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A New Kumaraswamy Class of Generalized Distributions with Applications to Exponential Model

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Abstract: In this paper, a new class of generalized distributions, so-called the alpha power Kumaraswamy (*AK*) class, is derived, three important classes of distributions are nested by the *AK* class. Some mathematical properties are studied and a parameters estimation method using maximum likelihood (*MLE*) is obtained. A simulation study using bootstrapping approach is applied to study the alpha power Kumaraswamy-exponential (*AKE*) distribution estimators' behavior. A real data set is used to investigate the *AKE* distribution flexibility.
Keywords: the alpha power Kumarasamy distribution; moments; order statistics; maximum likelihood estimation; bootstrapping approach.

1. Introduction

Adding parameters using alpha power transformation (*APT*) is a flexible method depending on the alpha power function, [1], having the following cumulative distribution function (*CDF)* and probability density function (*PDF)* for a continuous random variable *X*, respectively

$$F_{APT}(x)=\begin{cases}\dfrac{\alpha^{F(x)}-1}{\alpha-1};\alpha>0,\alpha\neq 1,\\ F(x);\alpha=1,\end{cases}$$

and

$$f_{APT}(x)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}f(x)\alpha^{F(x)};\alpha>0,\alpha\neq 1,\\ f(x);\alpha=1.\end{cases}$$

Ahmed [2] presented for the first time the *AK* distribution having the following *CDF* and *PDF*, respectively,

$$F_{\alpha K}(x)=\frac{\alpha^{1-\left(1-x^{\beta}\right)^{\theta}}-1}{\alpha-1},0<x<1;\alpha,\beta,\theta>0;\alpha\neq 1, \quad (1)$$

and

$$f_{\alpha K}(x)=\frac{\beta\theta\log\alpha}{\alpha-1}\alpha^{1-\left(1-x^{\beta}\right)^{\theta}}x^{\beta-1}\left(1-x^{\beta}\right)^{\theta-1}. \tag{2}$$

The main object of this manuscript is to derive an extended class of generalized distributions naming the *AK* class of generalized distributions depending on the *AK* distribution, Ahmed [2], as a simple generator, also it aims to study some properties of the class and gives some applications to *AKE* distribution.

The rest of this paper is organized as follows: In section 2, the generalized class is presented. In section 3, some properties are derived. In section 4, the Hazard function is given. In section 5, order statistics are obtained. In Section 6, the *MLE* method is used. In section 7, a simulation study using bootstrapping is performed for the *AKE* distribution. Finally, in Section 8, an application is investigated, practically, for the *AKE* distribution.

2. The New Class of *AK* Distributions

Many classes of generalized distributions are derived based on generating method, Wahed [3], the Kumaraswamy (*KW*) class [4] and [5], the Kummer beta class [6], the McDonald class [7] and [8], the Kumaraswamy- Kumaraswamy (*KW-KW*) class [9] and [10]. Replacing x in (1) with the generalized parent $G(x;\Lambda)$ gives

$$F(x)=\frac{\alpha^{1-\left(1-G^{\beta}(x;\Lambda)\right)^{\theta}}-1}{\alpha-1};\alpha,\beta,\theta>0;\alpha\neq1, \tag{3}$$

differentiating (3) with respect to x yields

$$f(x)=\frac{\beta\theta}{\alpha-1}\left(\log\alpha\right)\alpha^{1-\left(1-G^{\beta}(x;\Lambda)\right)^{\theta}}g(x;\Lambda)\,G^{\beta-1}(x;\Lambda)\left(1-G^{\beta}(x;\Lambda)\right)^{\theta-1}, \tag{4}$$

where $G(x;\Lambda)$ and $g(x;\Lambda)$ are the *CDF* and *pdf* of the parent distribution, Λ is the parameter vector of the parent distribution. When $\alpha=1$, the *AK* class gives Kumaraswamy (*Kw*) class [4] and [5], setting $\theta=1$ gives the alpha power (*AP*) class and setting $\alpha=1$, $\theta=1$ gives the power function (*P*) class [11]. Many distributions can be derived via the class of *AK* as the alpha power Kumaraswamy exponential (*AKE*) distribution, the *CDF* and *PDF* of the *AKE* distribution, respectively, can be given by

$$F(x)=\frac{\alpha^{1-\left[1-(1-e^{-\lambda x})^{\beta}\right]^{\theta}}-1}{\alpha-1};\alpha,\beta,\theta,\lambda>0;\alpha\neq1,$$

and

$$f(x)=\frac{\beta\theta\lambda}{\alpha-1}\left(\log\alpha\right)\alpha^{1-\left(1-(1-e^{-\lambda x})^{\beta}\right)^{\theta}}e^{-\lambda x}\,(1-e^{-\lambda x})^{\beta-1}\left(1-(1-e^{-\lambda x})^{\beta}\right)^{\theta-1},$$

some density function shapes for the *AKE* distribution are given in figure 1.

2.1. The CDF Expansion

Applying the exponential expansion for (3) leads to

$$F(x)=\frac{1}{1-\alpha}\left\{1-\sum_{i=0}^{\infty}\frac{\left[1-\left(1-G^{\beta}(x;\Lambda)\right)^{\theta}\right]^{i}\left(\log\alpha\right)^{i}}{i\,!}\right\},\alpha,\beta,\theta>0;\alpha\neq1,$$

then, using binomial expansion gives

$$F(x)=\frac{1}{1-\alpha}\left\{1-\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i}}{i!}\sum_{j=0}^{i}(-1)^{j}\binom{i}{j}\left(1-G^{\beta}(x;\Lambda)\right)^{\theta j}\right\},$$

replacing $\sum_{i=0}^{\infty}\sum_{j=0}^{i}$ with $\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}$ yields

$$F(x)=\frac{1}{1-\alpha}\left\{1-\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\frac{(\log\alpha)^{i}}{i!}(-1)^{j}\binom{i}{j}\left(1-G^{\beta}(x;\Lambda)\right)^{\theta j}\right\},$$

using binomial expansion, again, leads to

$$F(x)=\frac{1}{1-\alpha}\left\{1-\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{(\log\alpha)^{i}}{i!}\binom{i}{j}\binom{\theta j}{k}G^{\beta k}(x;\Lambda)\right\},$$

where β is an integer, when β is real non integer yields

$$F(x)=\frac{1}{1-\alpha}\left\{1-\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{(\log\alpha)^{i}}{i!}\binom{i}{j}\binom{\theta j}{k}\left[1-\left(1-G(x;\Lambda)\right)\right]^{\beta k}\right\},$$

then, using binomial expansion two times gives

$$F(x)=\frac{1}{1-\alpha}\left\{1-\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{p}(-1)^{j+k+p+q}\frac{(\log\alpha)^{i}}{i!}\binom{i}{j}\binom{\theta j}{k}\binom{\beta k}{p}\binom{p}{q}G^{q}(x;\Lambda)\right\},$$

replacing $\sum_{p=0}^{\infty}\sum_{q=0}^{p}$ with $\sum_{q=0}^{\infty}\sum_{p=q}^{\infty}$ leads to

$$F(x)=\frac{1}{1-\alpha}\left\{1-\sum_{q=0}^{\infty}w_{q}G^{q}(x;\Lambda)\right\}, \tag{5}$$

where

$$w_{q}=\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\sum_{k=0}^{\infty}\sum_{p=q}^{\infty}(-1)^{j+k+p+q}\frac{(\log\alpha)^{i}}{i!}\binom{i}{j}\binom{\theta j}{k}\binom{\beta k}{p}\binom{p}{q}.$$

