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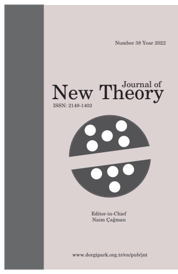

Internal Cat-1 and XMod

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Research Article

Abstract − In this study, internal categories in the category of cat-1 groups, 1-cat, are determined, and it is investigated whether there is a natural equivalence between the category of these categories and the category of internal categories within the category of crossed modules of groups, XMod.

Keywords − *Crossed module, cat-1 group, internal category, natural equivalence*

Mathematics Subject Classification (2020) − 18D35, 18G45

1. Introduction

The equivalence between the homotopy category of connected CW-complexes X whose homotopy groups $\pi_i(X)$ are trivial for $i > 1$ and the category of groups is well known. In [1] it is given that an analogous equivalence for $i > n+1$ (where n is a constant natural number). Whitehead invented the concept of a crossed module for $n = 1$. This notion replaces that of a group and gives a satisfactory answer. Loday reformulates the concept of crossed module to produce an "n-cat-group", which is a generalization to any n.

The notion of a cat-1 group is merely another method to express the axioms of a strict two-group. Nevertheless the type of characterization used for cat-1 groups, and, modified for cat-n groups is in strictly group theoretic terms and so is frequently better to check than the more categorically defined variant. As an example, getting a cat-1 group structure from a simplicial group, or a cat-n group structure from an n-fold simplicial group is typically easy. The notion of a category can be formulated internal to any other category with enough pullbacks. Since algebraic structures can be defined in a category by giving suitable objects and morphisms, we can sometimes construct categories within a category, $\mathfrak{C}$. In order to form a category (with objects O and morphisms A) inside $\mathfrak{C}$, we need to define a composition $m : A_t \times_s A \to A$ which is associative and respects identities; note in particular that m is also a morphism in $\mathfrak{C}$.

A cat-1 group is essentially another way of expressing an internal category in the category of Groups, Grp, where the kernel commutator condition determines the interchange law. It is well-known that these latter objects are equivalent to crossed modules, and so it's not surprising to see an equivalence between the category of cat-1 group and that of crossed modules in Loday's study.

Alp and Wensley present a share package XMOD consisting functions for computing with finite, permutation crossed modules, cat-1 groups and their morphisms, written using the GAP group theory

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programming language [2]. Also in [3,4], Porter generalized the category of categorical groups to that of categories of groups with operations.

The main object of this paper is to formulate internal categories for the category of cat-1 group in which the objects contain elements we can describe pullbacks in terms of. Also, it is observed that the equivalence of between internal 1-Cat group and internal crossed modules. As we know the category of cat-1 group is equivalent to that of crossed modules, we expect to be able to go between internal category of cat-1 groups and that of crossed modules without hindrance, and we can prove that the equivalence between the category of cat-1 group and that of crossed modules is also preserved for their internal categories.

2. Preliminaries

Definition 2.1. Let $\mathfrak{C}$ be a category with finite products. The internal category $\mathcal{C}$ in $\mathfrak{C}$ consists of the objects A, O with the morphisms $s, t : A \longrightarrow O$, e: O $\longrightarrow A$, $m : A \times A \longrightarrow A$. The diagram of the morphisms

$$\begin{tikzcd} A \arrow[r, shift left=2, "t"] \arrow[r, "s"'] & O \arrow[l, bend left, "e"] \end{tikzcd}$$

has the following equalities:

$$i.se = te = id_O$$
$$ii.sm = s\pi_2, tm = t\pi_1$$
$$iii.m(1_A \times m) = m(m \times 1_A)$$
$$iv.m(es, 1_A) = m(1_A, et) = 1_A$$

where π_1 and π_2 are the projections. In the internal category s, t, e and m are denoted the source, target, identity and composition morphism, respectively.

(A, O, s, t, e, m) is called the internal category of $\mathcal{C}$ in $\mathfrak{C}$, [5].

