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A General Fixed Point Theorem for Mappings Satisfying An ϕ - Implicit Relation in Complete *G* - Metric Spaces

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ABSTRACT
In this paper a general fixed point theorem for mappings satisfying an ϕ - implicit relation is proved, which generalize the results from [3] and [14].

Keywords: fixed point, G - metric space, implicit relation, ϕ - implicit relation.

1. INTRODUCTION

In [4], [5] Dhage introduced a new class of generalized metric space, called D - metric space. Mustafa and Sims [12], [13] proved that most of the claims concerning the fundamental topological structures of D - metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named G - metric space. In fact, Mustafa and other authors [3], [7] – [15], [18] studied many fixed point results for self mappings in a G - metric space under certain conditions. In [6] and [18] some fixed point theorems for mappings satisfying ϕ - maps are proved. In [16], [17], Popa initiated the study of fixed points for mappings satisfying implicit relations. In [2] Altun and Turkoglu introduced a new type of implicit relations satisfying a ϕ - map. The purpose of this paper is to prove a general fixed point theorem in G - metric spaces for mappings satisfying an ϕ - implicit relation which generalize the results from [3] and [14].

2. PRELIMINARIES

Definition 2.1 ([13]) *Let X be a nonempty set and $G : X^3 \to \mathbf{R}_+$ satisfying the following properties:*

$(G_1) : G(x, y, z) = 0$ *if* $x = y = z$ *;*

$(G_2) : 0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$ *;*

$(G_3) : G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $y \neq z$ *;*

$(G_4) : G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ *(symmetry in all three variables);*

$(G_5) : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ *(rectangle inequality).*

Then, the function G is called a G - metric on X and (X, G) is called a G - metric space.

Note that if $G(x, y, z) = 0$ then $x = y = z$.

Definition 2.2 ([13]) *Let (X, G) be a G - metric space. A sequence (x_n) in X is said to be*

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• *G - convergent if for* $\varepsilon > 0$, *there exists an* $x \in X$ *and* $k \in \mathbf{N}$ *such that for all* $m, n \geq k$, $G(x, x_n, x_m) < \varepsilon$.

• *G - Cauchy if for each* $\varepsilon > 0$, *there exists* $k \in \mathbf{N}$ *such that for all* $m, n, p \geq k$, $G(x_n, x_m, x_p) < \varepsilon$, *that is* $G(x_n, x_m, x_p) \to 0$ *as* $n, m, p \to \infty$.

A space (X, G) *is called G - complete if every G - Cauchy sequence in* (X, G) *is G - convergent.*

Lemma 2.1 ([13]) *Let* (X, G) *be a G - metric space. Then the following properties are equivalent:*
1) (x_n) *is G - convergent to x;*
2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$;
3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$;
4) $G(x_m, x_n, x) \to 0$ *as* $n, m \to \infty$.

Lemma 2.2 ([13]) *If* (X, G) *is a G - metric space, then the following are equivalent:*
1) The sequence (x_n) *is G - Cauchy;*
2) For every $\varepsilon > 0$, *there exists* $k \in \mathbf{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$ *for* $n, m > k$.

Lemma 2.3 ([13]) *Let* (X, G) *be a G - metric space. Then, the function* $G(x, y, z)$ *is jointly continuous in all three of its variables.*

3. IMPLICIT RELATIONS

Definition 3.1. *A function* $f : [0, \infty) \to [0, \infty)$ *is a* ϕ *- function,* $f \in \phi$ *if f is a nondecreasing function such that* $\sum_{n=1}^{\infty} f^n(t) < \infty$, *for all* $f(t) < t$ *for* $t > 0$ *and* $f(0) = 0$.

Definition 3.2. *Let* $\mathbf{F}_\phi$ *be the set of all continuous functions* $F(t_1, ..., t_6) : \mathbf{R}_+^6 \to \mathbf{R}$ *such that:*
(F_1): *F is nonincreasing in* t_5,
(F_2): *There exists a function* $\phi_1 \in \phi$ *such that for all* $u, v \geq 0$, $F(u, v, v, u, u+v, 0) \leq 0$ *implies* $u \leq \phi_1(v)$,
(F_3): *There exists a function* $\phi_2 \in \phi$ *such that for all* $t, t' > 0$, $F(t, t, 0, 0, t, t') \leq 0$ *implies* $t \leq \phi_2(t')$.