2.2. The PDF Expansion

Differentiating (5) with respect to x gives

$$f(x)=\frac{1}{1-\alpha}\left\{\sum_{q=1}^{\infty}w_{q}\,q\,G^{q-1}(x;\Lambda)\,g(x;\Lambda)\right\},$$

shifting q leads to

$$f(x)=\frac{1}{1-\alpha}\left\{\sum_{q=0}^{\infty}w_{q+1}(q+1)\,G^{q}(x;\Lambda)\,g(x;\Lambda)\right\},$$

then,

$$f(x)=\sum_{q=0}^{\infty} w_q^{*}\, G^{q}(x;\Lambda)\, g(x;\Lambda), \tag{6}$$

where

$$w_q^{*}=\frac{1}{1-\alpha}\, w_{q+1}(q+1). \tag{7}$$

The Condition for the *PDF* Expansion

since,

$$\sum_{q=0}^{\infty} w_q^{*} \int_{-\infty}^{\infty} G^{q}(x;\Lambda) g(x;\Lambda)dx = 1,$$

then,

$$\sum_{q=0}^{\infty} w_q^{*} \left[\frac{G^{q+1}(x;\Lambda)}{(q+1)}\right]_{-\infty}^{\infty} = 1,$$

hence,

$$\sum_{q=0}^{\infty} \frac{w_q^{*}}{q+1} = 1. \tag{8}$$

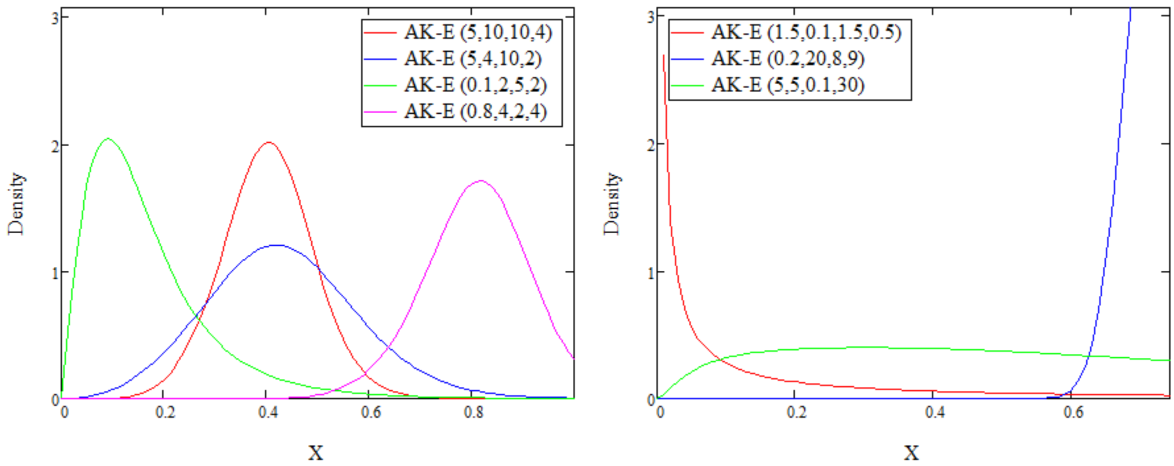

Figure 1: The *AKE* density functions

3. The *AK* Class of Distributions Properties

In this section some properties of the *AK* class of distributions will be given as follows:

3.1. The r-th Moment

Basically, a continuous random variable X has the following *r-th* moment [12]

$$E\left(X^{r}\right)=\int_{x} x^{r} f(x)dx,$$

substituting (6) into last equation yields

$$E\left(X^{r}\right)=\sum_{q=0}^{\infty} w_{q}^{*} \int_{-\infty}^{\infty} x^{r} g(x ; \Lambda) G^{q}(x ; \Lambda) d x, \quad (9)$$

then,

$$E\left(X^{r}\right)=\sum_{q=0}^{\infty} w_{q}^{*} \tau_{r, q, 0}, \quad (10)$$

where τ is the probability weighted moment (*PWM*), Greenwood *et. al.* [13].
Obviously, setting *r=0* and using (9) leads to

$$E\left(X^{0}\right)=\sum_{q=o}^{\infty} w_{q}^{*} \int_{-\infty}^{\infty} g(x ; \Lambda) G^{q}(x ; \Lambda) d x,$$

then,

$$E\left(x^{0}\right)=\sum_{q=o}^{\infty} w_{q}^{*}\left[\frac{G^{q+1}(x ; \Lambda)}{q+1}\right]_{-\infty}^{\infty},$$

substituting (8) into last equation gives

$$E\left(X^{0}\right)=1.$$

Using the Parent Quantile Function

Setting $G(x ; \Lambda)=u$, $x=Q(u)$ and substituting into (9) gives

$$E\left(X^{r}\right)=\sum_{q=o}^{\infty} w_{q}^{*} \int_{0}^{1} Q^{r}(u) u^{q} d u,$$

then,

$$E\left(X^{r}\right)=\sum_{q=o}^{\infty} w_{q}^{*} \tau_{r, q, o}.$$

3.2. The PWM

Basically, the *PWM* of a continuous random variable *X*, Greenwood *et. al.* [13], is given by

$$\tau_{r, s, 0}=\int_{x} x^{r} f(x) F^{s}(x) d x,$$

substituting (5) and (6) into last equation leads to

$$\tau_{r, s, o}=\left(\frac{1}{1-\alpha}\right)^{s} \int_{-\infty}^{\infty} x^{r} g(x ; \Lambda)\left[\sum_{q=o}^{\infty} w_{q}^{*} G^{q}(x ; \Lambda)\right]\left[1-\sum_{q=0}^{\infty} w_{q} G^{q}(x ; \Lambda)\right]^{s} d x,$$

