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Approximation for q-Chlodowsky Operators via Statistical Convergence with Respect to Power Series Method *

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Abstract
Many results which are obtained or unable to obtained by classical calculus have also been studied by q-calculus. It is effective to use q-calculus since it acts as a bridge between mathematics and physics. The q-analog of Chlodowsky operators has been introduced and the approximation properties of these operators have been studied in [12]. Then in [23], the q-analog of Stancu-Chlodowsky operators has been introduced and some approximation results of these operators have been studied via A-statistical convergence which is a more general setting. In this paper, we present the approximation properties of q-Chlodowsky operators via statistical convergence with respect to power series method. It is noteworthy to mention that statistical convergence and statistical convergence with respect to power series method are incompatible.

Keywords: *q-calculus; Chlodowsky operators; approximation theory; power series method; statistical convergence.*

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1. Introduction and Preliminaries

In approximation theory, Bernstein operators have different applications. With the use of these operators, it is possible to give an understandable and easy proof of Weierstrass's theorem. This is the most important application of these operators. The classical Bernstein operators have been introduced and discussed in detailed in [1, 2, 5, 13, 15, 25]. Since q-calculus acts as a bridge between mathematics and physics, the q-analog of Bernstein operators have been introduced by Lupaş [16]. Different type of q-Bernstein operators has also been introduced by Phillips [21] and Ostrovska [19] have investigated the approximation properties of these operators. Karsli and Gupta [12] have introduced q-Chlodowsky operators which extend q-Bernstein operators to an unbounded interval.

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The important results in approximation theory have also been studied by using different concepts of convergences such as statistical convergence, ideal convergence, summation process [3, 4, 8, 11, 20, 24]. It is effective to use these concepts since they make a nonconvergent sequence to converge.

In the present paper, we study the approximation properties of q-Chlodowsky operators via statistical convergence with respect to power series method. In [26], such examples have been provided to show that statistical convergence and statistical convergence with respect to power series method do not imply each other.
This paper is organized as follows:
The first section is devoted to basic definitions, notations and also well known results. The second section is devoted to our main results and in the third section we will give an application.

Now, let us recall basic definitions, notations and also the well known results which we need throughout the paper.

The density of the subset $E \subseteq \mathbb{N}_0$ is given by

$$\delta(E) := \lim_{n\to\infty} \frac{1}{n+1} |\{j \leqslant n : j \in E\}|$$

whenever the limit exists where the vertical bars indicate the cardinality of enclosed set and $\mathbb{N}_0$ is the set of all nonnegative integers. A sequence $x = (x_j)$ is called statistically convergent to L [9, 10, 22] if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n+1} |\{j \leqslant n : |x_j - L| \geqslant \varepsilon\}| = 0$$

that is, $\delta(E_\varepsilon) = 0$ for any $\varepsilon > 0$ where $E_\varepsilon = \{j \in \mathbb{N}_0 : |x_j - L| \geqslant \varepsilon\}$.

By assuming that (p_j) is nonnegative real sequence such that $p_0 > 0$ and the corresponding power series $p(t) := \sum_{j=0}^{\infty} p_j t^j$ has radius of convergence R with $0 < R \leqslant \infty$. Now the definition of power series method is as follows :

Let

$$C_p := \left\{ f : (-R, R) \to \mathbb{R} | \lim_{0<t\to R^-} \frac{1}{p(t)} f(t) \quad \text{exists} \right\}$$

and

$$C_{P_p} := \left\{ x = (x_k) | \; p_x(t) := \sum_{j=0}^{\infty} p_j t^j x_j \quad \text{has radius of convergence} \geqslant R \quad \text{and} \quad p_x \in C_p \right\}.$$

The functional $P_p - \lim : C_{P_p} \to \mathbb{R}$ (for short P_p) defined by

$$P_p - \lim x = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j x_j$$

is called a power series method and x is said to be P_p-convergent [6], [14].

A power series method P_p is said to be regular if $P_p - \lim x = L$ provided that $\lim x = L$ [6].