Definition 2.2. A cat-1 group consists of a group G with a normal subgroup N and the morphisms s, t from G to N satisfied the following conditions:

- $s|_N = t|_N = id_N$
- $[Kers, Kert] = 1$

A cat-1 group is denoted by (G, N, s, t), [1], [6].

Definition 2.3. Let (G, N, s, t) and (G', N', s', t') be cat-1 groups. A cat-1 group morphism $(G, N, s, t) \rightarrow (G', N', s', t')$ is an $\alpha : G \rightarrow G'$ group homomorphism satisfied the below equations:

Example 2.4. Let G be a group with a normal subgroup $N = G$. We get a cat-1 group (G, G, id_G, id_G) for $s = t = id_G$.

Example 2.5. Let G be an abelian group with a normal subgroup $N = \{1\}$. $(G, \{1\}, s, t)$ is a cat-1 group with $s(g) = t(g) = 1$ for $g \in G$.

Definition 2.6. Let G and N be two groups, $\partial : N \longrightarrow G$ a group homomorphism and G acts on N on the left. So (G, N, ∂) is a crossed module if and only if

(CM1) $\partial(g \cdot n) = g + \partial(n) - g$
(CM2) $\partial(n) \cdot n_1 = n + n_1 - n$

for $\forall n, n_1 \in N$ and $g \in G$, [7].

Definition 2.7. Let (G_1, N_1, ∂_1) and (G_0, N_0, ∂_0) be crossed modules. The crossed module morphism

$$(\alpha, \beta) : (G_0, N_0, \partial_0) \rightarrow (G_1, N_1, \partial_1)$$

is a pair of homomorphisms $\alpha : G_0 \rightarrow G_1$ and $\beta : N_0 \rightarrow N_1$ such that

- $\partial_1\beta(n) = \alpha\partial_0(n)$, for all $n \in N_0$,
- $\beta({}^{g_0}n_0) = {}^{\alpha(g_0)}\beta(n_0)$, for all $g_0 \in G_0, n_0 \in N_0$, [8].

Definition 2.8. Let $XMod$ be a category of crossed modules over groups and $\mathfrak{X}$ be an internal category. So $\mathfrak{X}$ includes $X_1 = (A_1, B_1, \partial_1)$ and $X_0 = (A_0, B_0, \partial_0)$ with source $s = (s_A, s_B)$, target $t = (t_A, t_B)$, identity $e = (e_A, e_B)$ and composition $m = (m_A, m_B)$ morphism defined by $m_A(a_1, a_1') = a_1 \circ a_1'$, $m_B(b_1, b_1') = b_1 \circ b_1'$ with $s_1(a_1) = t_1(a_1')$ and $s_0(b_1) = t_0(b_1')$. Then we have the following features:

$$i. s_A e_A = t_A e_A = id_{A_0}, s_B e_B = t_B e_B = id_{B_0}$$
$$ii. s_A m_A = s_A \pi_2, t_A m_A = t_A \pi_1, s_B m_B = s_B \pi_2, t_B m_B = t_B \pi_1$$
$$iii. m(1_{X_1} \times m) = m(m \times 1_{X_1})$$
$$iv. m(e_A s_A, 1_{X_1}) = m(1_{X_1}, e_A t_A) = 1_{X_1}, m(e_B s_B, 1_{X_1}) = m(1_{X_1}, e_B t_B) = 1_{X_1}$$

The condition iii can be expressed the following diagram:

$$\begin{tikzcd} X_1 \times X_1 \times X_1 \arrow[r, "id_{X_1} \times m"] \arrow[d, "m \times id_{X_1}"'] & X_1 \times X_1 \arrow[d, "m"] \\ X_1 \times X_1 \arrow[r, "m"'] & X_1 \end{tikzcd}$$

$\mathcal{X} = (X_1, X_0, s, t, e, m)$ is called the internal category of crossed modules over groups, [9], [10], [11].