using binomial expansion yields

$$\tau_{r,s,o}=\left(\frac{1}{1-\alpha}\right)^{s}\sum_{k=0}^{s}(-1)^{k}\binom{s}{k}\int_{-\infty}^{\infty}x^{r}g\left(x;\Lambda\right)\left[\sum_{q=o}^{\infty}w_{q}^{*}G^{q}\left(x;\Lambda\right)\right]\left[\sum_{q=0}^{\infty}w_{q}G^{q}(x;\Lambda)\right]^{k}dx,$$

since, $\left[\sum_{q=0}^{\infty}w_{q}G^{q}(x;\Lambda)\right]^{k}=\left[\sum_{q=0}^{\infty}c_{q}G^{q}(x;\Lambda)\right]$, Gradshteyn and Ryzhik [14], then,

$$\tau_{r,s,o}=\left(\frac{1}{1-\alpha}\right)^{s}\sum_{k=0}^{s}(-1)^{k}\binom{s}{k}\int_{-\infty}^{\infty}x^{r}g\left(x;\Lambda\right)\left[\sum_{q=o}^{\infty}w_{q}^{*}G^{q}\left(x;\Lambda\right)\right]\left[\sum_{q=0}^{\infty}c_{q}G^{q}(x;\Lambda)\right]dx,$$

where

$$c_{0}=w_{0}^{k},\ c_{m}=\frac{1}{m\,w_{0}}\sum_{q=1}^{m}(qk-m+q)w_{q}c_{m-q};m\geq 1,$$

since, $\left[\sum_{q=o}^{\infty}w_{q}^{*}G^{q}\left(x;\Lambda\right)\right]\left[\sum_{q=0}^{\infty}c_{q}G^{q}(x;\Lambda)\right]=\sum_{q=o}^{\infty}d_{q}G^{q}\left(x;\Lambda\right)$, Gradshteyn and Ryzhik [14],

hence,

$$\tau_{r,s,o}=\left(\frac{1}{1-\alpha}\right)^{s}\sum_{k=0}^{s}(-1)^{k}\binom{s}{k}\int_{-\infty}^{\infty}x^{r}g\left(x;\Lambda\right)\left[\sum_{q=o}^{\infty}d_{q}G^{q}\left(x;\Lambda\right)\right]dx,$$

where

$$d_{m}=\sum_{q=0}^{m}w_{q}^{*}c_{m-q},$$

since,

$$\tau_{r,s,o}=\sum_{q=o}^{\infty}h_{q}\int_{-\infty}^{\infty}x^{r}g\left(x;\Lambda\right)G^{q}\left(x;\Lambda\right)dx, \tag{11}$$

where

$$h_{q}=\left(\frac{1}{1-\alpha}\right)^{s}\sum_{k=0}^{s}\sum_{q=o}^{\infty}d_{q}\left(-1\right)^{k}\binom{s}{k},$$

then,

$$\tau_{r,s,o}=\sum_{q=o}^{\infty}h_{q}\,\tau_{r,q,o}.$$

Using the Parent Quantile Function

Setting $G\left(x;\Lambda\right)=u$, $y=Q\left(u\right)$ and substituting into (11) yields

$$\tau_{r,s,o}=\sum_{q=o}^{\infty}h_{q}\int_{0}^{1}Q^{r}\left(u\right)u^{q}du,$$

then,

$$\tau_{r,s,o} = \sum_{q=o}^{\infty} h_q \, \tau_{r,q,o} \,.$$

3.3. The Moment Generating Function

A continuous random variable *X* moment generating function (*MGF*) can be written as

$$M_x\left(t\right) = E\left(e^{tx}\right) = \int_x e^{tx} f\left(x\right) dx \,, \qquad (12)$$

applying the exponential expansion yields

$$E\left(e^{tx}\right) = E\left(\sum_{r=0}^{\infty} \frac{t^r x^r}{r!}\right),$$

then,

$$E\left(e^{tx}\right) = \sum_{r=0}^{\infty} \frac{t^r E\left(x^r\right)}{r!},$$

substituting (10) into last equation gives

$$E\left(e^{tx}\right) = \sum_{r=0}^{\infty} \sum_{q=0}^{\infty} \frac{t^r}{r!} w_q^* \tau_{r,q,0} \,.$$

Using the Parent Quantile Function

Substituting (6) into (12) leads to

$$E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} \sum_{q=0}^{\infty} w_q^* \, G^q(x;\Lambda) \, g(x;\Lambda) dx \,,$$

then,

$$E\left(e^{tx}\right) = \sum_{q=0}^{\infty} w_q^* \int_{-\infty}^{\infty} e^{tx} \, G^q(x;\Lambda) g(x;\Lambda) dx \,,$$

setting $G\left(x;\Lambda\right) = u$, $x = Q\left(u\right)$ and substituting into last equation yields

$$E\left(e^{tx}\right) = \sum_{q=o}^{\infty} w_q^* \int_0^1 e^{t\,Q(u)} u^q du \,,$$

using exponential expansion gives

$$E\left(e^{tx}\right) = \sum_{q=o}^{\infty} w_q^* \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^1 Q^r\left(u\right) u^q du \,,$$

hence,

$$E\left(e^{tx}\right)=\sum_{q=o}^{\infty}w_q^*\sum_{r=0}^{\infty}\frac{t^r}{r!}\tau_{r,q,0}.$$

3.4. The Mean Deviation

Basically, a random variable X having the mean deviation about mean and median, respectively, can be written as

$$S_1(x)=\int_x |x-\mu|\, f(x)dx \quad \text{and} \quad S_2(x)=\int_x |x-M|\, f(x)dx,$$

which is given by, Ali Ahmed [15],

$$S_1(x)=2\mu F(\mu)-2t(\mu) \text{ and } S_2(x)=\mu-2t(M),$$

where $T(z)=\int_{-\infty}^{z} x\, f(x)dx$ is the linear incomplete moment.

