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ON A NEW FAMILY OF THE GENERALIZED GAUSSIAN K-PELL-LUCAS NUMBERS AND THEIR POLYNOMIALS

Hayrullah ÖZİMAMOĞLU[1] and Ahmet KAYA[2]

[1,2]Department of Mathematics, Nevşehir Hacı Bektaş Veli University, Nevşehir, TÜRKİYE

Abstract. In this paper, we generalize the known Gaussian Pell-Lucas numbers, and call such numbers as the generalized Gaussian k-Pell-Lucas numbers. We obtain relations between the family of the generalized Gaussian k-Pell-Lucas numbers and the known Gaussian Pell-Lucas numbers. We generalize the known Gaussian Pell-Lucas polynomials, and call such polynomials as the generalized Gaussian k-Pell-Lucas polynomials. We obtain relations between the family of the generalized Gaussian k-Pell-Lucas polynomials and the known Gaussian Pell-Lucas polynomials. In addition, we present the new generalizations of these numbers and polynomials in matrix form. Then, we get Cassini's identities for these numbers and polynomials.

1. Introduction

Fibonacci and Lucas numbers have gained popularity in recent years, and they are now used in a variety of branches of mathematics, including linear algebra, applied mathematics, and calculus. In 1832, Gauss discovered Gaussian numbers, which are complex numbers $z = x+yi, x, y \in \mathbb{Z}$. These numbers were used to generalize special sequences by numerous researchers. Therefore, the study of Gaussian numbers is a very interesting academic area and several studies have been done on it. Horadam [7] introduced the complex Fibonacci numbers that is Gaussian Fibonacci numbers in 1963. Then Jordan [8] investigated Gaussian Fibonacci numbers and Lucas numbers. These numbers are defined by $GF_{n+1} = GF_n + GF_{n-1}$, where $GF_0 = i, GF_1 = 1$ and $GL_{n+1} = GL_n + GL_{n-1}$, where $GL_0 = 2 - i, GL_1 = 1 + 2i$, respectively. Also, many authors [1–3, 5, 6, 12, 15] have studied Gaussian Fibonacci, Gaussian Lucas, Gaussian Jacobsthal, Gaussian Jacobsthal-Lucas, Gaussian Pell,

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[1] h.ozimamoglu@nevsehir.edu.tr-Corresponding author; 0000-0001-7844-1840
[2] ahmetkaya@nevsehir.edu.tr; 0000-0001-5109-8130 .

Gaussian Pell-Lucas etc. numbers and their polynomials. A new family of k-Gaussian Fibonacci numbers is given by Taş [13] and a new family of Gaussian k-Fibonacci polynomials are defined by Taştan and Özkan [14]. Moreover they [10,11] presented a new families of Gaussian k-Jacobsthal numbers, Gaussian k-Jacobsthal-Lucas numbers and their polynomials and a new family of Gaussian k-Lucas numbers and their polynomials. In [9], Kaya and Özimamoğlu generalized the Gaussian Pell numbers and Gauss Pell polynomials, and defined generalized Gauss k-Pell numbers and generalized Gaussian k-Pell polynomials. They obtained Cassini's identities for these numbers and polynomials.

Next, we give the structure of the paper. In Section 2, we demonstrate several well-known definitions and characteristics. In Section 3.1, we define a new family of the generalized Gaussian k-Pell-Lucas numbers. These numbers are a generalization of the Gaussian Pell-Lucas numbers in [6]. We give relations between the generalized Gaussian k-Pell-Lucas numbers and the Gaussian Pell-Lucas numbers. Also, we determine the new generalization of these numbers in matrix form. Then we demonstrate Cassini's identity for these numbers.

In Section 3.2, we define a new family of the generalized Gaussian k-Pell-Lucas polynomials. These polynomials are a generalization of the Gaussian Pell-Lucas polynomials in [15]. We give relations between the generalized Gaussian k-Pell-Lucas polynomials and the Gaussian Pell-Lucas polynomials. Moreover, we determine the new generalization of these polynomials in matrix form. Then we demonstrate Cassini's identity for these polynomials. In Section 4, we conclude the paper.

2. Material and Methods

We provide the Gaussian Pell-Lucas numbers GQ_n, the Gaussian Pell-Lucas polynomials $GQ_n(x)$, and the Gaussian Pell-Lucas polynomial matrix $gq_n(x)$ in this section.