3. Internal of cat-1

Let C be an internal category in the category 1-Cat of cat-1 groups. Then $\mathcal{C}$ consists of two cat-1 groups $X_1 = (G_1, N_1, s, t)$, $X_0 = (G_0, N_0, s', t')$ with $s^* = (s_1, s_0)$, $t^* = (t_1, t_0)$, $e^* = (e_1, e_0)$ illustrated below diagram

$$\begin{tikzcd} G_1 \arrow[r, shift left, "t_1"] \arrow[r, shift right, "s_1"'] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & G_0 \arrow[l, bend left, "e_1"] \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] \\ N_1 \arrow[r, shift left, "t_0"] \arrow[r, shift right, "s_0"'] & N_0 \arrow[l, bend left, "e_0"] \end{tikzcd}$$

and $m = (m_G, m_N) : X_1 \times X_1 \longrightarrow X_1$ morphisms. $(G_1, G_0, s_1, t_1, m_1)$ and $(N_1, N_0, s_0, t_0, m_0)$ are the internal of groups.

$$i. s_1 e_1 = t_1 e_1 = id_{G_0}, s_0 e_0 = t_0 e_0 = id_{N_0}$$
$$ii. s_1 m_G = s_1 \pi_2, t_1 m_G = t_1 \pi_1, s_0 m_N = s_0 \pi_2, t_0 m_N = t_0 \pi_1$$
$$iii. m(1_{G_1} \times m) = m(m \times 1_{G_1})$$
$$iv. m(e_1 s_1, 1_{G_1}) = m(1_{G_1}, e_1 t_1) = 1_{G_1}, m(e_0 s_0, 1_{N_1}) = m(1_{N_1}, e_0 t_0) = 1_{N_1}$$

where $m = (m_G, m_N)$ is the composition map for the internal of the category Grp of groups and $m_G(g_1, g_1') = g_1 \circ g_1'$, $m_N(n_1, n_1') = n_1 \circ n_1'$ with $s_1(g_1) = t_1(g_1')$ and $s_0(n_1) = t_0(n_1')$. The category of internal categories within the category of cat-1 groups is denoted by C(Int(cat-1)).

Proposition 3.1. Let (G_1, N_1, s, t) be a cat-1 group. Then $(G_1 \times G_1, N_1 \times N_1, (s, s), (t, t))$ is a cat-1 group and $(X_1, X_0, s^*, t^*, e^*, m^*)$ becomes an internal category in 1-Cat where $X_1 = (G_1 \times G_1, N_1 \times N_1, (s, s), (t, t)), X_0 = (G_1, N_1, s, t), s^* = (s_G, s_N), t^* = (t_G, t_N), e^* = (e_G, e_N), m = (m_G, m_N)$.

Proof. We show that $(G_1 \times G_1, N_1 \times N_1, (s,s), (t,t))$ is a cat-1 group.

$$\begin{aligned} Ker(s,s) &= \{(g,g') \in G_1 \times G_1 | (s,s)(g,g') = 1_{N_1 \times N_1}\} \\ &= \{(g,g') \in G_1 \times G_1 | (sg, sg') = (1,1)\} \\ &= Kers \times Kers \end{aligned}$$

As the same way, $Ker(t,t) = Kert \times Kert$.
Since for $(n_1, n_1') \in N_1 \times N_1, (g_1, g_2) \in Kers, (g_3, g_4) \in Kert$ the equations

$$(s,s)|_{N_1 \times N_1}(n_1, n_1') = (s(n_1), s(n_1')) = (n_1, n_1') = Id_{N_1 \times N_1}(n_1, n_1'),$$

$$(t,t)|_{N_1 \times N_1}(n_1, n_1') = (t(n_1), t(n_1')) = (n_1, n_1') = Id_{N_1 \times N_1}(n_1, n_1')$$

and

$$\begin{aligned} (g_1, g_2)(g_3, g_4)(g_1^{-1}, g_2^{-1})(g_3^{-1}, g_4^{-1}) &= (g_1 g_3 g_1^{-1} g_3^{-1}, g_2 g_4 g_2^{-1} g_4^{-1}) \\ &= (1,1) \end{aligned}$$

are valid, the cat-1 group conditions are satisfied.
Also, we get

$$s_N(s,s)(g_1, g_1') = s_N(sg_1, sg_1') = s(g_1)s(g_1') = s(g_1 g_1') = ss_G(g_1, g_1')$$

