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Subdivision of the spectra for the generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c

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Abstract

There are many different ways to subdivide the spectrum of a bounded linear operator; some of them are motivated by applications to physics (in particular, quantum mechanics). In this study, we determine the approximate point spectrum, compression spectrum and defect spectrum of the generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c.

Keywords: Upper triangular double-band matrices, Approximate point spectrum, Compression spectrum, Defect spectrum.

Genelleştirilmiş üst üçgensel double-bant matrisi Δ^{uv} nin c_0 ve c dizi uzayları üzerindeki spektral ayrışımı

Özet

Bir sınırlı lineer operatörün spektrumunun çok farklı yollarla ayrışımı vardır; bunlardan bazıları fiziğin uygulamalarına uyarlanmıştır (özellikle, kuantum mekaniği). Bu çalışmada genelleştirilmiş üst üçgensel double-bant matrisi Δ^{uv} nin c_0 ve c dizi

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uzayları üzerindeki yaklaşık nokta spektrumunu, sıkıştırma spektrumunu ve eksik spektrumunu belirledik.

Anahtar Kelimeler: Üst üçgensel double-bant matrisi, Yaklaşık nokta spektrum, Sıkıştırma spektrum, Eksik spektrum.

Introduction

Spectral theory is an important branch of mathematics due to its application in other branches of science. As it is well known, the matrices play an important role in operator theory. The spectrum of an operator generalizes the notion of eigenvalues for matrices. It has been proved to be a standard tool of mathematical sciences because of its usefulness and application oriented scope in different fields. In numerical analysis, the spectral values may determine whether a discretization of a differential equation will get the right answer or how fast a conjugate gradient iteration will converge. In ecology, the spectral values may determine whether a food web will settle into a steady equilibrium. In aeronautics, the spectral values may determine whether the flow over a wing is laminar or turbulent. In electrical engineering, it may determine the frequency response of an amplifier or the reliability of a power system etc.

In the past decades, the spectrum of linear operators defined by some particular limitation matrices over some sequence spaces has been considered by many authors, say for example, Akhmedov and El-Shabrawy [1], [2], Yildirim [3], [4], [5], and B. Altay and F. Başar [6] etc.

In this work, our purpose is to determine the approximate point spectrum, compression spectrum and defect spectrum of the generalized upper triangular double-band matrices Δ^{uv} as an operator over the sequence spaces c_0 and c.

1.1 Preliminaries, Background and Notation

Let X and Y be the Banach spaces, and $L: X \to Y$ also be a bounded linear operator. By $R(L)$, we denote the range of L, i.e.,

$$R(L) = \{y \in Y : y = Lx,\ \ x \in X\}$$

by $B(X)$, we denote the set of all bounded linear operators on X into itself. If X is any Banach space and $L \in B(X)$ then the adjoint L^* of L is a bounded linear operator on the dual X^* of X defined by $(L^*f)(x) = f(Lx)$ for all $f \in X^*$ and $x \in X$.

Given an operator $L \in B(X)$, the set

$$\rho(L) := \{\lambda \in \mathbb{C} : \lambda I - L \ \ \text{bijection}\} \tag{1.1}$$

is called the resolvent set of L and its complement with respect to the complex plain

$$\sigma(L) := \mathbb{C} \backslash \rho(L) \tag{1.2}$$

is called the spectrum of L. By the closed graph theorem, the inverse operator

$$R(\lambda; L) := (\lambda I - L)^{-1} \qquad (\lambda \in \rho(L)) \tag{1.3}$$

is always bounded; this operator is usually called resolvent operator of L at λ.

Let X be a Banach space over $\mathbb{C}$ and $L \in B(X)$. Recall that a number $\lambda \in \mathbb{C}$ is called eigenvalue of L if the equation

$$Lx = \lambda x \tag{1.4}$$

has a nontrivial solution $x \in X$. Any such x is then called eigenvector, and the set of all eigenvectors is a subspace of X called eigenspace.