Definition 1. *The Gaussian Pell-Lucas numbers* $\{GQ_n\}_{n=0}^{\infty}$ *are defined by the following recurrence relation:*

$$GQ_{n+1} = 2GQ_n + GQ_{n-1}, n \geq 1 \tag{1}$$

with initial conditions $GQ_0 = 2 - 2i$ *and* $GQ_1 = 2 + 2i$ *[6].*

The Binet formulas for GQ_n are given as follows:

$$GQ_n = (\alpha^n + \beta^n) - i(\alpha\beta^n + \beta\alpha^n), \tag{2}$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ [6].

The Cassini's identity [6] for the Gaussian Pell-Lucas numbers are given as follows:

$$GQ_{n+1}GQ_{n-1} - GQ_n^2 = (-1)^{n+1} 16(1 - i), \ n \geq 1. \tag{3}$$

Definition 2. *The Gaussian Pell-Lucas polynomials* $\{GQ_n(x)\}_{n=0}^{\infty}$ *are defined by the recurrence relation shown below:*

$$GQ_{n+1}(x) = 2xGQ_n(x) + GQ_{n-1}(x), n \geq 1 \tag{4}$$

with initial conditions $GQ_0(x) = 2 - 2xi$ *and* $GQ_1 = 2x + 2i$ *[15].*

The following are the Binet formulas for $GQ_n(x)$:

$$GQ_n(x) = (\alpha^n(x) + \beta^n(x)) - i(\alpha(x)\beta^n(x) + \beta(x)\alpha^n(x)), \tag{5}$$

where $\alpha(x) = x + \sqrt{1+x^2}$ and $\beta(x) = x - \sqrt{1+x^2}$ [15].

The Cassini's identity [15] for the Gaussian Pell-Lucas polynomials are given as follows:

$$GQ_{n+1}(x)GQ_{n-1}(x) - GQ_n^2(x) = 8(-1)^{n-1}\left(1+x^2\right)(1-xi), \ n \geq 1. \tag{6}$$

In [15], The Gaussian Pell-Lucas polynomial matrix $gq_n(x)$ is defined by

$$gq_n(x) = \begin{bmatrix} GQ_{n+2}(x) & GQ_{n+1}(x) \\ GQ_{n+1}(x) & GQ_n(x) \end{bmatrix}, n \geq 1.$$

3. Main Results

3.1. The generalized Gaussian k-Pell-Lucas numbers.

Definition 3. *There are unique numbers* m *and* r *such that* $n = mk + r$ *and* $0 \leq r < k$, *for* $n, k \in \mathbb{N}\,(k \neq 0)$. *Then we define the generalized Gaussian* k*-Pell-Lucas numbers* $GQ_n^{(k)}$ *by*

$$\begin{aligned} GQ_n^{(k)} \; := \; & [(\alpha^m + \beta^m) - i(\alpha\beta^m + \beta\alpha^m)]^{k-r} \\ & \times \left[\left(\alpha^{m+1} + \beta^{m+1}\right) - i\left(\alpha\beta^{m+1} + \beta\alpha^{m+1}\right)\right]^r, \end{aligned}$$

where $\alpha = 1 + \sqrt{2}$ *and* $\beta = 1 - \sqrt{2}$.

Furthermore, using the matrix methods, we can derive the generalized Gaussian k-Pell-Lucas number. Clearly, it can be said that

$$GQ_n^{k-1} gq_n = \begin{bmatrix} GQ_{kn+1}^{(k)} & GQ_{kn}^{(k)} \\ GQ_{kn}^{(k)} & GQ_{kn-1}^{(k)} \end{bmatrix},$$

where $n > 0$ and

$$gq_n = \begin{bmatrix} GQ_{n+1} & GQ_n \\ GQ_n & GQ_{n-1} \end{bmatrix}.$$

Various values for the generalized Gaussian k-Pell-Lucas numbers are given in Table 1. From (2) and Definition 3, we get the following relationship between the generalized Gaussian k-Pell-Lucas numbers and the Gaussian Pell-Lucas numbers.

$$GQ_n^{(k)} := (GQ_m)^{k-r}(GQ_{m+1})^r, \ n = mk + r. \tag{7}$$

If we take $k = 1$ in (7), then we have that $m = n$ and $r = 0$ so $GQ_n^{(1)} = GQ_n$. Throughout this article, let $k, m \in \{1, 2, 3, \ldots\}$.