Substituting (6) into *T(.)* yields

$$T(z)=\int_{-\infty}^{z} x\sum_{q=o}^{\infty}w_q^*\, g(x;\Lambda)G^q(x;\Lambda)dy,$$

then,

$$T(z)=\sum_{q=o}^{\infty}w_q^*\int_{-\infty}^{z} x\, g(x;\Lambda)G^q(x;\Lambda)dy.$$

Using the Parent Quantile Function

Setting $G(x;\Lambda)=u$, $x=Q(u)$ and substituting into last equation gives

$$T(z)=\sum_{q=o}^{\infty}w_p^*\int_{0}^{G(z)} Q(u)u^q du.$$

4. The Hazard Function of the *MLN* Class of Distributions

A random variable X survival function [16] can be written as

$$S(x)=1-F(x),$$

substituting (3) into last equation gives

$$S(x)=\frac{\alpha-\alpha^{1-\left(1-G^{\beta}(x;\Lambda)\right)^{\theta}}}{\alpha-1};\alpha,\beta,\theta>0;\alpha\neq 1, \quad (13)$$

moreover, the Hazard function [16] can be written as

$$H(x)=\frac{f(x)}{S(x)},$$

substituting (4) and (13) into last equation yields

$$H(x)=\frac{\beta\theta(\log\alpha)\alpha^{1-\left(1-G^{\beta}(x;\Lambda)\right)^{\theta}}g(x;\Lambda)\,G^{\beta-1}(x;\Lambda)\left(1-G^{\beta}(x;\Lambda)\right)^{\theta-1}}{\alpha-\alpha^{1-\left(1-G^{\beta}(x;\Lambda)\right)^{\theta}}}.$$

The *AKE* Hazard function can be given by

$$H(x)=\frac{\beta\theta\lambda(\log\alpha)\alpha^{1-\left(1-(1-e^{-\lambda x})^{\beta}\right)^{\theta}}e^{-\lambda x}(1-e^{-\lambda x})^{\beta-1}\left(1-(1-e^{-\lambda x})^{\beta}\right)^{\theta-1}}{\alpha-\alpha^{1-\left(1-(1-e^{-\lambda x})^{\beta}\right)^{\theta}}},$$

some Hazard function shapes for the *AKE* distribution are given in figure 2.

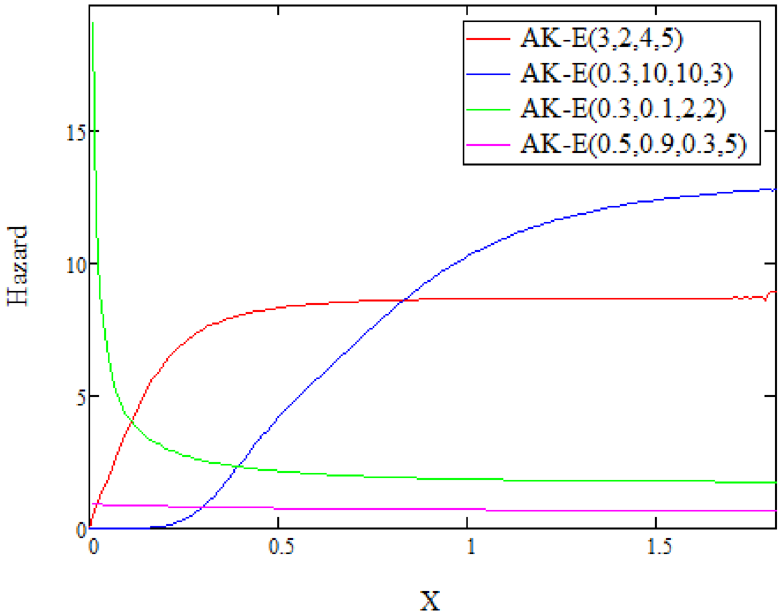

Figure 2: The *AKE* Hazard functions

In figure 2, Hazard functions curves four types for the *AKE* distribution are illustrated as follows: A decreasing then constant Hazard curve, a constant Hazard curve, an increasing then constant Hazard curve and a constant then increasing then constant Hazard curve.

5. Order Statistics of the *AK* Class of Distributions

The *u-th* order statistics density function $f(x_{u:v})$ for $u = 1,2,\ldots, v$ from *iid* random variables $X_1, X_2, \ldots, X_v$ following any *AK* generalized distribution [17] can be written as

$$f(x_{u:v})=\frac{f(x_u)}{B(u,v-u+1)}F^{u-1}(x_u)\left\{1-F(x_u)\right\}^{v-u},$$

applying binomial expansion for the last equation leads to

$$f\left(x_{u,v}\right)=\sum_{p=0}^{v-u}\frac{(-1)^{p}\binom{v-u}{p}}{B\left(u,v-u+1\right)}f\left(x_{u}\right)F^{u+p-1}\left(x_{u}\right),$$

substituting (5) and (6) into last equation leads to

$$f\left(x_{u,v}\right)=\sum_{p=0}^{v-u}\frac{(-1)^{p}\binom{v-u}{p}}{B\left(u,v-u+1\right)}\left[\sum_{q=0}^{\infty}w_{q}^{*}G^{q}(x_{u};\Lambda)g(x_{u};\Lambda)\right]\times\left[\frac{1}{1-\alpha}\left(1-\sum_{q=0}^{\infty}w_{q}G^{q}(x_{u};\Lambda)\right)\right]^{u+p-1},$$

then,

$$f\left(x_{u,v}\right)=\sum_{p=0}^{v-u}\frac{(-1)^{p}\binom{v-u}{p}\left(\frac{1}{1-\alpha}\right)^{u+p-1}}{B\left(u,v-u+1\right)}\left[\sum_{q=0}^{\infty}w_{q}^{*}G^{q}(x_{u};\Lambda)g(x_{u};\Lambda)\right]\left[1-\sum_{q=0}^{\infty}w_{q}G^{q}(x_{u};\Lambda)\right]^{u+p-1},$$

using binomial expansion gives

$$f\left(x_{u,v}\right)=\sum_{p=0}^{v-u}\frac{(-1)^{p}\binom{v-u}{p}\left(\frac{1}{1-\alpha}\right)^{u+p-1}}{B\left(u,v-u+1\right)}\left[\sum_{q=0}^{\infty}w_{q}^{*}G^{q}(x_{u};\Lambda)g(x_{u};\Lambda)\right]$$
$$\times\sum_{s=0}^{u+p-1}(-1)^{s}\binom{u+p-1}{s}\left(\sum_{q=0}^{\infty}w_{q}G^{q}(x_{u};\Lambda)\right)^{s},$$

since, $\left(\sum_{q=0}^{\infty}w_{q}G^{q}(x_{u};\Lambda)\right)^{s}=\left[\sum_{q=0}^{\infty}a_{q}G^{q}(x_{u};\Lambda)\right]$, Gradshteyn and Ryzhik [14], then,

$$f\left(x_{u,v}\right)=\sum_{p=0}^{v-u}\frac{(-1)^{p}\binom{v-u}{p}\left(\frac{1}{1-\alpha}\right)^{u+p-1}}{B\left(u,v-u+1\right)}\left[\sum_{q=0}^{\infty}w_{q}^{*}G^{q}(x_{u};\Lambda)g(x_{u};\Lambda)\right]$$
$$\times\sum_{s=0}^{u+p-1}(-1)^{s}\binom{u+p-1}{s}\left(\sum_{q=0}^{\infty}a_{q}G^{q}(x_{u};\Lambda)\right),$$

where

$$a_{0}=w_{0}^{s},\ a_{m}=\frac{1}{m\,w_{0}}\sum_{q=1}^{m}\left(qs-m+q\right)w_{q}\,a_{m-q};m\geq1,$$

