**e-ISSN:** 2564-7954 CUJSE 19(02): 113-123 (2022) Research Article

Çankaya University

**Journal of Science and Engineering**

https://dergipark.org.tr/cankujse

CUJSE

# Analytical Study of Fractional Reaction-Diffusion Brusselator System

Mohamed Zellal *[1] and Kacem Belghaba [2]

[1]Department of Common Core in Exact Sciences and Informatics, Hassiba Benbouali University of Chlef, Algeria
[1, 2] Laboratory of Mathematic and Its Applications (LAMAP), University of Oran1, Algeria.

**Keywords**

Reaction-diffusion Brusselator system, Fractional derivative in Caputo sence, Laplace Adomian decomposition method, Approximate solutions.

**Abstract**

In this paper, a time fractional reaction-diffusion Brusselator system with the Caputo type fractional derivatives, is solved by Laplace Adomian decomposition method (LADM). Two numerical examples supported by graphics, tables, and discussion are provided, by comparing the numerical results obtained with the exact solution when $\alpha = \beta = 1$, it is observed that they are in perfect agreement, this confirms the accuracy and smoothness of the current method.

## 1. Introduction

Fractional differential equations have gained much attention recently due to be an important and useful tool to show the hidden aspects in many phenomena occurring from real world, including mathematics [1–3], continuum mechanics [4], magnetohydrodynamic [5], and other areas. One of the most important reaction and diffusion equations is called the Brusselator system, which is used to describe the mechanism of chemical reaction-diffusion with nonlinear oscillations [6, 7]. In recent years, many researchers have resorted to using modern and different methods to solve this system or find approximate solutions, such a second-order scheme by Twizell, Gumel and Cao in [8], authors of [9–11] employed the Adomian decomposition method, Why-Teong Ang in [12] used the dual-reciprocity boundary element method, in [13] by insert combination of collocation method using the radial basis functions (RBFs) with first order accurate forward difference approximation, by Laplace transform method and the new homotopy perturbation method in [14], authors of [15] resorted to a computational study with applications in chemical processes etc, the solutions to the system were also studied fractional order, such as q-homotopy analysis transform method in [16], by fractional reduced differential transform method in [17]. Our main concern in this work, is to apply Laplace Adomian decomposition method (LADM) [18–22], to find approximate solutions to the reaction-diffusion Brusselator system (RDBS), witch has the following form [16]

$$\begin{aligned}
&\psi_{\zeta}^{\alpha}(x,\eta,\zeta) = b-(a+1)\psi+\psi^{2}\phi+\gamma(\psi_{xx}+\psi_{\eta\eta}), \quad 0<\alpha\leq 1,\\
&\phi_{\zeta}^{\beta}(x,\eta,\zeta) = a\psi-\psi^{2}\phi+\gamma(\phi_{xx}+\phi_{\eta\eta}), \quad 0<\beta\leq 1,
\end{aligned} \tag{1}$$

with the following initial conditions

$$\psi(x,\eta,0) = h(x,\eta), \ \ \phi(x,\eta,0) = j(x,\eta), \tag{2}$$

*Corresponding author: m.zellal@yahoo.fr
Received: June 26, 2022, Accepted: September 26, 2022

both $(\psi,\phi)$ refer to the dimensionless concentrations of two reactants; $a$ and $b$ are positive real constants (constants concentrations), $\gamma$ is a constant values for the diffusion coefficient and, $0<\alpha,\beta\leq 1$, are parameters representing the order of the time fractional derivatives in Caputo sence.

## 2. Preliminaries and notations

These are some basic definitions and properties of fractional calculus, for more details, see [1–3].

**Definition 1** *( [2, 23] **Riemann-Liouville integral**) Let $\psi\in L^1(a,b), a,b\in\mathbb{R}$. The left sided Riemann–Liouville fractional integral operator of order $\alpha\geq 0$, of a function $\psi$ is defined by*

$$\begin{cases} I^{\alpha}\psi(\zeta) &= \frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-\tau)^{\alpha-1}\psi(\tau)d\tau, \qquad \alpha>0,\\ I^{0}\psi(\zeta) &= \psi(\zeta), \qquad \alpha=0, \end{cases} \tag{3}$$

*where ,*

$$\Gamma(\omega)=\int_0^{\infty}\zeta^{\omega-1}e^{-\zeta}d\zeta,\;\; Re(\omega)>0. \tag{4}$$

*The operator $I_t^{\alpha}$ satisfy the following properties*

$$1) I_{\zeta}^{\alpha}I_{\zeta}^{\beta}\psi(\zeta)=I_{\zeta}^{\alpha+\beta}\psi(\zeta)=I_{\zeta}^{\beta}I_{\zeta}^{\alpha}\psi(\zeta),$$
$$2) I_{\zeta}^{\alpha}\zeta^{\rho}=\frac{\Gamma(\rho+1)}{\Gamma(\alpha+\rho+1)}\zeta^{\alpha+\rho}, \rho>-1.$$

**Definition 2** *( [2, 23] **Caputo derivative**) Let $\alpha\geq 0$, and $n=[\alpha]+1$. If $\psi\in AC^n[a,b], a,b\in\mathbb{R}$, then the Caputo fractional derivative operator $D_{\zeta}^{\alpha}\psi(\zeta)$ exist almost everywhere on $[a,b]$, is defined as*

$$D_{\zeta}^{\alpha}\psi(\zeta)=\begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\int_0^{\zeta}(\zeta-\tau)^{n-\alpha-1}\psi^{(n)}(\tau)\mathrm{d}\tau, \; n-1<\alpha<n,\\ \psi^{(n)}(\zeta), \qquad \alpha=n, n\in\mathbb{N}. \end{cases} \tag{5}$$

