3}{2} & \frac{17}{10} & \frac{25}{12} & \frac{137}{60} \end{bmatrix}$$

and

$$\mathfrak{B} = \begin{bmatrix} 1 & H_1 & H_1 & H_1 & H_1 \\ 1 & 1 & H_2 & H_2 & H_2 \\ 1 & 1 & 1 & H_3 & H_3 \\ 1 & 1 & 1 & 1 & H_4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 1 & 1 & 1 & \frac{11}{6} & \frac{11}{6} \\ 1 & 1 & 1 & 1 & \frac{25}{12} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The maximum row norm of the matrix $\mathfrak{A}$ is

$$r_1(\mathfrak{A}) = 4.0206$$

and the maximum column norm of the matrix $\mathfrak{B}$ is

$$c_1(\mathfrak{B}) = 3.4571.$$

Then we get

$$||H||_2 \leq 13.8994.$$

The Hadamard inverse of H and

$$H^{\circ(-1)} = \begin{bmatrix} \frac{1}{H_1} & \frac{1}{H_1} & \frac{1}{H_1} & \frac{1}{H_1} & \frac{1}{H_1} \\ \frac{1}{H_1} & \frac{1}{H_2} & \frac{1}{H_2} & \frac{1}{H_2} & \frac{1}{H_2} \\ \frac{1}{H_1} & \frac{1}{H_2} & \frac{1}{H_3} & \frac{1}{H_3} & \frac{1}{H_3} \\ \frac{1}{H_1} & \frac{1}{H_2} & \frac{1}{H_3} & \frac{1}{H_4} & \frac{1}{H_4} \\ \frac{1}{H_1} & \frac{1}{H_2} & \frac{1}{H_3} & \frac{1}{H_4} & \frac{1}{H_5} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 1 & \frac{2}{3} & \frac{6}{11} & \frac{6}{11} & \frac{10}{17} \\ 1 & \frac{2}{3} & \frac{6}{11} & \frac{12}{25} & \frac{60}{107} \\ 1 & \frac{2}{3} & \frac{6}{11} & \frac{12}{25} & \frac{60}{137} \end{bmatrix},$$

respectively. From Theorem 2.9, the determinant of $H^{\circ(-1)}$ is

$$\det(H^{\circ(-1)}) = \frac{5}{44968} \approx 1.1119 \times 10^{-4}.$$

The Hadamard exponential matrix of H is constructed as

$$e^{\circ H} = \begin{bmatrix} e & e & e & e & e \\ e & e^{\frac{3}{2}} & e^{\frac{3}{2}} & e^{\frac{3}{2}} & e^{\frac{3}{2}} \\ e & e^{\frac{3}{2}} & e^{\frac{11}{6}} & e^{\frac{11}{6}} & e^{\frac{11}{6}} \\ e & e^{\frac{3}{2}} & e^{\frac{11}{6}} & e^{\frac{25}{12}} & e^{\frac{25}{12}} \\ e & e^{\frac{3}{2}} & e^{\frac{11}{6}} & e^{\frac{25}{12}} & e^{\frac{137}{60}} \end{bmatrix}.$$

So, the determinant of $e^{\circ H}$ can be calculated from Theorem 2.10

$$\det(e^{\circ H}) \approx 26.8464.$$

The inverse of the matrix $e^{\circ H}$ can be written approximately as follow:

$$(e^{\circ H})^{-1} = \begin{bmatrix} \frac{115}{123} & -\frac{317}{559} & 0 & 0 & 0 \\ -\frac{317}{559} & \frac{371}{328} & -\frac{163}{289} & 0 & 0 \\ 0 & -\frac{163}{289} & \frac{293}{260} & -\frac{85}{151} & 0 \\ 0 & 0 & -\frac{85}{151} & \frac{476}{423} & -\frac{320}{569} \\ 0 & 0 & 0 & -\frac{320}{569} & \frac{320}{569} \end{bmatrix}.$$