Throughout the following, we will call the set of eigenvalues

$$\sigma(L) := \{\lambda \in \mathbb{C} : Lx = \lambda x \ \ \text{for some } x \neq 0\}. \tag{1.5}$$

We say that $\lambda \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_c(L)$ of L if the resolvent operator (1.3) is defined on a dense subspace of X and is unbounded. Furthermore, we say that $\lambda \in \mathbb{C}$ belongs to the residual spectrum $\sigma_r(L)$ of L if the resolvent operator (1.3) exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $(\lambda I - L)$ is not dense in X; in this case $R(\lambda; L)$ may be bounded or unbounded.

Given a bounded linear operator L in a Banach space X, we call a sequence (x_k) in X a Weyl sequence for L if $\|x_k\| = 1$ and $\|Lx_k\| \to 0$ as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(L) := \{\lambda \in \mathbb{C} : \text{there is a Weyl sequence for } \lambda I - L\} \tag{1.6}$$

the approximate point spectrum of L. Moreover, the subspectrum

$$\sigma_{\delta}(L) := \{\lambda \in \mathbb{C} : \lambda I - L \text{ is not surjective}\} \tag{1.7}$$

is called defect spectrum of L.

The two subspectra (1.6) and (1.7) form a (not necessarily disjoint) subdivision

$$\sigma(L) = \sigma_{ap}(L) \cup \sigma_{\delta}(L) \tag{1.8}$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(L) := \{\lambda \in \mathbb{C} : \overline{R(\lambda I - L)} \neq X\} \tag{1.9}$$

which is often called compression spectrum in the literature and which gives rise to another (not necessarily disjoint) decomposition

$$\sigma(L) = \sigma_{ap}(L) \cup \sigma_{co}(L) \tag{1.10}$$

of the spectrum. Clearly, $\sigma_p(L) \subseteq \sigma_{ap}(L)$ and $\sigma_{co}(L) \subseteq \sigma_{\delta}(L)$. Moreover, comparing these subspectra with those in (1.5) we note that

$$\sigma_r(L) = \sigma_{co}(L) \setminus \sigma_p(L) \tag{1.11}$$

and

$$\sigma_c(L) = \sigma(L) \setminus [\sigma_p(L) \cup \sigma_{co}(L)] \tag{1.12}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

Proposition 1.1 ([7], Proposition 1.3). The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:

(a) $\sigma(L^*) = \sigma(L)$,

(b) $\sigma_c(L^*) \subseteq \sigma_{ap}(L)$,

(c) $\sigma_{ap}(L^*) = \sigma_\delta(L)$,

(d) $\sigma_\delta(L^*) = \sigma_{ap}(L)$,

(e) $\sigma_p(L^*) = \sigma_{co}(L)$,

(f) $\sigma_{co}(L^*) \supseteq \sigma_p(L)$,

(g) $\sigma(L) = \sigma_{ap}(L) \cup \sigma_p(L^*) = \sigma_p(L) \cup \sigma_{ap}(L^*)$.

1.2. Goldberg's Classification of Spectrum

If X is a Banach space, $B(X)$ denotes the collection of all bounded linear operators on X and $T \in B(X)$, then there are three possibilities for $R(T)$, the range of T:

(I) $R(T) = X$

(II) $\overline{R(T)} = X$, but $R(T) \neq X$,

(III) $\overline{R(T)} \neq X$.

and three possibilities for T^{-1}:

(1) T^{-1} exists and continuous,

(2) T^{-1} exists but discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3$. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous (see [8]).