The operator $D_{\zeta}^{\alpha}$ satisfy the following properties

$$1) D_{\zeta}^{\alpha+\beta}\psi(\zeta)=D_{\zeta}^{\beta+\alpha}\psi(\zeta),$$
$$2) D_{\zeta}^{\alpha}\zeta^{\rho}=\frac{\Gamma(1+\rho)}{\Gamma(1+\rho-\alpha)}\zeta^{\rho-\alpha}, \rho>-1,$$
$$3) D_{\zeta}^{\alpha}I^{\alpha}\psi(\zeta)=\psi(\zeta),$$
$$4) I^{\alpha}D_{\zeta}^{\alpha}\psi(\zeta)=\psi(\zeta)-\sum_{k=0}^{n-1}\psi^{(k)}(0^+)\frac{\zeta^k}{k!}.$$

**Definition 3** *[2, 23] The Mittag-Leffler type of two-parameters function $E_{\lambda,\rho}(.)$ is presented as*

$$E_{\lambda,\rho}(\omega)=\sum_{k=0}^{+\infty}\frac{\omega^k}{\Gamma(\lambda k+\rho)},\;\; \lambda,\rho>0,\;\; \omega\in\mathbb{C}. \tag{6}$$

## 3. Laplace transform and important properties

**Definition 4** *[24] Let $\psi(\zeta)$ be defined for $\zeta\in(0,+\infty)$. Then, when the improper integral exists, the Laplace transform $\Psi(s)$ of $\psi(\zeta)$ is defined by*

$$\mathscr{L}[\psi(\zeta)]=\Psi(s)=\int_0^{+\infty}e^{-s\zeta}\psi(\zeta)d\zeta. \tag{7}$$

*Note that $\Psi$ is a function of the new variables $s$, while the original function $\psi$ is a function of the variable $\zeta$.*

### 3.1. Properties:

These are the basic properties of the Laplace transform, which are used in this work.

1. $\mathscr{L}[\zeta^{\rho}] = \frac{\Gamma(\rho+1)}{s^{\rho+1}}, \qquad \mathscr{L}^{-1}\left(\frac{1}{s^{\rho+1}}\right) = \frac{\zeta^{\rho}}{\Gamma(\rho+1)}, \qquad \rho > -1.$
   ($\mathscr{L}^{-1}$ denotes the inverse of $\mathscr{L}$).
2. $\mathscr{L}[I^{\alpha}\psi(\zeta)] = s^{-\alpha}\Psi(s).$
3. For $m-1 < \alpha \leq m, m \in \mathbb{N}^*$
$$\mathscr{L}[D_{\zeta}^{\alpha}\psi(\zeta)] = s^{\alpha}\Psi(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}\psi^{(k)}(0). \tag{8}$$
4. When $0 < \alpha \leq 1$
$$\mathscr{L}[D_{\zeta}^{\alpha}\psi(\zeta)] = s^{\alpha}\Psi(s) - s^{\alpha-1}\psi(0). \tag{9}$$

## 4. Analysis of (LADM)

This section describes the implementation of (LADM) as shown below [18–22], we consider a general fractional partial differential equation with initial conditions of the form

$$D_{\zeta}^{\alpha}\psi(x,\zeta) + R\psi(x,\zeta) + N\psi(x,\zeta) = w(x,\zeta), \quad m-1 < \alpha \leq m,\ m \in \mathbb{N}^*. \tag{10}$$

$$\left[\frac{\partial^{m-1}\psi(x,\zeta)}{\partial\zeta^{m-1}}\right]_{\zeta=0} = \varphi_{m-1}(x), m = 1,2,\dots\ .$$

$D_{\zeta}^{\alpha}\psi(x,\zeta) = \psi_{\zeta}^{\alpha}(x,\zeta)$ denotes the Caputo fractional derivative of order $\alpha$, $R$ is the linear differential operator of order less than order of $D$, $N$ represents the general nonlinear differential operator and $w$ is the source term. Operating the Laplace transform on both sides of Eq.(10),

$$\mathscr{L}[D_{\zeta}^{\alpha}\psi(x,\zeta)] + \mathscr{L}[R\psi(x,\zeta)] + \mathscr{L}[N\psi(x,\zeta)] = \mathscr{L}[w(x,\zeta)]. \tag{11}$$

By using the formula (8) in the above Eq.(11), we have

$$s^{\alpha}\mathscr{L}[\psi(x,\zeta)] - \sum_{k=0}^{m-1} s^{\alpha-k-1}\psi^{(k)}(x,0) + \mathscr{L}[R\psi(x,\zeta)] + \mathscr{L}[N\psi(x,\zeta)] = \mathscr{L}[w(x,\zeta)], \tag{12}$$

on simplifying

$$\mathscr{L}[\psi(x,\zeta)] = \sum_{k=0}^{m-1}\frac{\psi^{(k)}(x,0)}{s^{k+1}} + \frac{1}{s^{\alpha}}\mathscr{L}[w(x,\zeta)] - \frac{1}{s^{\alpha}}\mathscr{L}[R\psi(x,\zeta)] - \frac{1}{s^{\alpha}}\mathscr{L}[N\psi(x,\zeta)]. \tag{13}$$