By the light of Theorem 2.13, one can calculate the Euclidean norm of the matrix $e^{\circ H}$ as

$$\|e^{\circ H}\|_{\mathbb{E}} \approx 26.1137.$$

For the spectral norm of $e^{\circ H}$, we can write the following inequality from Corollary 2.14:

$$5.2227 \leq \|e^{\circ \mathfrak{P}}\|_2 \leq 26.1137.$$

Let us consider the following matrices

$$\mathfrak{F} = \begin{bmatrix} e^{H_1} & 1 & 1 & 1 & 1 \\ e^{H_1} & e^{H_2} & 1 & 1 & 1 \\ e^{H_1} & e^{H_2} & e^{H_3} & 1 & 1 \\ e^{H_1} & e^{H_2} & e^{H_3} & e^{H_4} & 1 \\ e^{H_1} & e^{H_2} & e^{H_3} & e^{H_4} & e^{H_5} \end{bmatrix} = \begin{bmatrix} e & 1 & 1 & 1 & 1 \\ e & e^{\frac{3}{2}} & 1 & 1 & 1 \\ e & e^{\frac{3}{2}} & e^{\frac{11}{6}} & 1 & 1 \\ e & e^{\frac{3}{2}} & e^{\frac{11}{6}} & e^{\frac{25}{12}} & 1 \\ e & e^{\frac{3}{2}} & e^{\frac{11}{6}} & e^{\frac{25}{12}} & e^{\frac{137}{60}} \end{bmatrix}$$

and

$$\mathfrak{G} = \begin{bmatrix} 1 & e^{H_1} & e^{H_1} & e^{H_1} & e^{H_1} \\ 1 & 1 & e^{H_2} & e^{H_2} & e^{H_2} \\ 1 & 1 & 1 & e^{H_3} & e^{H_3} \\ 1 & 1 & 1 & 1 & e^{H_4} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & e & e & e & e \\ 1 & 1 & e^{\frac{3}{2}} & e^{\frac{3}{2}} & e^{\frac{3}{2}} \\ 1 & 1 & 1 & e^{\frac{11}{6}} & e^{\frac{11}{6}} \\ 1 & 1 & 1 & 1 & e^{\frac{25}{12}} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

From Theorem 2.15, the maximum row norm of the matrix $\mathfrak{F}$ can be found as

$$r_1(\mathfrak{F}) \approx 15.0771$$

and the maximum column norm of the matrix $\mathfrak{G}$ can be calculated as

$$c_1(\mathfrak{G}) \approx 11.4933.$$

Hence, an upper bound for the spectral norm can be found from Theorem 2.15 as follows:

$$\|e^{\circ H}\|_2 \leq 173.2856$$

4. Conclusion

In this paper, we define a particular $n\times n$ matrix $H=[H_{k_{i,j}}]_{i,j=1}^{n}$ and its Hadamard exponential matrix $e^{\circ H}=[e^{H_{k_{i,j}}}]$, where $k_{i,j}=min(i,j)$ and H_n is the n^{th} harmonic number. The determinants and inverses of these matrices are investigated. Moreover Euclidean norm and two upper bounds and lower bounds for the spectral norm of these matrices are represented. Finally, we derive some identities about principal minors of these matrices.

Also, the particular $n\times n$ matrix $H=[H_{k_{i,j}}]_{i,j=1}^{n}$, where $k_{i,j}=min(i,j)$ and H_n is the n^{th} harmonic number, can be calculated by the help of the MATLAB-R2016a code as follows:

```
clc
clear all
n=input('n=?');
syms k
f(k) = 1/k
b = f(1:n)
t=cumsum(b)
for i=1:n
     for j=1:n
         if i==j
              a(i,j)=t(i)
         elseif i<j
             a(i,j)=t(i)
         elseif i>j
             a(i,j)=t(j)
         end
     end
end
disp(a)
```

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