since, $\left(\sum_{q=0}^{\infty}w_{q}^{*}G^{q}(x_{u};\Lambda)\right)\left(\sum_{q=0}^{\infty}a_{q}G^{q}(x_{u};\Lambda)\right)=\left[\sum_{q=0}^{\infty}b_{q}G^{q}(x_{u};\Lambda)\right]$, Gradshteyn and Ryzhik [14],

then,

$$f\left(x_{u,v}\right)=\sum_{p=0}^{v-u}\frac{(-1)^{p}\binom{v-u}{p}\left(\frac{1}{1-\alpha}\right)^{u+p-1}}{B\left(u,v-u+1\right)}\sum_{s=0}^{u+p-1}(-1)^{s}\binom{u+p-1}{s}g(x_{u};\Lambda)\left[\sum_{q=0}^{\infty}b_{q}\,G^{q}(x_{u};\Lambda)\right],$$

where

$$b_m = \sum_{q=0}^{m} w_q^* a_{m-q},$$

hence,

$$f\left(x_{u:v}\right) = \sum_{q=0}^{\infty} d_q\, g(x_u;\Lambda) G^q(x_u;\Lambda), \tag{14}$$

where

$$d_q = \sum_{p=0}^{v-u} \frac{(-1)^p \binom{v-u}{p} \left(\frac{1}{1-\alpha}\right)^{u+p-1}}{B\left(u, v-u+1\right)} \sum_{s=0}^{u+p-1} (-1)^s \binom{u+p-1}{s} b_q.$$

The order statistics *r-th* moment of the *AK* class of distributions can be written as

$$E\left(x_{u:v}^r\right) = \int_{x_u} x_u^r\, f\left(x_{u:v}\right) dx_u,$$

substituting (14) into last equation yields

$$E\left(X_{u:v}^r\right) = \sum_{q=o}^{\infty} d_q \int_{-\infty}^{\infty} X_u^r\, g(x_u;\Lambda) G^q(x_u;\Lambda) dx_u,$$

hence,

$$E\left(X_{u:v}^r\right) = \sum_{q=o}^{\infty} d_q\, \tau_{r,q,o}.$$

6. Estimation for Parameters of the *AK* Class of Distributions Using *MLE* Method

Let $X_1, X_2, \ldots, X_n$ be *iid* random variables following any *AK* generalized distribution $(x;\Delta)$ and the vector of parameter $\Delta = (\alpha, \beta, \theta, \Lambda)$ likelihood function [18] can be written as

$$L(x) = \left(\frac{\beta\theta}{\alpha-1}\right)^n (\log\alpha)^n \prod_{i=1}^{n} g\left(x_i;\Lambda\right) \prod_{i=1}^{n} G^{\beta-1}\left(x_i;\Lambda\right) \prod_{i=1}^{n} \left(1-G^{\beta}\left(x_i;\Lambda\right)\right)^{\theta-1} \prod_{i=1}^{n} \alpha^{1-\left(1-G^{\beta}\left(x_i;\Lambda\right)\right)^{\theta}},$$

the log likelihood function is given by

$$\ell(x) = n\log\frac{\beta\theta}{\alpha-1} + n\log\left(\log\alpha\right) + \sum_{i=1}^{n}\log g\left(x_i;\Lambda\right) + (\beta-1)\sum_{i=1}^{n}\log G\left(x_i;\Lambda\right)$$
$$+ (\theta-1)\sum_{i=1}^{n}\log\left(1-G^{\beta}\left(x_i;\Lambda\right)\right) + \left(\log\alpha\right)\sum_{i=1}^{n}\left[1-\left(1-G^{\beta}\left(x_i;\Lambda\right)\right)^{\theta}\right],$$

the score functions for the parameters α, β, θ and Λ can be obtained by

$$\frac{\partial \ell(x)}{\partial \alpha} = \frac{n}{1-\alpha} + \frac{n}{\alpha \log \alpha} + \frac{1}{\alpha}\sum_{i=1}^{n}\left[1-\left(1-G^{\beta}(x_i;\Lambda)\right)^{\theta}\right],$$

$$\frac{\partial \ell(x)}{\partial \beta} = \frac{n}{\beta} + (\log\alpha)\theta\sum_{i=1}^{n} G^{\beta}(x_i;\Lambda)\left[1-G^{\beta}(x_i;\Lambda)\right]^{\theta-1}\left[\log G(x_i;\Lambda)\right] + \sum_{i=1}^{n}\log G(x_i;\Lambda)$$
$$-(\theta-1)\sum_{i=1}^{n}\frac{\left[G^{\beta}(x_i;\Lambda)\log G(x_i;\Lambda)\right]}{1-G^{\beta}(x_i;\Lambda)},$$

$$\frac{\partial \ell(x)}{\partial \theta} = \frac{n}{\theta} - (\log\alpha)\sum_{i=1}^{n}\left[1-G^{\beta}(x_i;\Lambda)\right]^{\theta}\log\left[1-G^{\beta}(x_i;\Lambda)\right] + \sum_{i=1}^{n}\log\left[1-G^{\beta}(x_i;\Lambda)\right]$$

and

$$\frac{\partial \ell(x)}{\partial \Lambda} = (\log\alpha)\beta\theta\sum_{i=1}^{n}\left[1-G^{\beta}(x_i;\Lambda)\right]^{\theta-1} G^{\beta-1}(x_i;\Lambda)\frac{\partial G(x_i;\Lambda)}{\partial \Lambda} + \sum_{i=0}^{n}\frac{1}{g(x_i;\Lambda)}\frac{\partial g(x_i;\Lambda)}{\partial \Lambda}$$
$$+\sum_{i=0}^{n}\frac{(\beta-1)}{G(x_i;\Lambda)}\frac{\partial G(x_i;\Lambda)}{\partial \Lambda} - \sum_{i=0}^{n}\beta(\theta-1)\frac{G^{\beta-1}(x_i;\Lambda)}{\left(1-G^{\beta}(x_i;\Lambda)\right)}\frac{\partial G(x_i;\Lambda)}{\partial \Lambda}.$$

7. A Simulation Study

In this study, *MLEs* for parameters of the *AKE* distribution are obtained using random numbers to study the *MLEs* finite sample behavior via the bootstrapping resample approach. Obtaining parameters estimates algorithm is detailed in the following steps:

Step (1): Generating a random sample $X_1, X_2, \ldots, X_n$ of sizes *n=(5,15,30,50,100,300)* using the *AKE* distribution.