By combining these concepts, Ünver and Orhan [26] have recently introduced P_p-statistical convergence and have proved a Korovkin type theorem for a sequence of positive linear operators defined on $C[0,1]$, the space of all continuous functions on the interval $[0,1]$.
Now let us recall the statistical convergence with respect to power series method, i.e., P_p-statistical convergence.
Let P_p be a regular power series method and $E \subseteq \mathbb{N}_0$. If the limit

$$\delta_{P_p}(E) := \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j\in E} p_j t^j$$

exists then $\delta_{P_p}(E)$ is called the P_p-density of E. Notice that by the definition of a power series method and P_p-density it is obvious that $0 \leqslant \delta_{P_p}(E) \leqslant 1$ whenever it exists [26].
Let $x = (x_j)$ be a real sequence and let P_p be a regular power series method. Then x is said to be P_p-statistically convergent to L if for any $\varepsilon > 0$

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j\in E_\varepsilon} p_j t^j = 0$$

that is, $\delta_{P_p}(E_\varepsilon)=0$ for any $\varepsilon>0$. In the case we write $st_{P_p}-\lim x=L$ [26].

Before recalling the q-Chlodowsky operators, it is useful to mention certain properties of q-calculus. For any fixed real number $q>0$ and nonnegative integer r, the q-integer of the number r is defined by

$$[r]_q=\begin{cases}(1-q^r)/(1-q), & q\neq 1,\\ r, & q=1.\end{cases}$$

The q-factorial is defined by

$$[r]_q!=\begin{cases}[r]_q[r-1]_q\cdots[1]_q, & r=1,2,\ldots,\\ 1, & r=0\end{cases}$$

and q-binomial coefficient can be defined as

$$\begin{bmatrix}k\\ r\end{bmatrix}_q=\frac{[k]_q!}{[r]_q![k-r]_q!}$$

for integers $k\geqslant r\geqslant 0$ and q-binomial coefficients satisfy the following properties:

$$\begin{bmatrix}k+1\\ r\end{bmatrix}_q=q^{k-r+1}\begin{bmatrix}k\\ r-1\end{bmatrix}_q+\begin{bmatrix}k\\ r\end{bmatrix}_q$$

and

$$\begin{bmatrix}k+1\\ r\end{bmatrix}_q=\begin{bmatrix}k\\ r-1\end{bmatrix}_q+q^r\begin{bmatrix}k\\ r\end{bmatrix}_q.$$

The q-analog of $(1-a)^n$ is the polynomial

$$(1-a)_q^n=\begin{cases}1, & n=0\\ \prod_{s=0}^{n-1}(1-q^s a), & n\geqslant 1.\end{cases}$$

The Bernstein-Chlodowsky operators were defined by Chlodowsky on an unbounded set in 1937 [7] as follows:

$$C_n(f;x)=\sum_{k=0}^{n}f\left(\frac{k}{n}b_n\right)\begin{bmatrix}n\\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

where $0\leq x\leq b_n$, (b_n) is a positive increasing sequence with the properties that, $\lim\limits_{n\to\infty}b_n=\infty$ and $\lim\limits_{n\to\infty}\frac{b_n}{n}=0$.
The q-Bernstein operators have also been defined by [21] and have been studied by many researchers (see e.g. [17, 18, 27] etc.)

$$B_{n,q}(f;x)=\sum_{k=0}^{n}f\left(\frac{[k]_q}{[n]_q}\right)\begin{bmatrix}n\\ k\end{bmatrix}_q x^k\prod_{s=0}^{n-k-1}(1-q^s x).$$

With the same motivation in the classical procedure, the q-Chlodowsky operators have been defined as:

$$C_{n,q}(f;x)=\sum_{k=0}^{n}f\left(\frac{[k]_q}{[n]_q}b_n\right)\begin{bmatrix}n\\ k\end{bmatrix}_q\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n-k-1}\left(1-q^s\frac{x}{b_n}\right)$$

where $0\leq x\leq b_n$, (b_n) is a positive increasing sequence with the property that $\lim\limits_{n\to\infty}b_n=\infty$.
In [12] the following theorem has been obtained for q-Chlodowsky operators.

Theorem 1. *$C_{n,q}$ operators satisfy the following equalities*

$$\begin{aligned}C_{n,q}(1;x)=&1,\\ C_{n,q}(t;x)=&x,\\ C_{n,q}(t^2;x)=&x^2+\frac{x(b_n-x)}{[n]_q}.\end{aligned}$$

From Theorem 1 and by direct computations, we have the following equalities:

$$C_{n,q}((t-x)^2;x) = \frac{x(b_n - x)}{[n]_q}, \quad C_{n,q}((t-x);x) = 0.$$

One can easily observe that $[n]_q \to \frac{1}{1-q}$ as $n \to \infty$ for $0 < q < 1$ and it implies that $C_{n,q}(t^2;x)$ and $C_{n,q}((t-x)^2;x)$ do not converge to x^2 and 0 respectively, as $n \to \infty$.