$t_N(s,s) = st_G, tt_G = t_N(t,t)$ and $st_G = t_N(s,s)$ for the commutativity of below diagram. Thus, s^* and t^* are cat-1 group morphisms.

$$\begin{tikzcd}
(G_1 \times G_1) \times (G_1 \times G_1) \arrow[r, "m_G"] \arrow[d, shift right, "\widetilde{s}"'] \arrow[d, shift left, "\widetilde{t}"] & G_1 \times G_1 \arrow[r, shift left=2, "t_G"] \arrow[r, "s_G"'] \arrow[d, shift right, "{(s,s)}"'] \arrow[d, shift left, "{(t,t)}"] & G_1 \arrow[l, bend left, "e_G"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
(N_1 \times N_1) \times (N_1 \times N_1) \arrow[r, "m_N"] & N_1 \times N_1 \arrow[r, shift left=2, "t_N"] \arrow[r, "s_N"'] & N_1 \arrow[l, bend left, "e_N"]
\end{tikzcd}$$

(m_G, m_N) is a morphism of cat-1 groups because of the diagram's commutativity. □

4. Natural Equivalence

Theorem 4.1. The category of internal categories within the category of cat-1 groups is natural equivalent to the category of internal categories within the category of crossed modules over groups.

Proof. Let F be a functor

$$\mathcal{C}(Int(cat\text{–}1)) \xrightarrow{F} C(Int(XMod))$$

$$\begin{tikzcd}
G_1 \arrow[r, shift left=2, "s_1"] \arrow[r, "t_1"'] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & G_0 \arrow[l, bend left, "e_1"] \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] \\
N_1 \arrow[r, shift left=2, "s_0"] \arrow[r, "t_0"'] & N_0 \arrow[l, bend left, "e_0"]
\end{tikzcd}
\overset{F}{\longmapsto}
\begin{tikzcd}
Kers \arrow[r, shift left=2, "s_1|_{Kers}"] \arrow[r, "t_1|_{Kers}"'] \arrow[d, "t|_{Kers}"'] & Kers' \arrow[l, bend left, "e_1|_{Kers'}"] \arrow[d, "t'|_{Kers'}"] \\
Ims \arrow[r, shift left=2, "t_0|_{Ims}"] \arrow[r, "s_0|_{Ims}"'] & Ims' \arrow[l, bend left, "e_0|_{Ims'}"]
\end{tikzcd}$$

where $s_1|_{Kers}, t_1|_{Kers}, s_0|_{Ims}$ and $t_0|_{Ims}$ are well-defined morphisms with the following relations:

$$s't_1|_{Kers}(x) = t_0s(x) = t_0(1) = 1$$

and

$$s's_1|_{Kers}(x) = s'_0s(x) = s'_0(1) = 1$$

for $x \in Kers$,

$$s_0|_{Ims}(n) \in Ims'$$

and

$$t_0|_{Ims}(n) \in Ims'$$

with $n = s(g), s_0s(g) = s's_1(g)$ and $t_0|_{Ims}(n) = t_0|_{Ims}s(g) = s't_1(g)$ for $g \in G$ and $n \in Ims$.