Step (2): Selecting parameters different set values as: set (1): *(α=2, β=0.5, θ=0.5, λ=0.2),* set (2): *(α=2, β=0.5, θ=0.5, λ=0.5)* and set (3): *(α=2, β=0.5, θ=0.5, λ=2).*

Step (3): Solving normal equations of the *AKE* distribution via iteration to estimate distribution parameters.

Step (4): Replacing set (1), set(2) and set (3) with its estimators and repeating step (3) to compute, biases, *MLEs, RMSE* (the root of mean squared error) and the Pearson type [19] of parameters estimators of the *AKE* distribution.

Step (5): Repeating step (1) to step (4), 10000 times.

In this study, the conjugate gradient iteration method is performed in order to generate random numbers samples using Mathcad package v15. All results are included in tables and indicated in the appendix.

From study results, indicated in the appendix, one can see that: When sample size increases, biases, estimators, and *RMSEs* decrease. As sample size increases, the estimators can be consistent. $\hat{\alpha}$ and $\hat{\lambda}$ sampling distributions can be the Pearson type IV distribution in all times, $\hat{\beta}$ and $\hat{\theta}$ sampling distributions differ according to sample size. When λ increases, for fixed values of α and β, the biases and *MSEs* of $\hat{\alpha}$ and $\hat{\beta}$ decrease.

8. Application

A real data set is given using the *MLE* method to study the new model performance by Mathematica package version 10, some distributions are used as: The *AKE* distribution, the gamma distribution (2 parameters), the normal distribution, the Weibull distribution, the Kumaraswamy exponential (*KE*) distribution, the alpha power – power exponential (*APE*) distribution and the power exponential *(PE)* distribution. The following data set represents the classic lamps lifetime (Hours) for 40 devices, the data are given from the *UK* National Physical Laboratory, http://www.npl.co.uk/,

1.104, 0.285, 0.627, 0.282, 0.542, 0.439, 0.884, 1.021, 0.281, 0.737, 0.504, 0.713, 1.186, 0.380, 0.583, 0.769, 0.334, 1.030, 0.242, 0.386, 0.936, 1.313, 0.263, 0.508, 0.654, 0.481, 0.353, 0.614, 0.736, 0.727, 0.264, 0.528, 0.588, 0.535, 0.590, 0.420, 0.458, 0.278, 0.405, 0.381

Some goodness of fit measures results are indicated in the table (1), the likelihood ratio tests results are indicated in the table (2), the figure (3) describes probability density functions for some distributions having skewness and kurtosis values similar to the *AKE* distribution (the gamma distribution with 2 parameters, the Normal distribution, the Weibull distribution), the figure (4) shows the empirical *CDF* compared to *CDFs* for some distributions (the gamma distribution with 2 parameters, the normal distribution, the Weibull distribution) and the figure (5) describes the probability density functions for special cases from the *AKE* distribution (the Kumaraswamy exponential (*KE*) distribution, the alpha power – power exponential (*APE*) distribution and the power exponential *(PE)* distribution).

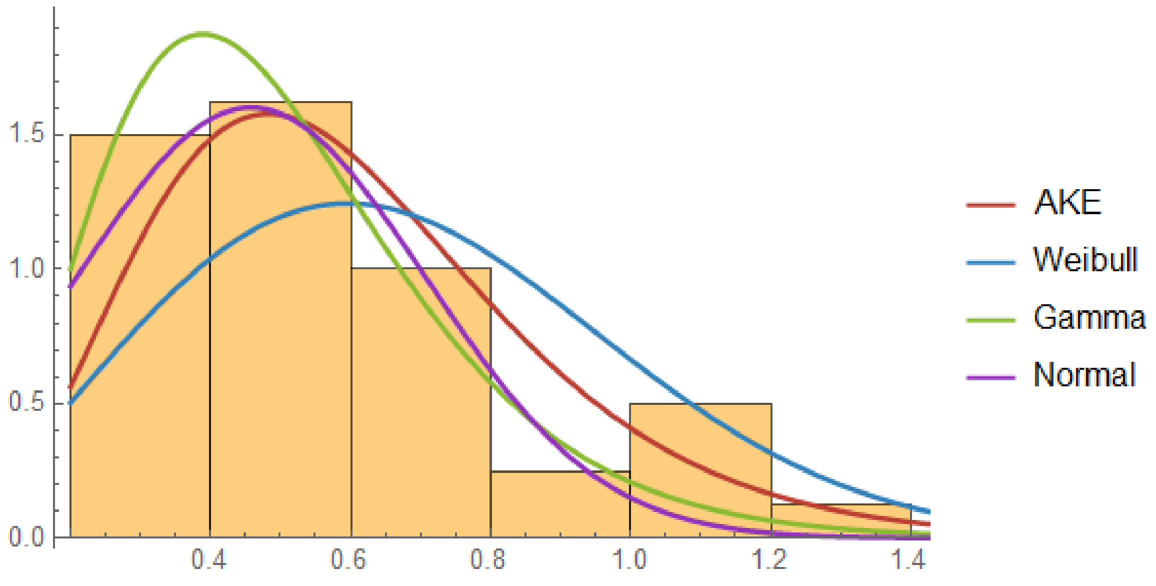

Figure 3: Probability density functions for some distributions having skewness and kurtosis Values similar to *AKE*

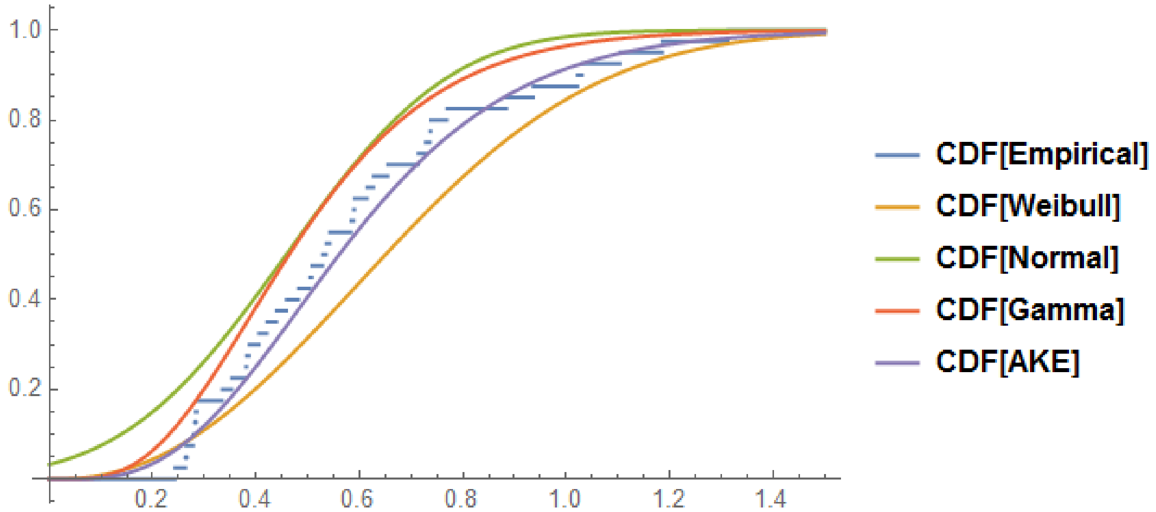

Figure 4: The empirical *CDF* compared to *CDF*s for some distributions

Table 1: The parameter(s) *MLE* and the associated *AIC* and *BIC* values.