Application of $\mathscr{L}^{-1}$, on the above equation Eq. (13), we then obtain

$$\psi(x,\zeta) = W(x,\zeta) - \mathscr{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathscr{L}\left[R\psi(x,\zeta) + N\psi(x,\zeta)\right]\right\}, \tag{14}$$

where $W(x,\zeta)$ is caused by $w(x,\zeta)$ and the given initial conditions. Now, by (ADM) technique [25–28]

$$\psi(x,\zeta) = \sum_{n=0}^{+\infty}\psi_n(x,\zeta). \tag{15}$$

The nonlinear term in the problem can be decomposed as

$$N\psi(x,\zeta) = \sum_{n=0}^{+\infty}\mathscr{A}_n(\psi_0,\psi_1,\cdots,\psi_n), \tag{16}$$

and $\mathscr{A}_n$ is the Adomian polynomials shown below see [25, 26]

$$\mathscr{A}_n(\psi_0,\psi_1,\cdots,\psi_n)=\frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[N\left(\sum_{i=0}^{+\infty}(\lambda^i\psi_i)\right)\right]_{\lambda=0}\quad n=0,1,2,3,\cdots. \tag{17}$$

Substituting Eq. (15) and Eq. (16) in Eq. (14) we get

$$\sum_{n=0}^{+\infty}\psi_n(x,\zeta)=W(x,\zeta)-\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left[R\left(\sum_{n=0}^{+\infty}\psi_n(x,\zeta)\right)+\left(\sum_{n=0}^{+\infty}\mathscr{A}_n\right)\right]\right\}. \tag{18}$$

We obtain $\psi_0(x,\zeta)=W(x,\zeta)$, and a recurrence relation as shown below

$$\psi_{n+1}(x,\zeta)=-\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}[R\psi_n(x,\zeta)+\mathscr{A}_n]\right\},\ n\geq 0. \tag{19}$$

## 5. Implementation (LADM) to (RDBS)

[16] Consider the (RDBS) (1) .

$$\begin{aligned}&\psi_\zeta^\alpha(x,\eta,\zeta)=b-(a+1)\psi+\psi^2\phi+\gamma(\psi_{xx}+\psi_{\eta\eta}),\quad 0<\alpha\leq 1,\\&\phi_\zeta^\beta(x,\eta,\zeta)=a\psi-\psi^2\phi+\gamma(\phi_{xx}+\phi_{\eta\eta}),\quad 0<\beta\leq 1,\end{aligned} \tag{20}$$

with the following initial conditions

$$\psi(x,\eta,0)=h(x,\eta),\ \ \phi(x,\eta,0)=j(x,\eta). \tag{21}$$

Operating $\mathscr{L}$ on both sides of eqs. (20), we get

$$\begin{aligned}&\mathscr{L}[D_\zeta^\alpha\psi(x,\eta,\zeta)]=\mathscr{L}[b-(a+1)\psi+\psi^2\phi]+\mathscr{L}[\gamma(\psi_{xx}+\psi_{\eta\eta})],\\&\mathscr{L}[D_\zeta^\beta\phi(x,\eta,\zeta)]=\mathscr{L}[a\psi-\psi^2\phi]+\mathscr{L}[\gamma(\phi_{xx}+\phi_{\eta\eta})].\end{aligned} \tag{22}$$

Now, by using the formula (9), in the above eqs. (22), we have

$$\begin{aligned}&s^\alpha\mathscr{L}[\psi(x,\eta,\zeta)]-s^{\alpha-1}\psi(x,\eta,0)=\mathscr{L}[b-(a+1)\psi+\psi^2\phi]+\mathscr{L}[\gamma(\psi_{xx}+\psi_{\eta\eta})],\\&s^\beta\mathscr{L}[\phi(x,\eta,\zeta)]-s^{\beta-1}\phi(x,\eta,0)=\mathscr{L}[a\psi-\psi^2\phi]+\mathscr{L}[\gamma(\phi_{xx}+\phi_{\eta\eta})].\end{aligned} \tag{23}$$

Or

$$\begin{aligned}&\mathscr{L}[\psi(x,\eta,\zeta)]=\left[\frac{\psi(x,\eta,0)}{s}+\frac{b}{s^{\alpha+1}}\right]+\frac{1}{s^\alpha}\mathscr{L}[-(a+1)\psi+\psi^2\phi]+\frac{1}{s^\alpha}\mathscr{L}[\gamma(\nabla^2\psi)],\\&\mathscr{L}[\phi(x,\eta,\zeta)]=\left[\frac{\phi(x,\eta,0)}{s}\right]+\frac{1}{s^\beta}\mathscr{L}[a\psi-\psi^2\phi]+\frac{1}{s^\beta}\mathscr{L}[\gamma(\nabla^2\phi)].\end{aligned} \tag{24}$$

Thus, we apply $\mathscr{L}^{-1}$, to the above equation (24) we get

$$\begin{aligned}&u(x,\eta,\zeta)=\psi(x,\eta,0)+\frac{b\zeta^\alpha}{\Gamma(\alpha+1)}+\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left[-(a+1)\psi+\psi^2\phi+\gamma(\nabla^2\psi)\right]\right\},\\&\phi(x,\eta,\zeta)=\phi(x,\eta,0)+\mathscr{L}^{-1}\left\{\frac{1}{s^\beta}\mathscr{L}\left[a\psi-\psi^2\phi+\gamma(\nabla^2\phi)\right]\right\}.\end{aligned} \tag{25}$$