Since $t't_1(g) = t_0t(g)$ and $s's_1(g) = s_0s(g)$ for $g \in G_1$, we get

$$t'|_{Kers'}t_1|_{Kers}(x) = t_0|_{Ims}t|_{Kers}(x)$$

for $x \in Kers$. $t|_{Kers} : Kers \to Ims$ is a crossed module with the conjugation action of Ims on $Kers$, [1]. The composition $m_1 : Kers \times Kers \to Kers$

$$m_1(x, y) = m_G|_{Kers}(x, y) = m_N(s, s)(x, y)$$

is well-defined group homomorphism for $x, y \in Kers$, since

$$sm_1(x, y) = sm_G|_{Kers}(x, y) = m_N(s, s)(x, y) = m_N(sx, sy) = m_N(1, 1) = 1.$$

It is clear that $m_0 : Ims \times Ims \to Ims$

$$m_0(a, b) = m_N(a, b)$$

is also well-defined for $a, b \in Ims$.

$$\begin{tikzcd}
Kers \times Kers \arrow[r, "m_1"] \arrow[d, "(t|_{Kers}, t|_{Kers})"'] & Kers \arrow[d, "t|_{Kers}"] \\
Ims \times Ims \arrow[r, "m_0"] & Ims
\end{tikzcd}$$

The above diagram is commutative:

$$\begin{aligned}
t|_{Kers}m_1(x, y) &= t|_{Kers}(m_G|_{Kers}(x, y)) = t|_{Kers}m_N(s(x), s(y)) \\
m_0(t|_{Kers}, t|_{Kers})(x, y) &= m_0(t|_{Kers}(x), t|_{Kers}(y)) = m_N(t|_{Kers}(x), t|_{Kers}(y)) = t|_{Kers}m_G(x, y)
\end{aligned}$$

for $x, y \in Kers$. Also,

$$\begin{aligned}
{}^{m_0(a,b)}m_1(x, y) &= m_N(a, b)m_1(x, y)m_N(a, b)^{-1} \\
&= m_G(a, b)m_G(x, y)m_G(a, b)^{-1} \\
&= m_G((a, b)(x, y)(a, b)^{-1}) \\
&= m_G(axa^{-1}, byb^{-1}) \\
&= m_G({}^ax, {}^by) \\
&= m_1({}^{(a,b)}(x, y))
\end{aligned}$$

for $a, b \in Ims$. Thus (m_0, m_1) is a crossed module morphism.

Let G be a functor as the following

$$C(Int(XMod)) \xrightarrow{G} C(Int(cat{-}1))$$

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "s_A"] \arrow[r, shift right, "t_A"'] \arrow[d, "\partial_1"'] & A_0 \arrow[l, bend left, "e_A"] \arrow[d, "\partial_0"] & & A_1 \rtimes B_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] \arrow[d, shift right, "t_1^+"'] \arrow[d, shift left, "s_1^+"] & A_0 \rtimes B_0 \arrow[l, bend left, "e_1"] \arrow[d, shift right, "s_0^+"'] \arrow[d, shift left, "t_0^+"] \\
B_1 \arrow[r, shift left, "s_B"] \arrow[r, shift right, "t_B"'] & B_0 \arrow[l, bend left, "e_B"] & \overset{G}{\longmapsto} & B_1 \arrow[r, shift left, "s_B"] \arrow[r, shift right, "t_B"'] & B_0 \arrow[l, bend left, "e_B"]
\end{tikzcd}$$

where

$$\begin{aligned}
s_1^+(a_1, b_1) &= b_1 \\
s_0^+(a_0, b_0) &= b_0
\end{aligned}$$

and

$$\begin{aligned}
t_1^+(a_1, b_1) &= \partial_1(a_1)b_1 \\
t_0^+(a_0, b_0) &= \partial_0(a_0)b_0
\end{aligned}$$

for $(a_1, b_1) \in A_1 \rtimes B_1$,$(a_0, b_0) \in A_0 \rtimes B_0$. It is clear that the first axiom of the definition of cat-1 group is satisfied.

$$\begin{aligned}
Kers_1^+ &= \{(a_1, b_1) \in A_1 \times B_1 | s_1^+(a_1, b_1) = b_1 = 1_{B_1}\} = A_1 \times 1_{B_1} \\
Kert_1^+ &= \{(a_1, b_1) \in A_1 \times B_1 | t_1^+(a_1, b_1) = \partial_1(a_1)b_1 = 1_{B_1}\} \\
&= \{(a_1^{-1}, \partial_1(a_1)) | a_1 \in A_1\}
\end{aligned}$$

Since ∂_1 is a crossed module, we have $a_1 {a'_1}^{-1} = {a'_1}^{-1}({}^{\partial_1(a'_1)}a_1)$.