Distribution	MLE_parameters				Skewness	Kurtosis	KS	P-value	Log Likel-ihood	AIC	BIC	CAIC
	α	β	θ	λ								
AK-E	2	4	3	2	0.058	2.403	0.08	0.938	-0.555	7.111	8.488	7.777
	(0.013)	(0.033)	(0.247)	(0.182)								
Weibull	2.307	0.762	—	—	0.451	2.961	0.203	0.062	-3.901	11.802	15.180	12.127
	(0.272)	(0.048)										
Gamma	4.503	0.111	—	—	0.942	4.332	0.144	0.338	-2.649	9.299	12.677	9.624
	(1.088)	(0.026)										
Normal	0.458	0.249	—	—	0	3	0.193	0.086	-10.05	24.115	27.492	24.439
	(0.042)	(0.0303)										

In table 1, the distributions parameters *MLE*s, parameters standard error (*SEs*), in parentheses, Kolmogorov-Smirnov (*KS*) test statistic, *AIC* (Akaike Information Criterion), *CAIC* (the consistent Akaike Information Criterion) and *BIC* (Bayesian information criterion), Merovcia and Puka [20], are computed for distributions having similar skewness and kurtosis values. Since, the *AKE* distribution has the smallest *KS*, *AIC*, *CAIC*, *BIC*, *SEs* and the largest p-value, hence the *AKE* distribution can be the best fitted distribution to the data compared with other distributions which have similar skewness and kurtosis values.

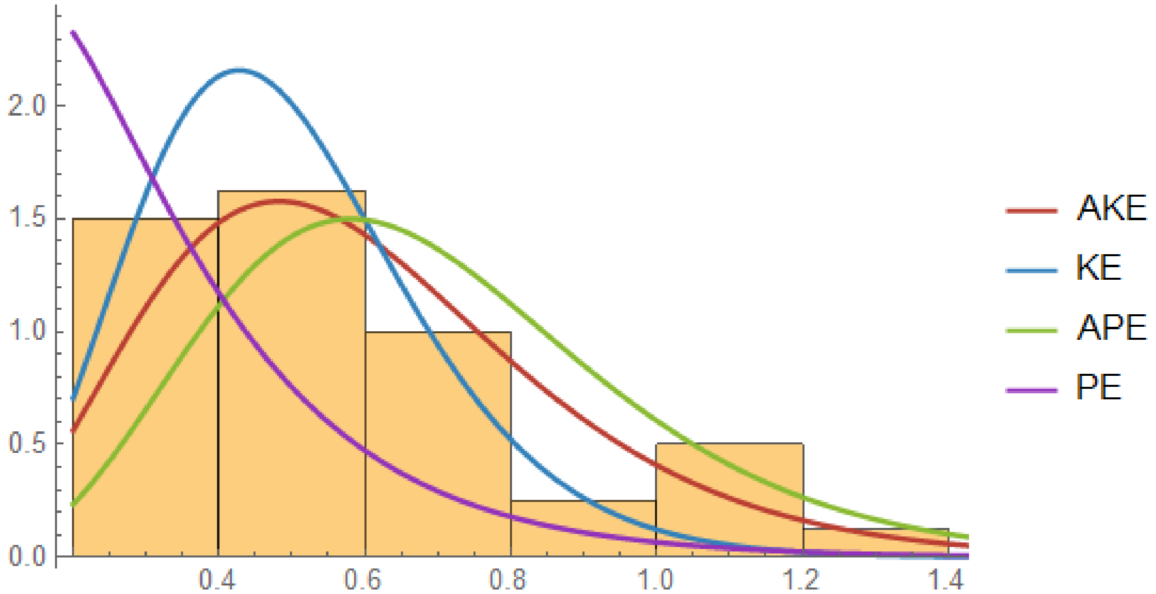

Figure 5: Probability density functions for special cases from the *AKE* distribution

Table 2: The log-likelihood function, the likelihood ratio tests statistic and p-values.

Distribution	Parameters				ℓ (log likelihood)	Λ (The likelihood ratio test statistic)	df (degrees of freedom)	p-value
	α	β	θ	λ				
KE	–	2.176	1.647	1.928	-8.214	15.318	1	9.085×10^{-5}
		(0.418)	(0.329)	(0.284)				
APE	1.275	3.217	–	1.099	-10.247	19.384	1	1.069×10^{-5}
	(0.214)	(1.517)		(0.314)				
PE	–	7.782	–	4.626	−14.869	28.628	2	6.074×10^{-7}
		(2.567)		(0.684)				

*Note that the *AKE* distribution log likelihood = - 0.555

In table 2, upon the likelihood ratio test, the null hypothesis is the data follow the nested model and the alternative is the data follow the full model, where the *KE* distribution, the *APE* distribution and the *PE* distribution are nested by *AKE* distribution, one can see that, all null hypotheses can be rejected at significance level is 0.05

9. Conclusion

The alpha power Kumaraswamy class has an explicit form gives more flexibility in mathematical properties and random number generating. The alpha power Kumaraswamy class generalizes some important classes of distributions as the Kumaraswamy class, the alpha power class and the power function class. The alpha power Kumaraswamy exponential distribution has wide applications in real data sets and in some cases it can be the best fitted distribution. Author encourages researchers to study more cases from that flexible class.