If λ is a complex number such that $T = \lambda I - L \in I_1$ or $T = \lambda I - L \in II_1$ then $\lambda \in \rho(L,X)$. All scalar values of λ not in $\rho(L,X)$ comprise the spectrum of L. The further classification of $\sigma(L,X)$ gives rise to the fine spectrum of L.That is, $\sigma(L,X)$ can be divided into the subsets

$I_2\sigma(L,X) = \emptyset$, $I_3\sigma(L,X)$, $II_2\sigma(L,X)$, $II_3\sigma(L,X)$, $III_1\sigma(L,X)$, $III_2\sigma(L,X)$, $III_3\sigma(L,X)$.
For example, if $T = \lambda I - L$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(L,X)$.

By the definitions given above, in [9], Durna and Yildirim have written following table:

Table 1.

		1	2	3
		$R(\lambda; L)$ exists and is bounded	$R(\lambda; L)$ exists and is unbounded	$R(\lambda; L)$ dos not exists
I	$R(\lambda I - L) = X$	$\lambda \in \rho(L)$	———	$\lambda \in \sigma_p(L)$ $\lambda \in \sigma_{ap}(L)$
II	$\overline{R(\lambda I - L)} = X$	$\lambda \in \rho(L)$	$\lambda \in \sigma_c(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$	$\lambda \in \sigma_p(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$
III	$\overline{R(\lambda I - L)} \neq X$	$\lambda \in \sigma_r(L)$ $\lambda \in \sigma_\delta(L)$ $\lambda \in \sigma_{co}(L)$	$\lambda \in \sigma_r(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$ $\lambda \in \sigma_{co}(L)$	$\lambda \in \sigma_p(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$ $\lambda \in \sigma_{co}(L)$

Let w, c_0, c, ℓ_p denote the set of all sequences; the space of all null sequences; convergent sequences; sequences such that $\sum_k |x_k|^p < \infty$, respectively.

Lemma 1.1 ([8], Theorem II 3.11) The adjoint operator T^* is onto if and only if T has a bounded inverse.

Lemma 1.2 ([8], Theorem II 3.7) A linear operator T has a dense range if and only if the adjoint operator T^* is one to one.

2. The fine spectrum of the operator Δ^{uv} on c_0 and c

In this paper, we introduce a class of a generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c. Let (u_k) be a sequence of positive real numbers such that $u_k \neq 0$ for each $k \in \mathbb{N}$ with $u = \lim_{k\to\infty} u_k \neq 0$ and (v_k) is either constant or strictly decreasing sequence of positive real numbers with $v = \lim_{k\to\infty} v_k \neq 0$, and $\sup_k v_k \leq u + v$. In [11], Fathi has defined the operator Δ^{uv} on squences space c_0 as follows:

$$\Delta^{uv}x = \Delta^{uv}(x_n) = (v_n x_n + u_{n+1}x_{n+1})_{n=0}^{\infty}.$$

It is easy to verify that the operator Δ^{uv} can be represented by the matrix,

$$\Delta^{uv} = \begin{pmatrix} v_0 & u_1 & 0 & 0 & 0 & \cdots \\ 0 & v_1 & u_2 & 0 & 0 & \cdots \\ 0 & 0 & v_2 & u_3 & 0 & \cdots \\ 0 & 0 & 0 & v_3 & u_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Note that, if (v_k) and $(u_k)$is a constant sequence, say $v_k = r \neq 0$ and $u_k = s \neq 0$ for all $k \in \mathbb{N}$, then the operator Δ^{uv} is reduced to the operator $U(r,s)$ and the results for fine spectra of upper triangular double-band matrices have been studied in [10].

2.1. Subdivision of the spectrum of Δ^{uv} on c_0

If $T: c_0 \to c_0$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^*: \ell_1 \to \ell_1$ is defined by the transpoze of the matrix A. It is well known that the dual space c_0^* of c_0 is isomorphic to ℓ_1.

The fine spectrum of the operator Δ^{uv} over the sequence space c_0 has been studied by Fathi [11]. In this subsection we summarize the main results.