Then, we apply the Adomian decomposition method in (25) we obtain

$$\begin{aligned}&\sum_{n=0}^{+\infty}\psi_n=h(x,\eta)+\frac{b\zeta^\alpha}{\Gamma(\alpha+1)}+\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left[-(a+1)\sum_{n=0}^{+\infty}\psi_n+\gamma(\nabla^2\sum_{n=0}^{+\infty}\psi_n)+\sum_{n=0}^{+\infty}\mathscr{A}_n(\psi,\phi)\right]\right\},\\&\sum_{n=0}^{+\infty}\phi_n=j(x,\eta)+\mathscr{L}^{-1}\left\{\frac{1}{s^\beta}\mathscr{L}\left[a\sum_{n=0}^{+\infty}\psi_n+\gamma(\nabla^2\sum_{n=0}^{+\infty}\phi_n)-\sum_{n=0}^{+\infty}\mathscr{A}_n(\psi,\phi)\right]\right\},\end{aligned} \tag{26}$$

where $\mathscr{A}_n$, are Adomian polynomials that represent nonlinear term $\psi^2\phi$, we obtain a recurrence relation to the approximations as shown below

$$\begin{aligned}
\psi_0(x,\eta,\zeta) &= h(x,\eta)+\frac{b\zeta^\alpha}{\Gamma(\alpha+1)},\\
\psi_{n+1}(x,\eta,\zeta) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left[-(a+1)\psi_n+\gamma(\nabla^2\psi_n)+\mathscr{A}_n\right]\right\},\ n\geq 0.\\
\phi_0(x,\eta,\zeta) &= j(x,\eta),\\
\phi_{n+1}(x,\eta,\zeta) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\beta}\mathscr{L}\left[a\psi_n+\gamma(\nabla^2\phi_n)-\mathscr{A}_n\right]\right\},\ n\geq 0.
\end{aligned} \tag{27}$$

These are the first values of $\mathscr{A}_n$

$$\begin{aligned}
\mathscr{A}_0 &= \psi_0^2\phi_0,\\
\mathscr{A}_1 &= 2\psi_0\psi_1\phi_0+\psi_0^2\phi_1,\\
\mathscr{A}_2 &= (2\psi_0\psi_2+\psi_1^2)\phi_0+2\psi_0\psi_1\phi_1+\psi_0^2\phi_2,\\
\mathscr{A}_3 &= (2\psi_1\psi_2+2\psi_0\psi_3)\phi_0+(2\psi_0\psi_2+\psi_1^2)\phi_1+2\psi_0\psi_1\phi_2+\psi_0^2\phi_3,\\
\mathscr{A}_4 &= (2\psi_0\psi_4+2\psi_1\psi_3+\psi_2^2)\phi_0+(2\psi_1\psi_2+2\psi_0\psi_3)\phi_1+(2\psi_0\psi_2+\psi_1^2)\phi_2+2\psi_0\psi_1\phi_3+\psi_0^2\phi_4,\\
&\vdots
\end{aligned}$$

## 6. Numerical results and discussion

**Example 1:** [15, 16] Consider the (RDBS) (1), for $a=1,\ b=0,\ and\ \gamma=\frac{1}{4}$.

$$\begin{aligned}
\psi_\zeta^\alpha &= -2\psi+\psi^2\phi+\frac{1}{4}(\psi_{xx}+\psi_{\eta\eta}),\\
\phi_\zeta^\beta &= \psi-\psi^2\phi+\frac{1}{4}(\phi_{xx}+\phi_{\eta\eta}),
\end{aligned} \tag{28}$$

with the following initial conditions

$$\psi(x,\eta,0)=e^{-(x+\eta)},\ \ \phi(x,\eta,0)=e^{(x+\eta)}. \tag{29}$$

Therefore from (26), the following procedure can be defined

$$\begin{aligned}
\sum_{n=0}^{+\infty}\psi_n &= e^{-(x+\eta)}+\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left[-2\sum_{n=0}^{+\infty}\psi_n+\frac{1}{4}(\nabla^2\sum_{n=0}^{+\infty}\psi_n)+\sum_{n=0}^{+\infty}\mathscr{A}_n(\psi,\phi)\right]\right\},\\
\sum_{n=0}^{+\infty}\phi_n &= e^{(x+\eta)}+\mathscr{L}^{-1}\left\{\frac{1}{s^\beta}\mathscr{L}\left[\sum_{n=0}^{+\infty}\psi_n+\frac{1}{4}(\nabla^2\sum_{n=0}^{+\infty}\phi_n)-\sum_{n=0}^{+\infty}\mathscr{A}_n(\psi,\phi)\right]\right\}.
\end{aligned} \tag{30}$$

Thus, we obtain $\psi_0(x,\eta,\zeta)=e^{-(x+\eta)}$, $\phi_0(x,\eta,\zeta)=e^{(x+\eta)}$, and a recurrence relation as shown below

$$\begin{aligned}
\psi_{n+1}(x,\eta,\zeta) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left[-2\psi_n+\frac{1}{4}(\nabla^2\psi_n)+\mathscr{A}_n\right]\right\},\ n\geq 0.\\
\phi_{n+1}(x,\eta,\zeta) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\beta}\mathscr{L}\left[\psi_n+\frac{1}{4}(\nabla^2\phi_n)-\mathscr{A}_n\right]\right\},\ n\geq 0.
\end{aligned} \tag{31}$$