In order to overcome this difficulty, we replace q by (q_n) where (q_n) is a sequence of real numbers such that $0 < q_n < 1$,

$$st_{P_p} - \lim q_n = 1$$

and

$$st_{P_p} - \lim \frac{b_n}{[n]_{q_n}} = 0.$$

Now let us recall the modulus of continuity of f, $\omega(f,\delta)$ is defined by

$$\omega(f,\delta) = \sup_{\substack{|x-y|\leq\delta \\ x,y\in[0,B]}} |f(x) - f(y)|.$$

It is well known that for a function $f \in C[0,B]$,

$$\lim_{\delta\to 0^+} \omega(f,\delta) = 0,$$

and for any $\lambda > 0$

$$\omega(f,\lambda\delta) \leq (1+\lambda)\omega(f,\delta).$$

2. Main Results

In this section, we present our main results which fill the gaps in the existing literature. First of all, we recall the following theorem which states the necessary and sufficient condition for the convergence of a sequence of positive linear operators on $C[0,1]$.

Theorem 2. *[26] Let P_p be a regular power series method and let (L_n) be a sequence of positive linear operators on $C[a,b]$ such that for $e_i(x) = x^i$, $i = 0,1,2$*

$$st_{P_p} - \lim \|L_n e_i - e_i\| = 0$$

then for any $f \in C[a,b]$, we have

$$st_{P_p} - \lim \|L_n f - f\| = 0.$$

Under the light of Theorem 2, we are ready to present and prove the following:

Theorem 3. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$, $st_{P_p} - \lim q_n = 1$. Then for any $f \in C[0,\infty)$, we have*

$$st_{P_p} - \lim \|C_{n,q_n}(f) - f\|_{C[0,B]} = 0$$

where B is positive real number.

Proof. Using Theorem 1 and Theorem 2 and since $st_{P_p} - \lim q_n = 1$ and $st_{P_p} - \lim \frac{b_n}{[n]_{q_n}} = 0$, we obtain the desired result. This completes the proof. □

Theorem 4. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and $st_{P_p} - \lim q_n = 1$. If $f \in C[0,\infty)$, then we have*

$$|C_{n,q_n}(f;x) - f(x)| \leq 2\omega\left(f, \sqrt{\frac{x(b_n - x)}{[n]_{q_n}}}\right).$$

Proof. By using simple calculations, we have

$$
\begin{aligned}
|C_{n,q_n}(f;x)-f(x)| &= \left|\sum_{k=0}^{n} f\left(\frac{[k]_{q_n}}{[n]_{q_n}}b_n\right)\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} - f(x)\right| \\
&= \left|\sum_{k=0}^{n}\left[f\left(\frac{[k]_{q_n}}{[n]_{q_n}}b_n\right)-f(x)\right]\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}\right| \\
&\leq \sum_{k=0}^{n}\left|f\left(\frac{[k]_{q_n}}{[n]_{q_n}}b_n\right)-f(x)\right|\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&\leq \sum_{k=0}^{n}\omega\left(f,\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|\right)\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&= \sum_{k=0}^{n}\omega\left(f,\frac{\delta\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|}{\delta}\right)\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&\leq \sum_{k=0}^{n}\left(1+\frac{\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|}{\delta}\right)\omega(f,\delta)\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&= \sum_{k=0}^{n}\omega(f,\delta)\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&\quad +\sum_{k=0}^{n}\frac{\omega(f,\delta)}{\delta}\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&= \omega(f,\delta)\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&\quad +\frac{\omega(f,\delta)}{\delta}\sum_{k=0}^{n}\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}.
\end{aligned}
$$