Theorem 2.1 ([11], Theorem 2.2) $\sigma_p(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C}: |\lambda - v| < u\} \cup M_1$, where

$$M_1 = \left\{\lambda \in \mathbb{C}: |\lambda - v| = u,\ \lim_{k\to\infty}\left(\prod_{i=1}^{k} \frac{v_{i-1}-\lambda}{u_i}\right) = 0\right\}.$$

Theorem 2.2 ([11], Corollary 2.5) $\sigma_r(\Delta^{uv}, c_0) = \emptyset$.

Corollary 2.1 $III_1\sigma(\Delta^{uv}, c_0) = III_2\sigma(\Delta^{uv}, c_0) = \emptyset$.

Proof. It is clear from Theorem 2.2, since from Table 1.,

$\sigma_r(\Delta^{uv}, c_0) = III_1\sigma(\Delta^{uv}, c_0) = III_2\sigma(\Delta^{uv}, c_0)$.

Theorem 2.3 ([11], Theorem 2.6) $\sigma(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C}: |\lambda - v| \le u\}$.

Theorem 2.4 ([11], Theorem 2.7) $\sigma_c(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C}: |\lambda - v| = u\} \setminus M_1$.

Theorem 2.5 ([11], Theorem 2.8) If $|\lambda - v| < u$, then $\lambda \in I_3\sigma(\Delta^{uv}, c_0)$.

Theorem 2.6 If $\lambda \in M_1$, then $\lambda \in II_3\sigma(\Delta^{uv}, c_0)$.

Proof. Let we find $\ker(\Delta^{uv} - \lambda I)^*$. If $(\Delta^{uv} - \lambda I)^* x = 0$, then we get

$$\begin{aligned} &(v_0 - \lambda)x_0 = 0 \\ u_1 x_0 \; + \; &(v_1 - \lambda)x_1 = 0 \\ u_2 x_1 \; + \; &(v_2 - \lambda)x_2 = 0 \\ &\dots \\ u_n x_{n-1} \; + \; &(v_n - \lambda)x_n = 0 \\ &\dots \end{aligned}$$

Hence we obtain that

$$x_n = x_0 \prod_{k=1}^{n} \frac{u_k}{\lambda - v_k}, \quad n \geq 1. \tag{2.1}$$

If $\lambda \in M_1$, then $\lambda \neq v_0$, since (v_k) is either constant or strictly decreasing sequence of positive real numbers with $v - \lim_{k\to\infty} v_k \neq 0$, and $\sup_k v_k \leq u + v$. Therefore x_0 must be zero and so $\ker(\Delta^{uv} - \lambda I)^* = \{\theta\}$. This means that for $\lambda \in M_1$, $(\Delta^{uv} - \lambda I)^*$ is one to one. Thus for $\lambda \in M_1$, $\lambda I - \Delta^{uv}$ has a dense range from Lemma 1.2. Therefore we have

$\lambda \in II_3\sigma(\Delta^{uv}, c_0)$.

Corollary 2.2 $III_3\sigma(\Delta^{uv}, c_0) = \emptyset$.

Proof. It is clear from Theorem 2.1, Theorem 2.5 and Theorem 2.6, since

$\sigma_p(\Delta^{uv}, c_0) = I_3\sigma(\Delta^{uv}, c_0) \cup II_3\sigma(\Delta^{uv}, c_0) \cup III_3\sigma(\Delta^{uv}, c_0) = M_1$ from Table 1. and $I_3\sigma(\Delta^{uv}, c_0) \cap II_3\sigma(\Delta^{uv}, c_0) \cap III_3\sigma(\Delta^{uv}, c_0) = \emptyset$.

Theorem 2.7 (a) $\sigma_{ap}(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C}: |\lambda - v| \leq u\}$,

(b) $\sigma_\delta(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C}: |\lambda - v| < u\}$,

(c) $\sigma_{co}(\Delta^{uv}, c_0) = \emptyset$.

Proof. (a) It is clear from Theorem 2.3 and Corollary 2.1, since

$\sigma_{ap}(\Delta^{uv}, c_0) = \sigma(\Delta^{uv}, c_0) \setminus III_1\sigma(\Delta^{uv}, c_0)$ from Table 1.