By applying (31), we obtain

$$\begin{aligned}
\psi_1(x,\eta,\zeta) &= -\frac{1}{2}\frac{e^{-(x+\eta)}\zeta^{\alpha}}{\Gamma(1+\alpha)},\\
\phi_1(x,\eta,\zeta) &= \frac{1}{2}\frac{e^{(x+\eta)}\zeta^{\beta}}{\Gamma(1+\beta)},\\
\psi_2(x,\eta,\zeta) &= \frac{1}{4}e^{-(x+\eta)}\left(-\frac{\zeta^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{2\zeta^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)}\right),\\
\phi_2(x,\eta,\zeta) &= \frac{1}{2}\frac{e^{-(x+\eta)}\zeta^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)}+\frac{1}{4}\frac{(-2e^{-(x+\eta)}+e^{(x+\eta)})\zeta^{2\beta}}{\Gamma(1+2\beta)},\\
\psi_3(x,\eta,\zeta) &= -\frac{1}{2}\frac{e^{-(x+\eta)}\Gamma(1+\alpha+\beta)\zeta^{2\alpha+\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+2\alpha+\beta)}+\frac{1}{4}\frac{e^{-(x+\eta)}\Gamma(1+2\alpha)\zeta^{3\alpha}}{(\Gamma(1+\alpha))^2\Gamma(1+3\alpha)}-\frac{1}{8}\frac{e^{-(x+\eta)}\zeta^{3\alpha}}{\Gamma(1+3\alpha)}\\
&\quad+\frac{1}{4}\frac{(-2e^{-(3x+3\eta)}+e^{-(x+\eta)})\zeta^{\alpha+2\beta}}{\Gamma(1+\alpha+2\beta)}+\frac{1}{4}\frac{(2e^{-(3x+3\eta)}+e^{-(x+\eta)})\zeta^{2\alpha+\beta}}{\Gamma(1+2\alpha+\beta)},\\
\phi_3(x,\eta,\zeta) &= \frac{1}{2}\frac{e^{-(x+\eta)}\Gamma(1+\alpha+\beta)\zeta^{\alpha+2\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\alpha+2\beta)}-\frac{1}{4}\frac{e^{-(x+\eta)}\Gamma(1+2\alpha)\zeta^{\beta+2\alpha}}{(\Gamma(1+\alpha))^2\Gamma(1+\beta+2\alpha)}+\frac{1}{4}\frac{e^{-(x+\eta)}\zeta^{\beta+2\alpha}}{\Gamma(1+\beta+2\alpha)}\\
&\quad+\frac{1}{8}\frac{(e^{(x+\eta)}+4e^{-(3x+3\eta)}-4e^{-(x+\eta)})\zeta^{3\beta}}{\Gamma(1+3\beta)}-\frac{1}{4}\frac{(2e^{-(3x+3\eta)}+e^{-(x+\eta)})\zeta^{\alpha+2\beta}}{\Gamma(1+\alpha+2\beta)},
\end{aligned}$$

in this way, the remaining components of the solution can be obtained.
The solution in series form when $\alpha=\beta=1$, is given by

$$\begin{aligned}
\psi(x,\eta,\zeta) &= e^{-x-\eta}\left(1-\frac{1}{2}\zeta+\frac{1}{8}\zeta^2-\frac{1}{48}\zeta^3+\cdots\right)=e^{-x-\eta-\frac{1}{2}\zeta},\\
\phi(x,\eta,\zeta) &= e^{x+\eta}\left(1+\frac{1}{2}\zeta+\frac{1}{8}\zeta^2+\frac{1}{48}\zeta^3+\cdots\right)=e^{x+\eta+\frac{1}{2}\zeta}.
\end{aligned} \tag{32}$$

**Example 2:** [10, 12] Consider the (RDBS) (1), for $a=\frac{1}{2},\ b=1,\ and\ \gamma=\frac{1}{500}$ :

$$\begin{aligned}
\psi_\zeta^\alpha &= 1-\frac{3}{2}\psi+\psi^2\phi+\frac{1}{500}(\psi_{xx}+\psi_{\eta\eta}),\\
\phi_\zeta^\beta &= \frac{1}{2}\psi-\psi^2\phi+\frac{1}{500}(\phi_{xx}+\phi_{\eta\eta}),
\end{aligned} \tag{33}$$

with the following initial conditions

$$\psi(x,\eta,0)=x^2,\ \phi(x,\eta,0)=\eta^2. \tag{34}$$

By using (26), we get

$$\begin{aligned}
\sum_{n=0}^{+\infty}\psi_n &= x^2+\frac{\zeta^\alpha}{\Gamma(\alpha+1)}+\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left[-\frac{3}{2}\sum_{n=0}^{+\infty}\psi_n+\frac{1}{500}(\nabla^2\sum_{n=0}^{+\infty}\psi_n)+\sum_{n=0}^{+\infty}\mathscr{A}_n(\psi,\phi)\right]\right\},\\
\sum_{n=0}^{+\infty}\phi_n &= \eta^2+\mathscr{L}^{-1}\left\{\frac{1}{s^\beta}\mathscr{L}\left[\frac{1}{2}\sum_{n=0}^{+\infty}\psi_n+\frac{1}{500}(\nabla^2\sum_{n=0}^{+\infty}\phi_n)-\sum_{n=0}^{+\infty}\mathscr{A}_n(\psi,\phi)\right]\right\}.
\end{aligned} \tag{35}$$

we obtain $\psi_0(x,\eta,\zeta)=x^2+\dfrac{\zeta^\alpha}{\Gamma(\alpha+1)}, \phi_0(x,\eta,\zeta)=\eta^2$ and a recurrence relation as shown below

$$\begin{aligned}
\psi_{n+1}(x,\eta,\zeta) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left[-\frac{3}{2}\psi_n+\frac{1}{500}(\nabla^2\psi_n)+\mathscr{A}_n\right]\right\},\ n\geq 0.\\
\phi_{n+1}(x,\eta,\zeta) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\beta}\mathscr{L}\left[\frac{1}{2}\psi_n+\frac{1}{500}(\nabla^2\phi_n)-\mathscr{A}_n\right]\right\},\ n\geq 0.
\end{aligned} \tag{36}$$

Substituting $\psi_0$ and $\phi_0$ into (36), we get

$$\psi_1(x,\eta,\zeta) = \frac{\Gamma(1+2\alpha)\eta^2\zeta^{3\alpha}}{(\Gamma(1+\alpha))^2\Gamma(1+3\alpha)} + \frac{1}{250}\frac{(1-375x^2+250x^4\eta^2)\zeta^{\alpha}}{\Gamma(1+\alpha)} + \frac{1}{2}\frac{(4x^2\eta^2-3)\zeta^{2\alpha}}{\Gamma(1+2\alpha)},$$

$$\phi_1(x,\eta,\zeta) = \frac{\Gamma(1+2\alpha)\eta^2\zeta^{2\alpha+\beta}}{(\Gamma(1+\alpha))^2\Gamma(1+2\alpha+\beta)} + \frac{1}{250}\frac{(1+250x^2-250x^4\eta^2)\zeta^{\beta}}{\Gamma(1+\beta)} + \frac{(-2x^2\eta^2+1)\zeta^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)},$$

and so on. In this manner the rest of the iterative components can be found.