It is also well known that

$$
\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} = \left[\frac{x}{b_n}+\left(1-\frac{x}{b_n}\right)\right]_{q_n}^{n} = 1
$$

and by using Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\sum_{k=0}^{n}\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&\leq \sum_{k=0}^{n}\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|\left\{\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}\right\}^{1/2}\left\{\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}\right\}^{1/2} \\
&\leq \left\{\sum_{k=0}^{n}\left(\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right)^2\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}\right\}^{1/2}\left\{\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}\right\}^{1/2} \\
&= \left\{\frac{x(b_n-x)}{[n]_{q_n}}\right\}^{1/2}.
\end{aligned}
$$

Then

$$
|C_{n,q_n}(f;x)-f(x)| \leq \omega(f,\delta)+\frac{\omega(f,\delta)}{\delta}\left\{\frac{x(b_n-x)}{[n]_{q_n}}\right\}^{1/2}
$$

holds. By taking $\delta = \left\{ \frac{x(b_n - x)}{[n]_{q_n}} \right\}^{1/2}$, we have

$$|C_{n,q_n}(f;x) - f(x)| \leq 2\omega(f,\delta) = 2\omega\left(f, \sqrt{\frac{x(b_n - x)}{[n]_{q_n}}}\right).$$

This completes the proof.

□

Theorem 5. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and $st_{P_p} - \lim q_n = 1$. If $f \in C[0, b_n]$, then we have*

$$|C_{n,q_n}(f;x_0) - f(x_0)| \leq 2\omega\left(f, \sqrt{\frac{x_0 b_n}{[n]_{q_n}}}\right)$$

where $x_0 \in [0, b_n]$ and x_0 is a fixed point.

Proof. The validity of the following is obvious:

$$\frac{x(b_n - x)}{[n]_{q_n}} \leq \frac{x_0 b_n}{[n]_{q_n}}$$

for any fixed point x_0. One can obtain the remaining part in a similar way in [12]. Therefore we omit the details here. □

Theorem 6. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and $st_{P_p} - \lim q_n = 1$. If $f \in C[0, \infty)$, then we have, for sufficiently large n*

$$\|C_{n,q_n}(f) - f\|_{C[0,b_n]} \leq 2\omega\left(f, \sqrt{\frac{B b_n}{[n]_{q_n}}}\right)$$

where $B > 0$ is a constant being appeared in Theorem 3.

In [23], the q-analog of the Stancu type Bernstein-Chlodowsky operators have been introduced as follows:

$$C_{n,q}^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q + [\alpha]_q}{[n]_q + [\beta]_q} b_n\right) \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}$$

where $0 \leq x \leq b_n$, (b_n) is a positive increasing sequence with the property that $\lim_{n\to\infty} b_n = \infty$ and α, β are positive integers such that $0 \leqslant \alpha \leqslant \beta$.

Observe that when we take $\alpha = \beta = 0$, q-analog of the Stancu type Bernstein-Chlodowsky operators coincide with q-Bernstein-Chlodowsky operators.

In [23] the following theorem has been obtained for the Stancu type Bernstein-Chlodowsky operators.

Theorem 7. *The followings are satisfied for $C_{n,q}^{\alpha,\beta}$;*

$$\begin{aligned} C_{n,q}^{\alpha,\beta}(1;x) =& 1, \\ C_{n,q}^{\alpha,\beta}(t;x) =& \frac{[n]_q}{[n]_q + [\beta]_q} x + \frac{[\alpha]_q}{[n]_q + [\beta]_q} b_n, \\ C_{n,q}^{\alpha,\beta}(t^2;x) =& \frac{{[n]_q}^2}{([n]_q + [\beta]_q)^2}\left(x^2 + \frac{x(b_n - x)}{[n]_q}\right) + \frac{2[\alpha]_q[\beta]_q}{([n]_q + [\beta]_q)^2} + \frac{{[n]_q}^2}{([n]_q + [\beta]_q)^2} {b_n}^2. \end{aligned}$$

Since

$$st_{P_p} - \lim \frac{b_n}{[n]_{q_n}} = 0,\ 0 < q_n < 1 \text{ and } st_{P_p} - \lim q_n = 1,$$

we can also obtain analogous results those given in [23] in a similar manner for the concept of P_p-statistical convergence.