Example3.1.
$F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, *where* $a > 0$, $b, c, d, e \geq 0$, $a + b + c + 2d + e < 1$.
(F_1): Obviously.
(F_2): Let $u, v \geq 0$ be and $F(u, v, v, u, u+v, 0) = u - av - bv - cu - d(u+v) \leq 0$ which implies $u \leq \frac{a+b+d}{1-c-d} v$ and (F_2) is satisfied for $\phi_1(t) = \frac{a+b+d}{1-(c+d)} t$.
(F_3): Let $t, t' > 0$ be and $F(t, t, 0, 0, t, t') = t - at - dt - et' \leq 0$ which implies $t \leq \frac{e}{1-(a+d)} t'$ and (F_3) is satisfied for $\phi_2(t) = \frac{e}{1-(a+d)} t$.

Example 3.2. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, ..., t_6\}$, *where* $k \in \left(0, \frac{1}{2}\right)$.
(F_1): Obviously.
(F_2): Let $u, v \geq 0$ be and $F(u, v, v, u, u+v, 0) = u - k \max\{u, v, u+v\} \leq 0$. Hence $u \leq \frac{k}{1-k} v$ and (F_2) is satisfied for $\phi_1(t) = \frac{k}{1-k} t$.
(F_3): Let $t, t' > 0$ be and $F(t, t, 0, 0, t, t') = t - k \max\{t, t'\} \leq 0$. If $t > t'$, then $t(1-k) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq kt'$ and (F_3) is satisfied for $\phi_2(t) = kt$.

Example3.3.
$F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}$, *where* $k \in (0, 1)$.
(F_1): Obviously.
(F_2): Let $u, v \geq 0$ be and $F(u, v, v, u, u+v, 0) = u - k \max\left\{u, v, \frac{u+v}{2}\right\} \leq 0$. If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence $u \leq v$ which implies $u \leq kv$ and (F_2) is satisfied for $\phi_1(t) = kt$.
(F_3): Let $t, t' > 0$ be and $F(t, t, 0, 0, t, t') = t - k \max\left\{t, \frac{t+t'}{2}\right\} \leq 0$. If $t > t'$, then $t(1-k) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq kt'$ and (F_3) is satisfied for $\phi_2(t) = kt$.

Example3.4.
$F(t_1, ..., t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5 t_6$, *where* $a > 0$, $b, c, d \geq 0$, $a + b + c < 1$.
(F_1): Obviously.
(F_2): Let $u, v \geq 0$ be and $F(u, v, v, u, u+v, 0) = u^2 - u(av + bv + cu) \leq 0$. If $u > 0$, then $u \leq \frac{a+b}{1-c} v$. If $u = 0$ then $u \leq \frac{a+b}{1-c} v$ and (F_2) is satisfied for $\phi_1(t) = \frac{a+b}{1-c} t$.

(F_3): Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t^2 - at^2 - ctt' \le 0$, which implies $t \le \frac{c}{1-a} t'$ and (F_3) is satisfied for $\phi_2(t) = \frac{c}{1-a} t.$

Example3.5.
$F(t_1,...,t_6) = t_1 - k \max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}$, *where* $k \in (0,1)$.

(F_1): Obviously.

(F_2): Let $u, v \ge 0$ be and $F(u,v,v,u,u+v,0) = u - k \max\left\{v, \frac{u+v}{2}\right\} \le 0$. If $u > v$, then $u(1-k) \le 0$, a contradiction. Hence $u \le v$ which implies $u \le kv$ and (F_2) is satisfied for $\phi_1(t) = kt.$

(F_3): Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t - k \max\left\{t, \frac{t+t'}{2}\right\} \le 0$. If $t > t'$ then $t(1-k) \le 0$, a contradiction, hence $t \le t'$, which implies $t \le kt'$ and (F_3) is satisfied for $\phi_2(t) = kt.$

Example 3.6. $F(t_1,...,t_6) = t_1^3 - c\frac{t_3^2 t_4^2 + t_5^2 t_6^2}{1 + t_2 + t_3 + t_4}$, *where* $c \in (0,1)$.

(F_1): Obviously.

(F_2): Let $u, v \ge 0$ be and $F(u,v,v,u,u+v,0) = u^3 - c\frac{u^2 v^2}{1+2v+u} \le 0$. If $u > 0$, then $u \le cv\frac{v}{1+2v+u} \le cv$. If $u = 0$, then $u \le cv$ and (F_2) is satisfied for $\phi_1(t) = ct.$

(F_3): Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t^3 - \frac{ct^2(t')^2}{1+t} \le 0$, which implies $t^2 \le c\frac{t}{1+t}(t')^2 \le c(t')^2$, from where $t \le \sqrt{c}t'$ and (F_3) is satisfied for $\phi_2(t) = \sqrt{c}t.$

Example 3.7. $F(t_1,...,t_6) = t_1^2 - at_2^2 - c\frac{t_5 t_6}{1 + t_3^2 + t_4^2}$, *where* $a > 0$, $c \ge 0$, $a + c < 1$.