## 7. Tables and Figures

Below we know the approximate solution of the four order $(\Psi_4, \Phi_4)$, which we use in the four molar tables to compare with the exact solution and calculate the absolute error for various values of $\alpha$ and $\beta$.

$$\Psi_4 = \psi_0 + \psi_1 + \psi_2 + \psi_3.$$

$$\Phi_4 = \phi_0 + \phi_1 + \phi_2 + \phi_3.$$

**Table 1:** Numerical results at the point $(0.4, 0.6)$.

| | | $\alpha=\beta=1$ | $\alpha=\beta=0.9$ | $\alpha=\beta=0.8$ |
|---|---|---|---|---|
| $\zeta$ | $\psi$ Exact | $\Psi_4$ | $\Psi_4$ | $\Psi_4$ |
| 0.1 | 0.3499377491111554 | 0.3499376542593101 | 0.3446579328808398 | 0.3381770131237322 |
| 0.2 | 0.3328710836980796 | 0.3328695810199602 | 0.3258880488044142 | 0.3181852991081172 |
| 0.3 | 0.3166367693790532 | 0.3166292365232458 | 0.3091774383430047 | 0.3016193376402662 |
| 0.4 | 0.3011942119122021 | 0.3011706358390207 | 0.2939999865300189 | 0.2873383399060295 |
| 0.5 | 0.2865047968601901 | 0.2864477940371386 | 0.2800836892671381 | 0.2748070199554759 |
| 0.6 | 0.2725317930340126 | 0.2724147261874530 | 0.2672528759832714 | 0.2637083996441059 |

**Table 2:** Numerical results at the point $(0.4, 0.6)$.

| | | $\alpha=\beta=1$ | $\alpha=\beta=0.9$ | $\alpha=\beta=0.8$ |
|---|---|---|---|---|
| $\zeta$ | $\phi$ Exact | $\Phi_4$ | $\Phi_4$ | $\Phi_4$ |
| 0.1 | 2.857651118063164 | 2.857650403038997 | 2.902768404574928 | 2.961911368085330 |
| 0.2 | 3.004166023946433 | 3.004154467418655 | 3.073643397868355 | 3.159324321821366 |
| 0.3 | 3.158192909689768 | 3.158133806826573 | 3.245886670787438 | 3.350234564546649 |
| 0.4 | 3.320116922736547 | 3.319928206491313 | 3.422208983952619 | 3.540549820548468 |
| 0.5 | 3.490342957461841 | 3.489877451641429 | 3.603890449326122 | 3.732756435818485 |
| 0.6 | 3.669296667619244 | 3.668321327505480 | 3.791718269201408 | 3.928207358812717 |

**Table 3:** Absolute error $|\psi_{Exact} - \Psi_4|$.

| $(x,\eta)$ | $\zeta$ | $\alpha=\beta=1$ | $\alpha=\beta=0.9$ | $\alpha=\beta=0.8$ |
|---|---|---|---|---|
| $(0.1, 0.9)$ | 0.1 | 0.0000000948518453 | 0.0052798162303156 | 0.0117607359874232 |
| $(0.2, 0.8)$ | 0.2 | 0.0000015026781194 | 0.0069830348936654 | 0.0146857845899624 |
| $(0.3, 0.7)$ | 0.3 | 0.0000075328558074 | 0.0074593310360485 | 0.0150174317387870 |
| $(0.4, 0.6)$ | 0.4 | 0.0000235760731814 | 0.0071942253821832 | 0.0138558720061726 |
| $(0.5, 0.5)$ | 0.5 | 0.0000570028230515 | 0.0064211075930520 | 0.0116977769047142 |
| $(0.6, 0.4)$ | 0.6 | 0.0001170668465596 | 0.0052789170507412 | 0.0088233933899067 |

**Table 4:** Absolute error $|\phi_{Exact} - \Phi_4|$.

| $(x,\eta)$ | $\zeta$ | $\alpha=\beta=1$ | $\alpha=\beta=0.9$ | $\alpha=\beta=0.8$ |
|---|---|---|---|---|
| $(0.1,0.9)$ | 0.1 | 0.000000715024167 | 0.045117286511764 | 0.104260250022166 |
| $(0.2,0.8)$ | 0.2 | 0.000011556527778 | 0.069477373921922 | 0.155158297874933 |
| $(0.3,0.7)$ | 0.3 | 0.000059102863195 | 0.087693761097670 | 0.192041654856881 |
| $(0.4,0.6)$ | 0.4 | 0.000188716245234 | 0.102092061216072 | 0.220432897811921 |
| $(0.5,0.5)$ | 0.5 | 0.000465505820412 | 0.113547491864281 | 0.242413478356644 |
| $(0.6,0.4)$ | 0.6 | 0.000975340113764 | 0.122421601582164 | 0.258910691193473 |