Recall that for $f \in C[a,b]$ and $t > 0$, the Peetre-K Functional is defined by

$$K(f,\delta) := \inf_{g \in C^2[a,b]} \{\|f-g\|_{C[a,b]} + t\|g\|_{C^2[a,b]}\}$$

where $C^2[a,b] = \{f \in C[a,b] : f', f'' \in C[a,b]\}$, with the norm

$$\|g\|_{C^2[a,b]} := \|g\|_{C[a,b]} + \|g'\|_{C[a,b]} + \|g''\|_{C[a,b]}.$$

It is obtained in [12] that for $g \in C^2[0,b_n]$, then we have

$$|C_{n,q}(g;x) - g(x)| \leqslant \frac{x(b_n - x)}{2[n]_q} \|g\|_{C^2[0,b_n]}.$$

Theorem 8. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and*

$$st_{P_p} - \lim q_n = 1.$$

If $f \in C[0,\infty)$ and $B > 0$ is a constant, then we have

$$\|C_{n,q_n}(f) - f\|_{C[0,b_n]} \leqslant 2K\left(f, \frac{Bb_n}{2[n]_{q_n}}\right).$$

Proof. From [12], it is known that

$$|C_{n,q_n}(f;x) - f(x)| \leqslant \|f-g\|_{C[0,b_n]} |C_{n,q_n}(1;x)| + \|f-g\|_{C[0,b_n]} + |C_{n,q_n}(g;x) - g(x)|$$

and

$$|C_{n,q_n}(f;x) - f(x)| \leqslant 2\|f-g\|_{C[0,b_n]} + \frac{x(b_n - x)}{2[n]_q} \|g\|_{C^2[0,b_n]}$$

and hence

$$|C_{n,q_n}(f;x) - f(x)| \leqslant 2\|f-g\|_{C[0,b_n]} + \frac{Bb_n}{2[n]_{q_n}} \|g\|_{C^2[0,b_n]}.$$

Taking infimum over all $g \in C^2[0,b_n]$, we obtain the desired result which completes the proof. □

Also recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be uniform Lipschitz continuous of order $\gamma > 0$ if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leqslant M|x-y|^\gamma$$

for any x and y in $\mathbb{R}$. In this case, we write $f \in Lip(\gamma, \mathbb{R})$.

Theorem 9. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and*

$$st_{P_p} - \lim q_n = 1.$$

If $f \in Lip_M[0,b_n]$ and $x \in [0,B]$, $B > 0$ is a constant, then we have

$$\|C_{n,q_n}(f) - f\|_{C[0,b_n]} \leqslant M\left\{\frac{Bb_n}{[n]_{q_n}}\right\}^{\frac{\alpha}{2}}.$$

Proof. The proof follows in a similar manner used in [12]. Therefore we omit the details here. □

Theorem 10. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and*

$$st_{P_p} - \lim q_n = 1.$$

Also let $\omega(\delta)$ is the modulus of continuity of f on $[0,B]$ and $f(x)$ has continuous derivative as $f'(x)$, then we have

$$|C_{n,q_n}(f;x) - f(x)| \leqslant M\sqrt{\frac{b_n}{[n]_{q_n}}}\,\omega\left(f, \sqrt{\frac{b_n}{[n]_{q_n}}}\right).$$

3. Conclusions

In this section, we provide an example such that the sequence (q_n) satisfies neither the conditions of the results obtained in [12] nor the conditions of the results obtained in [23].

Example 1. *Let the sequences (p_n) and (q_n) defined as follows:*

$$p_n = \begin{cases} 1 & , \quad n = 2k \\ 0 & , \quad n = 2k+1 \end{cases}, \qquad q_n = \begin{cases} 0 & , \quad n = 2k+1 \\ 1 - \frac{1}{n} & , \quad n = 2k \end{cases}.$$

It is easy to see that the method P_p is regular and one can easily see that

$$\delta_{P_p}(E_\varepsilon) = 0$$

where $E_\varepsilon = \{n \in \mathbb{N}_0 : |q_n - 1| \geqslant \varepsilon\}$ holds for every $\varepsilon > 0$. That is $st_{P_p} - \lim q_n = 1$. Notice that (q_n) is not convergent in the ordinary sense or statistically convergent.
In [26], a sequence of positive linear operators has been presented which satisfies neither the conditions of Theorem 1 of [11] nor the conditions of the classical Korovkin theorem (Theorem 4 of [11]) but it satisfies the conditions of Theorem 5 of [11].

Here it is remarkable to mention that our results cannot be deduced from the results in [23] since (q_n) is not convergent or statistically convergent.

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Author's contributions

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