(F_1): Obviously.

(F_2): Let $u, v \ge 0$ be and $F(u,v,v,u,u+v,0) = u^2 - av^2 \le 0$. Hence $u \le \sqrt{a}v$ and (F_2) is satisfied for $\phi_1(t) = \sqrt{a}t.$

(F_3): Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t^2 - at^2 - ctt' \le 0$, which implies $t \le \frac{c}{1-a} t'$ and (F_3) is satisfied for $\phi_2(t) = \frac{c}{1-a} t.$

Example3.8.
$F(t_1,...,t_6) = t_1 - at_2 - bt_3 - c\max\{2t_4, t_5 + t_6\}$ *where* $a > 0$, $b, c \ge 0$, $a + b + 2c < 1$.

(F_1): Obviously.

(F_2): Let $u, v \ge 0$ be and $F(u,v,v,u,u+v,0) = u - av - bv - c\max\{2u, u+v\} \le 0$. If $u > v$, then $u(1-(a+b+2c)) \le 0$, a contradiction. Hence $u \le v$ which implies $u \le \frac{a+b+c}{1-c} v$ and (F_2) is satisfied for $\phi_1(t) = \frac{a+b+c}{1-c} t.$

(F_3): Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t - at - c(t+t') \le 0$, which implies $t \le \frac{c}{1-(a+c)} t'$ and (F_3) is satisfied for $\phi_2(t) = \frac{c}{1-(a+c)} t.$

Example3.9.
$F(t_1,...,t_6) = t_1 - at_2 - bt_3 - c\max\{t_4 + t_5, 2t_6\}$, *where* $a > 0$, $b, c \ge 0$, $a + b + 3c < 1$.

(F_1): Obviously.

(F_2): Let $u, v \ge 0$ be and $F(u,v,v,u,u+v,0) = u - av - bv - c(2u+v) \le 0$ which implies $u \le \frac{a+b+c}{1-2c} v$ and (F_2) is satisfied for $\phi_1(t) = \frac{a+b+c}{1-2c} t.$

(F_3): Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t - at - c\max\{t, 2t'\} \le 0$. If $t > 2t'$ then $t(1-(a+c)) \le 0$, a contradiction. Hence $t \le 2t'$ which implies $t \le \frac{2c}{1-a} t'$ and (F_3) is satisfied for $\phi_2(t) = \frac{2c}{1-a} t.$

Example3.10.
$F(t_1,...,t_6) = t_1 - c\max\{t_2, t_3, \sqrt{t_4 t_6}, \sqrt{t_5 t_6}\}$, *where* $c \in (0,1)$.

(F_1): Obviously.

(F_2): Let $u, v \ge 0$ be and $F(u,v,v,u,u+v,0) = u - cv \le 0$ which implies $u \le cv$ and (F_2) is satisfied for $\phi_1(t) = ct.$

(F_3): Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t - c\max\{t, \sqrt{tt'}\} \le 0$. If $t > t'$ then $t(1-c) \le 0$, a contradiction. Hence $t \le t'$ which implies $t \le ct'$ and (F_3) is satisfied for $\phi_2(t) = ct$.

4. MAIN RESULTS

Theorem 4.1 *Let (X,G) be a G - metric space. Suppose that*

$$F(G(Tx,Ty,Ty),G(x,y,y),G(x,Tx,Tx),\\ G(y,Ty,Ty),G(x,Ty,Ty),G(y,Tx,Tx)) \le 0, \quad (4.1)$$

for all $x,y \in X$ where F satisfies condition (F_3). Then T has at most a fixed point.

Proof. Suppose that T has two distinct fixed points u, v. Then by (4.1) we have successively

$$F(G(Tu,Tv,Tv),G(u,v,v),G(u,Tu,Tu),\\ G(v,Tv,Tv),G(u,Tv,Tv),G(v,Tu,Tu)) \le 0,$$

$$F(G(u,v,v),G(u,v,v),0,0,G(u,v,v),G(v,u,u)) \le 0,$$

which implies by (F_3) that

$$G(u,v,v) \le \phi_2(G(v,u,u)).$$

Similarly, we have

$$G(v,u,u) \le \phi_2(G(u,v,v)).$$

Hence

$$G(u,v,v) \le \phi_2(G(v,u,u)) \le \phi_2^2(G(u,v,v)) < G(u,v,v),$$

a contradiction. Hence $u = v$.