(b) It is clear from Theorem 2.3 and Theorem 2.1, since $\sigma_\delta(\Delta^{uv}, c_0) = \sigma(\Delta^{uv}, c_0) \setminus I_3\sigma(\Delta^{uv}, c_0)$

from Table 1.

(c) It is clear from Theorem 2.2 and Corollary 2.2, since

$$\begin{aligned}\sigma_{co}(\Delta^{uv}, c_0) &= III_1\sigma(\Delta^{uv}, c_0) \cup III_2\sigma(\Delta^{uv}, c_0) \cup III_3\sigma(\Delta^{uv}, c_0)\\ &= \sigma_r(\Delta^{uv}, c_0) \cup III_3\sigma(\Delta^{uv}, c_0)\end{aligned}$$

from Table 1.

Corollary 2.3 (a) $\sigma_{ap}((\Delta^{uv})^*, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - v| < u\}$,

(b) $\sigma_\delta((\Delta^{uv})^*, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - v| \leq u\}$.

Proof. It is clear from Theorem 2.7 and Proposition 1.1 (c) and (d).

2.2. Subdivision of the spectrum of Δ^{uv} on c

If $T: c \to c$ is a bounded linear operator with matrix A, then the adjoint operator $T^*: c^* \to c^*$ acting on $\mathbb{C} \oplus \ell_1$ has a matrix representation of the form

$$\begin{pmatrix} \chi & 0 \\ b & A^t \end{pmatrix}$$

where χ is the limit of the sequence of row sums of A minus the sum of the limit of the columns of A, and b is the column vector whose k-th entry is the limit of the k-th column of A for each $k \in \mathbb{N}$. For $\Delta^{uv}: c \to c$, the matrix $(\Delta^{uv})^* \in B(\ell_1)$ is of the form

$$\begin{pmatrix} u+v & 0 \\ 0 & (\Delta^{uv})^t \end{pmatrix}.$$

It should be noted that the dual space c^* of c is isomorphic to the Banach space ℓ_1 of absolutely summable sequences normed by $\|x\|_{\ell_1} = \sum_k |x_k|$.

The fine spectrum of the operator Δ^{uv} over the sequence space c has been studied by Fathi [11]. In this subsection we summarize the main results.

Theorem 2.8 ([11], Theorem 3.2) $\sigma_p(\Delta^{uv}, c) = \{\lambda \in \mathbb{C} : |\lambda - v| < u\} \cup M_1$, where

$$M_1 = \left\{\lambda \in \mathbb{C} : |\lambda - v| = u,\ \lim_{k \to \infty} \left(\prod_{i=1}^{k} \frac{v_{i-1} - \lambda}{u_i}\right)\ exist\right\}.$$

Theorem 2.9 ([11], Corollary 3.5) $\sigma_r(\Delta^{uv}, c) = \emptyset$.

Corollary 2.4 $III_1\sigma(\Delta^{uv}, c) = III_2\sigma(\Delta^{uv}, c) = \emptyset$.

Proof. It is clear from Theorem 2.9, since from Table 1.,

$\sigma_r(\Delta^{uv}, c) = III_1\sigma(\Delta^{uv}, c) = III_2\sigma(\Delta^{uv}, c)$.

Theorem 2.10 ([11], Theorem 3.6) (a) $\sigma(\Delta^{uv}, c) = \{\lambda \in \mathbb{C}: |\lambda - v| \leq u\}$,

(b) $\sigma_c(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C}: |\lambda - v| = u\} \setminus M_2$.

Theorem 2.11 ([11], Theorem 3.7) If $|\lambda - v| < u$, then $\lambda \in I_3\sigma(\Delta^{uv}, c)$.

Theorem 2.12 If $\lambda \in M_2$, then $\lambda \in II_3\sigma(\Delta^{uv}, c)$.