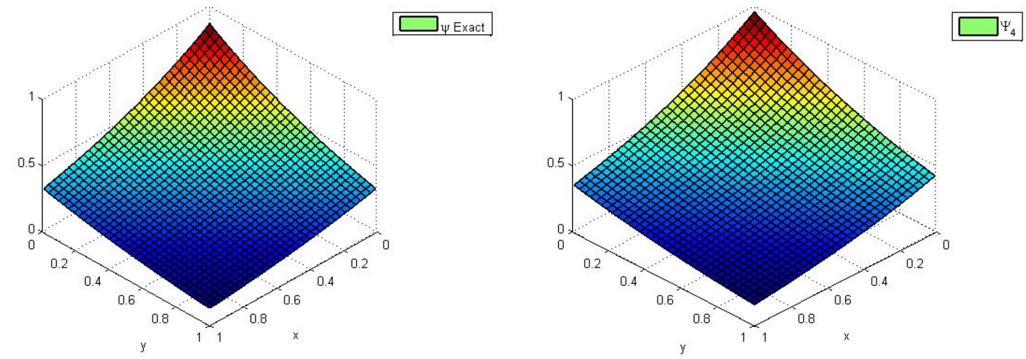


**Figure 1:** The behavior of $\psi$ and $\Psi_4$ for Example 1, when $\zeta = 0.3,\ \alpha = \beta = 1$.

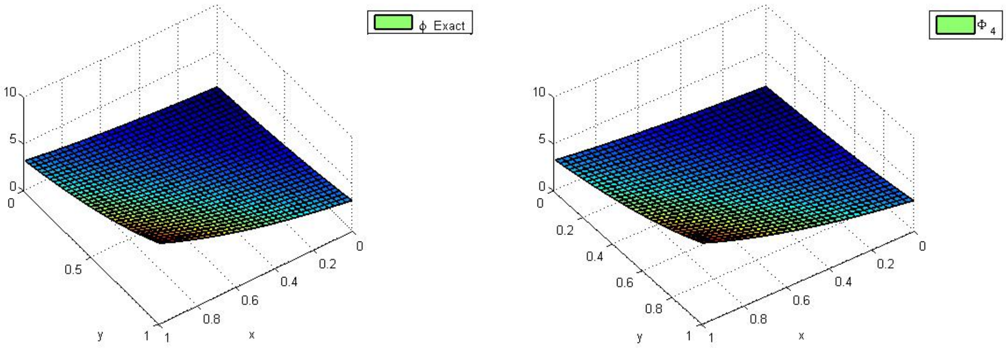


**Figure 2:** The behavior of $\phi$ and $\Phi_4$ for Example 1, when $\zeta = 0.3,\ \alpha = \beta = 1$.

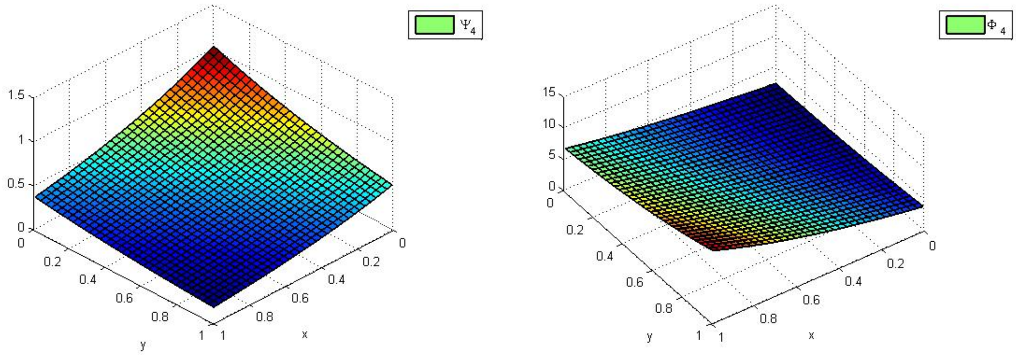


**Figure 3:** The behavior of $\Psi_4$ and $\Phi_4$ for Example 1, when $\zeta = 0.3,\ \alpha = \beta = 0.9$.

## Conclusion

The approximate solutions to time fractional reaction-diffusion Brusselator system with initial conditions were successfully and flexibly found using the effective method (LADM), although a case study, the order of the fractional time derivatives is different between $\psi$ and $\phi$, ie $\psi_\zeta^\alpha, 0 < \alpha \leq 1$, $\phi_\zeta^\beta, 0 < \beta \leq 1$ , however, good approximations were obtained gradually, it has been shown by comparing the numerical results obtained with the exact solution when $\alpha = \beta = 1$, in the first example, it is observed that they are in perfect agreement. In the second example, although there is no known analytic solution, but we were able to find approximate solutions using the present method without any transformation, discretization, perturbation, which proves that the (LADM) is very efficient and reliable, and can be used to solve a wide class of nonlinear fractional order differential equations, especially since most of them do not contain exact analytic solutions.

## Acknowledgements

The two author(s), sincerely are very grateful to the reviewers and the editor for their constructive comments and valuable suggestions to improve the presentation of the paper. This research work is also supported by the General Direction of Scientific Research and Technological Development (DGRSDT), Algeria.

## Declaration of Competing Interest

The author(s), declares that there is no competing financial interests or personal relationships that influence the work in this paper.

## Authorship Contribution Statement

**Mohamed Zellal:** Conceptualization, Methodology, Validation, Formal Analysis, Writing Original Draft.
**Kacem Belghaba:** Investigation, Resources, Visualization, Supervision.

## References

[1] K. S. Miller and B. Ross, ”An Introduction to the Fractional Calculus and Fractional Differential Equations,” Wiley, New York, 1993.