Theorem 4.2 *Let (X,G) be a complete G - metric space. Suppose that (4.1) holds for all $x,y \in X$ and $F \in \mathbf{F}_\phi$. Then T has a unique fixed point.*

Proof. By (4.1) for $y = Tx$ we obtain

$$F(G(Tx,T^2x,T^2x),G(x,Tx,Tx),G(x,Tx,Tx),\\ G(Tx,T^2x,T^2x),G(x,T^2x,T^2x),0) \le 0.$$

By rectangle inequality and (F_1) we have that

$$F(G(Tx,T^2x,T^2x),G(x,Tx,Tx),G(x,Tx,Tx),\\ G(Tx,T^2x,T^2x),G(x,Tx,Tx)+G(Tx,T^2x,T^2x),0) \le 0.$$

By (F_2) we obtain

$$G(Tx,T^2x,T^2x) \le \phi_1(G(x,Tx,Tx)). \quad (4.2)$$

Let $x_0 \in X$ be and $x_n = Tx_{n-1}, n = 0,1,2,\ldots$. Hence

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) &\le \phi_1(G(x_{n-1},x_n,x_n)) \\ &\le \phi_1^2(G(x_{n-2},x_{n-1},x_{n-1})) \\ &\le \ldots \le \phi_1^n(G(x_0,x_1,x_1)). \end{aligned}$$

By rectangle inequality we obtain

$$\begin{aligned} G(x_n,x_m,x_m) &\le G(x_n,x_{n+1},x_{n+1}) + \ldots + G(x_{m-1},x_m,x_m) \\ &\le \phi_1^n(G(x_0,x_1,x_1)) + \ldots + \phi_1^{m-1}(G(x_0,x_1,x_1)) \\ &= \sum_{k=n}^{m-1} \phi_1^k(G(x_0,x_1,x_1)). \end{aligned}$$

Let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} \phi_1^k(G(x_0,x_1,x_1)) < \infty$, there exists $k \in \mathbf{N}$ such that for $m > n \ge k$,

$$\sum_{k=n+1}^{m-1} \phi_1^k(G(x_0,x_1,x_1)) < \varepsilon.$$

It follows by Lemma 2.2 that $\{x_n\}$ is a G - Cauchy sequence in a complete G - metric space and so has a limit u. We prove that $u = Tu$. By (4.1) we have successively

$$F(G(Tx_n,Tu,Tu),G(x_n,u,u),G(x_n,Tx_n,Tx_n),\\ G(u,Tu,Tu),G(x_n,Tu,Tu),G(u,Tx_n,Tx_n)) \le 0,$$

$$F(G(x_{n+1},Tu,Tu),G(x_n,u,u),G(x_n,x_{n+1},x_{n+1}),\\ G(u,Tu,Tu),G(x_n,Tu,Tu),G(u,x_{n+1},x_{n+1})) \le 0.$$

Letting n tend to infinity we obtain

$$F(G(u,Tu,Tu),0,0,G(u,Tu,Tu),G(u,Tu,Tu),0) \le 0.$$

By (F_2) we obtain $G(u,Tu,Tu) \le \phi(0) = 0$. Hence $u = Tu$, and u is a fixed point of T. By Theorem 4.1 u is the unique fixed point of T.

Corollary 4.1 (Theorem 2.1 [14]) *Let (X,G) be a complete G - metric space and let $T: X \to X$ be a mapping which satisfy for all $x,y,z \in X$ the inequality*

$$G(Tx,Ty,Tz) \le k\max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),\\ G(x,Ty,Ty),G(y,Tz,Tz),G(z,Tx,Tx)\} \quad (4.3)$$

where $k \in \left(0,\frac{1}{2}\right)$. Then, T has a unique fixed point.

Proof. If $z = y$ by (4.3) we obtain

$$G(Tx,Ty,Ty) \le k\max\{G(x,y,y),G(x,Tx,Tx),\\ G(y,Ty,Ty),G(x,Ty,Ty),0,G(y,Tx,Tx)\}.$$

By Theorem 4.2 and Example 3.2, T has a unique fixed point.