Thus $[Kers_1^+, Kert_1^+] = 1$ because of

$$(a_1, 1)({a'_1}^{-1}, \partial_1(a'_1))((({a'_1}^{-1}, \partial_1(a'_1))(a_1, 1))^{-1} = (a_1{a'_1}^{-1}, \partial_1(a'_1))({a'_1}^{-1}({}^{\partial_1(a'_1)}a_1), \partial_1(a'_1))^{-1} = 1,$$

[1]. So, $(A_1 \rtimes B_1, B_1, s^*, t^*)$ is a cat-1 group.

The composition $m_A : (A_1 \rtimes B_1) \times (A_1 \rtimes B_1) \to A_1 \rtimes B_1$ given by

$$m_A(((a_1, b_1), (a'_1, b'_1)) = (m_1((a_1, a'_1)), m_0((b_1, b'_1)))$$

is a group homomorphism with

$$\begin{aligned}
m_A(((a_1, b_1), (a'_1, b'_1))((\overline{a_1}, \overline{b_1}), (\overline{a'_1}, \overline{b'_1}))) &= m_A((a_1, b_1) * (\overline{a_1}, \overline{b_1}), (a'_1, b'_1) * (\overline{a'_1}, \overline{b'_1})) \\
&= m_A((a_1^{b_1}\overline{a_1}, b_1\overline{b_1}), ({a'_1}^{b'_1}\overline{a'_1}, b'_1\overline{b'_1})) \\
&= (m_1(a_1^{b_1}\overline{a_1}, {a'_1}^{b'_1}\overline{a'_1}), m_0(b_1\overline{b_1}, b'_1\overline{b'_1})) \\
&= (m_1((a_1, a'_1)^{(b_1, b'_1)}(\overline{a_1}, \overline{a'_1})), m_0((b_1, b'_1)(\overline{b_1}, \overline{b'_1}))) \\
&= (m_1(a_1, a'_1)m_1({}^{(b_1, b'_1)}(\overline{a_1}, \overline{a'_1})), m_0(b_1, b'_1)m_0(\overline{b_1}, \overline{b'_1}))) \\
&= (m_1(a_1, a'_1)^{m_0(b_1, b'_1)}m_1(\overline{a_1}, \overline{a'_1}), m_0(b_1, b'_1)m_0(\overline{b_1}, \overline{b'_1})) \\
&= (m_1(a_1, a'_1), m_0(b_1, b'_1)) * (m_1(\overline{a_1}, \overline{a'_1}), m_0(\overline{b_1}, \overline{b'_1})) \\
&= m_A((a_1, b_1), (a'_1, b'_1)) * m_A((\overline{a_1}, \overline{b_1}), (\overline{a'_1}, \overline{b'_1})).
\end{aligned}$$

Also, since $(X_1, X_0, s, t, e, (m_1, m_2))$ in $C(Int(XMod))$ where $X_1 = (A_1, B_1, \partial_1), X_0 = (A_0, B_0, \partial_0), s = (s_A, s_B), t = (t_A, t_B), e = (e_A, e_B)$, we get group morphism

$$m_2 = m_B : B_1 \times B_1 \to B_1.$$

$$\begin{tikzcd}
(A_1 \rtimes B_1) \times (A_1 \rtimes B_1) \arrow[r, "m_A"] \arrow[d, shift right, "(s_1^+, s_1^+)"'] \arrow[d, shift left, "(t_1^+, t_1^+)"] & A_1 \rtimes B_1 \arrow[d, shift right, "s_1^+"'] \arrow[d, shift left, "t_1^+"] \\
B_1 \times B_1 \arrow[r, "m_B"'] & B_1
\end{tikzcd}$$

In this diagram,

$$\begin{aligned} s_1^+ m_A &= m_B(s_1^+, s_1^+) \\ t_1^+ m_A &= m_B(t_1^+, t_1^+) \end{aligned}$$

are hold. So, (m_A, m_B) is a cat-1 group morphism.