Proof. Let we find $\ker(\Delta^{uv} - \lambda I)^*$. If $(\Delta^{uv} - \lambda I)^* x = 0$, then we get

$$\begin{aligned}
(\lambda - u - v)x_0 &= 0\\
(\lambda - v_0)x_1 &= 0
\end{aligned}$$

$$\begin{aligned}
u_1x_1 + (\lambda - v_1)x_2 &= 0\\
u_2x_2 + (\lambda - v_2)x_3 &= 0
\end{aligned}$$

$$\ldots$$

$$u_{n-1}x_{n-1} + (\lambda - v_{n-1})x_n = 0$$

$$\ldots$$

Hence we get

$$x_n = x_1 \prod_{k=1}^{n-1} \frac{u_k}{\lambda - v_k}, \quad n \geq 2.$$

If $\lambda \in M_2$, then $\lambda \neq v_0$, and $\lambda \neq u + v$, since (v_k) is either constant or strictly decreasing sequence of positive real numbers with $v = \lim_{k\to\infty} v_k \neq 0$, and $\sup_k v_k \leq u + v$. From here x_0 and must x_1 be zero and so $\ker(\Delta^{uv} - \lambda I)^* = \{\theta\}$. This means that for $\lambda \in M_2$, $(\Delta^{uv} - \lambda I)^*$ is one to one. Thus for $\lambda \in M_2$, $\lambda I - \Delta^{uv}$ has a dense range from Lemma 1.2. Therefore we have

$$II_3\sigma(\Delta^{uv}, c_0) = M_2.$$

Corollary 2.5 $III_3\sigma(\Delta^{uv}, c) = \emptyset$.

Proof. It is clear from Theorem 2.8, Theorem 2.11 and Theorem 2.12, since

$\sigma_p(\Delta^{uv},c) = I_3\sigma(\Delta^{uv},c) \cup II_3\sigma(\Delta^{uv},c) \cup III_3\sigma(\Delta^{uv},c) = M_2$ from Table 1. and $I_3\sigma(\Delta^{uv},c) \cap II_3\sigma(\Delta^{uv},c) \cap III_3\sigma(\Delta^{uv},c) = \emptyset$.

Theorem 2.13 (a) $\sigma_{ap}(\Delta^{uv},c) = \{\lambda \in \mathbb{C}: |\lambda - v| \leq u\}$,

(b) $\sigma_\delta(\Delta^{uv},c) = \{\lambda \in \mathbb{C}: |\lambda - v| < u\}$,

(c) $\sigma_{co}(\Delta^{uv},c) = \emptyset$.

Proof. (a) It is clear from Theorem 2.10 (a) and Corollary 2.5, since

$\sigma_{ap}(\Delta^{uv},c) = \sigma(\Delta^{uv},c) \backslash III_1\sigma(\Delta^{uv},c)$ from Table 1.

(b) It is clear from Theorem 2.10 and Theorem 2.11, since $\sigma_\delta(\Delta^{uv},c) = \sigma(\Delta^{uv},c) \backslash I_3\sigma(\Delta^{uv},c)$

from Table 1.

(c) It is clear from Theorem 2.9 and Corollary 2.5, since

$\sigma_{co}(\Delta^{uv},c) = III_1\sigma(\Delta^{uv},c) \cup III_2\sigma(\Delta^{uv},c) \cup III_3\sigma(\Delta^{uv},c) = \sigma_r(\Delta^{uv},c) \cup III_3\sigma(\Delta^{uv},c)$

from Table 1.

Corollary 2.6 (a) $\sigma_{ap}((\Delta^{uv})^*,\ell_1) = \{\lambda \in \mathbb{C}: |\lambda - v| < u\}$,

(b) $\sigma_\delta((\Delta^{uv})^*,\ell_1) = \{\lambda \in \mathbb{C}: |\lambda - v| \leq u\}$.

Proof. It is clear from Theorem 2.13 and Proposition 1.1 (c) and (d).

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