Corollary 4.2 (Theorem 3.2 [14]) *Let (X,G) be a complete G - metric space and let $T: X \to X$ be a mapping satisfying the following inequality for all $x,y,z \in X$*

$$G(Tx,Ty,Tz) \le k\max\{G(x,Ty,Ty)+G(y,Tx,Tx),\\ G(y,Tz,Tz)+G(z,Ty,Ty),\\ G(x,Tz,Tz)+G(z,Tx,Tx)\} \quad (4.4)$$

where $k \in (0,1)$. Then, T has a unique fixed point.

Proof. If $z = y$ we obtain by (4.4) that

$$G(Tx,Ty,Ty) \le k\max\{G(x,Ty,Ty)+G(y,Tx,Tx),2G(y,Ty,Ty)\}.$$

By Theorem 4.2 and Example 3.9 with $a = b = 0$ and $c = k$, then T has a unique fixed point.

Corollary 4.3 (Theorem 2.6 [14]) *Let (X,G) be a complete G - metric space and let $T: X \to X$ be a mapping satisfying the following inequality for all $x,y,z \in X$*

$$G(Tx,Ty,Tz) \le k\max\{G(y,Ty,Ty)+G(x,Ty,Ty),2G(y,Tx,Tx)\}, \quad (4.5)$$

where $k \in \left(0, \frac{1}{3}\right)$. *Then, T has a unique fixed point.*

Proof. By Theorem 4.2 and Example 3.9 with $a = b = 0$ and $c = k$, then *T* has a unique fixed point.

Corollary 4.4 (Theorem 2.8 [14]) *Let* (X, G) *be a complete G - metric space and let* $T : X \to X$ *be a mapping satisfying the following inequality for all* $x, y, z \in X$

$$\begin{aligned} G(Tx,Ty,Tz) \le k \max\{&G(z,Tx,Tx) + G(y,Tx,Tx), \\ &G(y,Tz,Tz) + G(x,Tz,Tz), \\ &G(x,Ty,Ty) + G(y,Ty,Ty)\} \end{aligned} \tag{4.6}$$

where $k \in \left(0, \frac{1}{3}\right)$. *Then, T has a unique fixed point.*

Proof. If $z = y$ by (4.6) we obtain

$$G(Tx,Ty,Ty) \le k \max\{G(y,Ty,Ty) + G(x,Ty,Ty), 2G(y,Tx,Tx)$$

and the proof follows by Corollary 4.3.

Corollary 4.5 (Theorem 2.1 [3]) *Let* (X, G) *be a complete G - metric space and let* $T : X \to X$ *be a mapping satisfying the following inequality for all* $x, y, z \in X$

$$\begin{aligned} G(Tx,Ty,Tz) \le k \max\{&G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), \\ &\frac{G(x,Ty,Ty) + G(z,Tx,Tx)}{2}, \\ &\frac{G(y,Tz,Tz) + G(z,Ty,Ty)}{2}, \\ &\frac{G(x,Tz,Tz) + G(z,Tx,Tx)}{2}\}, \end{aligned} \tag{4.7}$$

where $k \in (0,1)$. *Then, T has a unique fixed point.*

Proof. If $z = y$ by (4.7) we obtain

$$\begin{aligned} G(Tx,Ty,Ty) \le k \max\{&G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), \\ &\frac{G(x,Ty,Ty) + G(y,Tx,Tx)}{2}\}. \end{aligned}$$

By Theorem 4.2 and Example 3.3, *T* has a unique fixed point.

Corollary 4.6 (Theorem 2.2 [3]) *Let* (X, G) *be a complete G - metric space and let* $T : X \to X$ *be a mapping satisfying the following inequality for all* $x, y, z \in X$

$$\begin{aligned} G(Tx,Ty,Tz) \le k \max\{&G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), \\ &G(x,Ty,Ty), G(z,Tx,Tx)\} \end{aligned} \tag{4.8}$$

where $k \in \left[0, \frac{1}{2}\right)$. *Then, T has a unique fixed point.*

Proof. If $y = z$ we obtain

$$\begin{aligned} G(Tx,Ty,Ty) \le k \max\{&G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), \\ &G(x,Ty,Ty), G(y,Tx,Tx)\}. \end{aligned}$$

By Theorem 4.2 and Example 3.2, *T* has a unique fixed point.

Remark 4.1.

1) In the proof of Theorem 2.2 [3], $k \in [0,1)$.

2) By Examples 3.1, 3.4, 3.5, 3.6, 3.7 and 3.10 we obtain new results.

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