Competing interests: The author declare that he has no competing interests

Data and Material Availability: One can find the date set at: http://www.npl.co.uk/

Code availability: Not applicable

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Appendix

Results of the simulation study for different data sets:

Set(1):(α=2, β=0.5, θ =0.5, λ=0.2)								
Sample Size	**Parameters**	**Mean of Estimators**	**Biases**	**Total Bias**	**RMSE**	**Total RMSE**	**Pearson System Coefficients**	**Pearson Type**
10	α=2	**2.262**	**0.262**	**1.474**	**1.258**	**2.185**	**0.202**	**IV**
	β=0.5	**0.759**	**0.259**		**0.75**		**0.27**	**IV**
	θ =0.5	**0.92**	**0.42**		**0.436**		**-0.27**	**I**
	λ=0.2	**1.565**	**1.365**		**1.563**		**0.576**	**IV**
20	α=2	**2.144**	**0.144**	**1.054**	**1.235**	**1.683**	**0.392**	**IV**
	β=0.5	**0.602**	**0.102**		**0.246**		**0.827**	**IV**
	θ =0.5	**0.844**	**0.344**		**0.36**		**0.054**	**IV**
	λ=0.2	**1.181**	**0.981**		**1.058**		**0.594**	**IV**
30	α=2	**2.14**	**0.14**	**0.898**	**1.103**	**1.462**	**0.461**	**IV**
	β=0.5	**0.568**	**0.068**		**0.173**		**-0.312**	**I**
	θ =0.5	**0.808**	**0.308**		**0.322**		**-0.496**	**I**
	λ=0.2	**1.03**	**0.83**		**0.888**		**0.14**	**IV**
50	α=2	**2.106**	**0.106**	**0.760**	**0.905**	**1.211**	**0.438**	**IV**
	β=0.5	**0.537**	**0.037**		**0.118**		**-2.505**	**I**
	θ =0.5	**0.776**	**0.276**		**0.289**		**0.021**	**IV**
	λ=0.2	**0.9**	**0.7**		**0.742**		**0.096**	**IV**
100	α=2	**2.073**	**0.073**	**0.224**	**0.368**	**0.391**	**0.44**	**IV**
	β=0.5	**0.511**	**0.011**		**0.085**		**-0.267**	**I**
	θ =0.5	**0.637**	**0.137**		**0.051**		**0.011**	**IV**
	λ=0.2	**0.362**	**0.162**		**0.091**		**0.666**	**IV**
300	α=2	**2.043**	**0.043**	**0.050**	**0.055**	**0.087**	**0.105**	**IV**
	β=0.5	**0.507**	**0.007**		**0.012**		**-0.043**	**I**
	θ =0.5	**0.506**	**0.006**		**0.018**		**-3.643**	**I**
	λ=0.2	**0.225**	**0.025**		**0.064**		**0.57**	**IV**

Set(2):(α=2, β=0.5, θ =0.5, λ=0.8)								
Sample Size	**Parameters**	**Mean of Estimators**	**Biases**	**Total Bias**	**RMSE**	**Total RMSE**	**Pearson System Coefficients**	**Pearson Type**
10	α=2	**2.239**	**0.239**	**3.506**	**1.234**	**6.406**	**0.203**	**IV**
	β=0.5	**0.743**	**0.243**		**0.460**		**0.268**	**IV**
	θ =0.5	**0.959**	**0.459**		**0.435**		**-0.372**	**I**
	λ=0.8	**4.26**	**3.46**		**6.255**		**0.583**	**IV**
20	α=2	**2.125**	**0.125**	**2.954**	**1.053**	**4.382**	**0.395**	**IV**
	β=0.5	**0.583**	**0.083**		**0.126**		**0.46**	**IV**
	θ =0.5	**0.894**	**0.394**		**0.391**		**0.062**	**IV**
	λ=0.8	**3.724**	**2.924**		**4.234**		**0.592**	**IV**
30	α=2	**2.117**	**0.117**	**2.352**	**1.002**	**3.709**	**0.461**	**IV**
	β=0.5	**0.549**	**0.049**		**0.101**		**-0.281**	**I**
	θ =0.5	**0.878**	**0.378**		**0.351**		**-0.221**	**I**
	λ=0.8	**3.118**	**2.318**		**3.553**		**0.692**	**IV**
50	α=2	**2.104**	**0.104**	**1.827**	**0.704**	**1.382**	**0.438**	**IV**
	β=0.5	**0.526**	**0.026**		**0.091**		**-4.266**	**I**
	θ =0.5	**0.796**	**0.296**		**0.20**		**0.018**	**IV**
	λ=0.8	**2.6**	**1.8**		**1.169**		**0.095**	**IV**
100	α=2	**2.053**	**0.053**	**1.25**	**0.171**	**0.470**	**0.439**	**IV**
	β=0.5	**0.509**	**0.009**		**0.064**		**-1.288**	**I**
	θ =0.5	**0.733**	**0.233**		**0.267**		**0.018**	**IV**
	λ=0.8	**2.036**	**1.236**		**0.341**		**0.046**	**IV**
300	α=2	**2.031**	**0.031**	**0.061**	**0.027**	**0.111**	**0.0018**	**IV**
	β=0.5	**0.504**	**0.004**		**0.009**		**-0.0021**	**I**
	θ =0.5	**0.521**	**0.021**		**0.037**		**0.00115**	**IV**
	λ=0.8	**0.848**	**0.048**		**0.101**		**0.529**	**IV**

Set(3):(α=2, β=0.5, θ =0.5, λ=2)								
Sample Size	**Parameters**	**Mean of Estimators**	**Biases**	**Total Bias**	**RMSE**	**Total RMSE**	**Pearson System Coefficients**	**Pearson Type**
10	α=2	**2.218**	**0.218**	**13.667**	**1.202**	**15.709**	**0.2**	**IV**
	β=0.5	**0.727**	**0.227**		**0.384**		**0.279**	**IV**
	θ =0.5	**0.972**	**0.472**		**0.518**		**-0.231**	**I**
	λ=2	**15.656**	**13.656**		**15.65**		**0.561**	**IV**
20	α=2	**2.113**	**0.113**	**9.814**	**1.004**	**10.634**	**0.393**	**IV**
	β=0.5	**0.561**	**0.061**		**0.117**		**0.308**	**IV**
	θ =0.5	**0.897**	**0.397**		**0.438**		**0.06**	**IV**
	λ=2	**11.806**	**9.806**		10.577		**0.605**	**IV**
30	α=2	**2.095**	**0.095**	**8.307**	**0.809**	**8.934**	**0.454**	**IV**

	β=0.5	0.524	0.024		0.096		-0.756	I
	θ =0.5	0.885	0.385		0.472		-0.4	I
	λ=2	10.298	8.298		8.885		0.479	IV
50	*α*=2	2.073	0.073	7.006	0.427	7.443	0.438	IV
	β=0.5	0.514	0.014		0.058		-4.187	I
	θ =0.5	0.799	0.299		0.342		0.02	IV
	λ=2	9	7		7.423		0.097	IV
100	*α*=2	2.029	0.029	5.519	0.062	5.756	0.975	IV
	β=0.5	0.508	0.0084		0.043		-1.329	I
	θ =0.5	0.753	0.253		0.290		0.0079	IV
	λ=2	7.514	5.514		5.749		0.013	IV
300	*α*=2	2.027	0.027	0.113	0.016	0.160	0.0026	IV
	β=0.5	0.5025	0.0025		0.007		-0.0031	I
	θ =0.5	0.515	0.015		0.096		-0.0053	I
	λ=2	2.109	0.109		0.128		0.35	IV