For any object $(X_1, X_0, s^*, t^*, e^*, m)$ in $C(Int(cat{-}1))$ with $X_1 = (G_1, N_1, s, t)$, $X_0 = (G_0, N_0, s', t'), s^* = (s_1, s_0), t^* = (t_1, t_0), e^* = (e_1, e_0)$ and $m = (m_G, m_N)$, we get the following diagrams

$$\begin{tikzcd}
G_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & G_0 \arrow[l, bend left, "e_1"] \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] \\
N_1 \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] & N_0 \arrow[l, bend left, "e_0"]
\end{tikzcd}
\quad \overset{F}{\longmapsto} \quad
\begin{tikzcd}
Kers \arrow[r, shift left, "s_1|_{Kers}"] \arrow[r, shift right, "t_1|_{Kers}"'] \arrow[d, "t|_{Kers}"'] & Kers' \arrow[l, bend left, "e_1|_{Kers}"] \arrow[d, "t'|_{Kers'}"] \\
Ims \arrow[r, shift left, "t_0|_{Ims}"] \arrow[r, shift right, "s_0|_{Ims}"'] & Ims' \arrow[l, bend left, "e_0|_{Ims}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
Kers \arrow[r, shift left, "s_1|_{Kers}"] \arrow[r, shift right, "t_1|_{Kers}"'] \arrow[d, "t|_{Kers}"'] & Kers' \arrow[l, bend left, "e_1|_{Kers}"] \arrow[d, "t'|_{Kers'}"'] \\
Ims \arrow[r, shift left, "t_0|_{Ims}"] \arrow[r, shift right, "s_0|_{Ims}"'] & Ims' \arrow[l, bend left, "e_0|_{Ims}"]
\end{tikzcd}
\quad \overset{G}{\longmapsto} \quad
\begin{tikzcd}
Kers \rtimes Ims \arrow[r, shift left, "t_A"] \arrow[r, shift right, "s_A"'] \arrow[d, shift right, "t_1^+"'] \arrow[d, shift left, "s_1{}^+"] & Kers' \rtimes Ims' \arrow[l, bend left, "e_A"] \arrow[d, shift right, "s_0^+"'] \arrow[d, shift left, "t_0^+"] \\
Ims \arrow[r, shift left, "s_B"] \arrow[r, shift right, "t_B"'] & Ims'. \arrow[l, bend left, "e_B"]
\end{tikzcd}$$

Thus, we have $GF(X_1, X_0, s^*, t^*, e^*, m)$ as an object in $C(Int(cat{-}1))$.

$$\begin{tikzcd}
Kers \rtimes Ims \arrow[r, shift left, "t_A"] \arrow[r, shift right, "s_A"'] \arrow[d, shift right, "t_1^+"'] \arrow[d, shift left, "s_1^+"] & Kers' \rtimes Ims' \arrow[l, bend left, "e_A"] \arrow[d, shift right, "s_0^+"'] \arrow[d, shift left, "t_0^+"] \\
Ims \arrow[r, shift left, "t_B"] \arrow[r, shift right, "s_B"'] & Ims' \arrow[l, bend left, "e_B"]
\end{tikzcd}$$