[2] I. Podlubny, ”Fractional Differential Equations,” Academic Press, New York, 1999.

[3] A. Kilbas, H.M. Srivastava and J.J. Trujillo, ”Theory and Application of Fractional Differential Equations,” Amsterdam: Elsevier, 2006.

[4] A. Carpinteri and F. Mainardi, ”Fractals and Fractional Calculus in Continuum Mechanics,” Springer Verlag, Wien, New York, 1997.

[5] A.U. Rehman, M.B. Riaz, A. Akgül, S.T. Saeed and D .Baleanu, ”Heat and mass transport impact on MHD second grade fluid: A comparative analysis of fractional operators,” Heat Transfer, vol. 50, no. 7, pp. 7042-7064, 2021.

[6] R. Lefever and G. Nicolis, ”Chemical instabilities and sustained oscillations,” J. Theor. Biol., vol. 30, no. 2, pp. 267–284, 1971.

[7] J. Tyson, ”Some further studies of nonlinear oscillations in chemical systems,” J. Chem. Phys., vol. 58, pp. 3919–3930, 1973.

[8] E. H. Twizell and A. B. Gumel,”A second-order scheme for the Brus-selator reaction–difusion system,” Q. Cao. J. Math. Chem., vol. 26, pp. 297–316, 1999.

[9] G. Adomian, ”The diffusion Brusselator equation,” Comput. Math. Appl., vol. 29, no. 2, pp. 1-3, 1995.

[10] J. Biazar and Z. Ayati, ”A numerical solution of reaction-diffusion brusselator system by A.D.M,” Journal of Nature Science and Sustainable Technology., vol. 1, no. 2, pp. 263-270, 2007.

[11] A. M. Wazwaz, ”The decomposition method applied to systems of partial differential equations and to the reaction–diffusion Brusselator model,” Appl. Math. Comput., vol. 110, no. (2-3), pp. 251–264, 2000.

[12] A. W. Teong, ”The two-dimensional reaction–diffusion Brusselator system: a dual-reciprocity boundary element solution,” Eng. Anal. Bound. Elem., vol. 27, pp. 897–903, 2003.

[13] S. Islam, A. Ali and S. Haq, ”A computational modeling of the behavior of the two-dimensional reaction-diffusion Brusselator system,” App. Math. Model., vol. 34, pp. 3896–3909, 2010.

[14] H. Aminikhan and A. Jamalian, ”An Efficient Method for Solving the Brusselator System,” Walailak J Sci and Tech., vol. 10, no. 5, pp. 449-465, 2013.

[15] H. Siraj, A. Ihteram and S. N. Kottakkaran, ”A computational study of two-dimensional reaction–diffusion Brusselator system with applications in chemical processes,” Alexandria Engineering Journal, vol. 60, pp. 4381–4392, 2021.

[16] J. Singh, M. M. Rashidi, D. Kumar and R. Swroop, ”A fractional model of a dynamical Brusselator reaction-difusion system arising in triple collision and enzymatic reactions,” Nonlinear Engineering., vol. 5, no. 4, pp. 277–285, 2016.

[17] R. M. Jena, S. Chakraverty, H. Rezazadeh and D. D. Ganji, ”On the solution of time-fractional dynamical model of Brusselator reaction-diffusion system arising in chemical reactions,” Math Meth Appl Sci., pp. 1–11, 2020.

[18] H. Jafari, C. M. Khalique and M. Nazari, ”Application of the Laplace decomposition method for solving linear and nonlinear fractional diffusion-wave equations,” Appl. Math. Lett., vol. 24, no. 11, pp. 1799–1805, 2011.

[19] H. Khan, R. Shah, P. Kumam, D. Baleanu and M. Arif, ”Laplace decomposition for solving nonlinear system of fractional order partial differential equations,” Advances in Difference Equations, vol. 2020:537, 2020.

[20] O. H. Mohammed and H.A. Salim, ”Computational methods based laplace decomposition for solving non-linear system of fractional order differential equations,” Alexandria Engineering Journal., vol. 57, no. 4, pp. 3549–3557, 2018.

[21] R. Shah, H. Khan, M. Arif, P. Kumam, ”Application of Laplace Adomian Decomposition Method for the Analytical Solution of Third-Order Dispersive Fractional Partial Differential Equations,” Entropy., vol. 21, no. 335, 2019.

[22] A. M. Wazwaz, ”The combined Laplace transform–Adomian decomposition method for handling nonlinear Volterra integro–differential equations,” Appl. Math. Comput., vol. 216, no. 4, pp. 1304–1309, 2010.

[23] D. Baleanu and H.K. Jassim, ”Exact Solution of Two-dimensional Fractional Partial Differential Equations,” Fractal Fract., vol. 4, no. 21, 2020

[24] V.F. Morales-Delgado, M.A. Taneco-Hern´andez, and J.F. Gómez Aguilar, ”On the solutions of fractional order of evolution equations,” Eur. Phys. J. Plus, vol. 132: 47, 2017.

[25] G. Adomian, ”A review of the decomposition method in applied mathematics,” Math. Anal. Appli., vol. 135, no. 2, pp. 501-544, 1988.

[26] G. Adomian, ”Solving Frontier Problems of Physics: The Decomposition Method,” Kluwer Academic, Boston. 1994.

[27] K. Abbaoui and Y. Cherrault, ”Convergence of Adomians Method Applied to Differential Equations,” Computers and Mathematics with Applications, vol. 28, pp. 103-109, 1994.

[28] K. Abbaoui, Y. Cherrault, ”Convergence of Adomians Method Applied Nonlinear Equations,” Mathematical and Computer Modelling, vol. 20, pp. 69-73, 1994.