Also, we have $G_1 \cong Kers \rtimes Ims, N_1 \cong Ims, G_0 \cong Kers' \rtimes Ims'$ and $N_0 \cong Ims'$ and the natural transformation

$$\zeta : 1_{C(Int(cat-1))} \Rightarrow GF$$

Conversely, for any object (X_1, X_0, s, t, e, m) in $C(Int(XMod))$ where $X_1 = (A_1, B_1, \partial_1), X_0 = (A_0, B_0, \partial_0)$, we get the following diagrams

$$
\begin{tikzcd}
A_1 \arrow[r, shift left, "s_A"] \arrow[r, shift right, "t_A"'] \arrow[d, "\partial_1"'] & A_0 \arrow[l, bend left, "e_A"] \arrow[d, "\partial_0"] \\
B_1 \arrow[r, shift left, "s_B"] \arrow[r, shift right, "t_B"'] & B_0 \arrow[l, bend left, "e_B"]
\end{tikzcd}
\quad \xmapsto{G} \quad
\begin{tikzcd}
A_1 \rtimes B_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] \arrow[d, shift right, "t_1^+"'] \arrow[d, shift left, "s_1^+"] & A_0 \rtimes B_0 \arrow[l, bend left, "e_1"] \arrow[d, shift right, "s_0^+"'] \arrow[d, shift left, "t_0^+"] \\
B_1 \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] & B_0 \arrow[l, bend left, "e_0"]
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
A_1 \rtimes B_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] \arrow[d, shift right, "t_1^+"'] \arrow[d, shift left, "s_1^+"] & A_0 \rtimes B_0 \arrow[l, bend left, "e_1"] \arrow[d, shift right, "t_0^+"'] \arrow[d, shift left, "s_0^+"] \\
B_1 \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] & B_0 \arrow[l, bend left, "e_0"]
\end{tikzcd}
\quad \xmapsto{F} \quad
\begin{tikzcd}
Kers_1^+ \arrow[r, shift left, "s_1|_{Kers_1^+}"] \arrow[r, shift right, "t_1|_{Kers_1^+}"'] \arrow[d, "t_1^+|_{Kers_1^+}"'] & Kers_0^+ \arrow[l, bend left, "e_1|_{Kers_0^+}"] \arrow[d, "t_0^+|_{Kers_0^+}"] \\
Ims_1^+ \arrow[r, shift left, "t_0|_{Ims_1^+}"] \arrow[r, shift right, "s_0|_{Ims_1^+}"'] & Ims_0^+ \arrow[l, bend left, "e_0|_{Ims_0^+}"]
\end{tikzcd}
$$

Therefore, we get $F(G(X_1, X_0, s, t, e))$

$$
\begin{tikzcd}
Kers_1^+ \arrow[r, shift left, "s_1|_{Kers_1^+}"] \arrow[r, shift right, "t_1|_{Kers_1^+}"'] \arrow[d, "t_1^+|_{Kers_1^+}"'] & Kers_0^+ \arrow[l, bend left, "e_1|_{Kers_0^+}"] \arrow[d, "t_0^+|_{Kers_0^+}"] \\
Ims_1^+ \arrow[r, shift left, "s_0|_{Ims_1^+}"] \arrow[r, shift right, "t_0|_{Ims_1^+}"'] & Ims_0^+ \arrow[l, bend left, "e_0|_{Ims_0^+}"]
\end{tikzcd}
$$

as an object in $C(Int(XMod))$. We can easily find

$$
\begin{aligned}
Kers_1^+ \cong A_1, Kers_0^+ \cong A_0 \\
Ims_1^+ \cong B_1, Ims_0^+ \cong B_0.
\end{aligned}
$$

So, there is a

$$\xi : 1_{C(Int(XMod))} \Rightarrow FG$$

natural transformation.

Finally, there is a natural equivalence between the category of internal categories in cat-1 and the category of internal categories in XMod. $\square$

5. Conclusion

It is possible that each category containing pullbacks can generate other categories inside that category. By this idea, we construct internal categories in the category of cat-1 groups. Since the category of crossed modules is equivalent to that of cat-1 groups, we conclude that this equivalence is also between their internal categories valid. This idea can be extended to other equivalent categories.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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