TABLE 1. The generalized Gaussian k-Pell-Lucas numbers $GQ_n^{(k)}$ for some k and n.

$GQ_n^{(k)}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$GQ_0^{(k)}$	$2-2i$	$-8i$	$-16-16i$	-64	$-128+128i$	$512i$
$GQ_1^{(k)}$	$2+2i$	8	$16-16i$	$-64i$	$-128-128i$	-512
$GQ_2^{(k)}$	$6+2i$	$8i$	$16+16i$	64	$128-128i$	$-512i$
$GQ_3^{(k)}$	$14+6i$	$8+16i$	$-16+16i$	$64i$	$128+128i$	512
$GQ_4^{(k)}$	$34+14i$	$32+24i$	$-16+48i$	-64	$-128+128i$	$512i$
$GQ_5^{(k)}$	$82+34i$	$72+64i$	$16+112i$	$-128+64i$	$-128-128i$	-512
$GQ_6^{(k)}$	$198+82i$	$160+168i$	$144+208i$	$-192+256i$	$-384-128i$	$-512i$
$GQ_7^{(k)}$	$478+198i$	$392+400i$	$304+528i$	$-128+704i$	$-896+128i$	$-512-1024i$
$GQ_8^{(k)}$	$1154+478i$	$960+952i$	$624+1328i$	$448+1536i$	$-1664+1152i$	$-2048-1536i$
$GQ_9^{(k)}$	$2786+1154i$	$2312+2304i$	$1232+3312i$	$768+3776i$	$-2176+3968i$	$-5632-1024i$
$GQ_{10}^{(k)}$	$6726+2786i$	$5568+5576i$	$3088+7952i$	$1088+9216i$	$-384+10112i$	$-12288+3584i$

For $k=2,3,4$ and n, we get the following relations between the generalized Gaussian k-Pell-Lucas numbers and the Gaussian Pell-Lucas numbers by (1) and (7):

(1) $GQ_{2n}^{(2)}=GQ_n^2$,
(2) $GQ_{2n+1}^{(2)}=GQ_nGQ_{n+1}$
(3) $GQ_{2n+1}^{(2)}=2GQ_{2n}^{(2)}+GQ_{2n-1}^{(2)}$,
(4) $GQ_{3n}^{(2)}=GQ_n^3$,
(5) $GQ_{3n+1}^{(3)}=GQ_n^2GQ_{n+1}$,
(6) $GQ_{3n+1}^{(3)}=2GQ_{3n}^{(3)}+GQ_{3n-1}^{(3)}$,
(7) $GQ_{3n+2}^{(3)}=GQ_nGQ_{n+1}^2$,
(8) $GQ_{4n}^{(4)}=GQ_n^4$,
(9) $GQ_{4n+1}^{(4)}=GQ_n^3GQ_{n+1}$,
(10) $GQ_{4n+1}^{(4)}=2GQ_{4n}^{(4)}+GQ_{4n-1}^{(4)}$,
(11) $GQ_{4n+2}^{(4)}=GQ_n^2GQ_{n+1}^2$,
(12) $GQ_{4n+3}^{(4)}=GQ_nGQ_{n+1}^3$.

Proposition 1. *For k and n, we have $GQ_{kn}^{(k)}=GQ_n^k$.*

Proof. By (7), we get $GQ_{kn}^{(k)}=GQ_n^kGQ_{n+1}^0=GQ_n^k$. □

Theorem 1. *For n and s such that $n+s\geq 2$, we have*

$$GQ_{n+s}GQ_{n+s-2}-GQ_{2(n+s-1)}^{(2)}=(-1)^{n+s}\,16\,(1-i)\,.$$

Proof. By Proposition 1 and (3), we get

$$\begin{aligned}GQ_{n+s}GQ_{n+s-2}-GQ_{2(n+s-1)}^{(2)} &= GQ_{n+s}GQ_{n+s-2}-GQ_{n+s-1}^2\\ &= (-1)^{n+s}\,16\,(1-i)\,.\end{aligned}$$

□

Theorem 2. *For k and s, we have*

$$GQ^k_{s+1} - GQ^k_s = GQ^{(k)}_{(s+1)k} - GQ^{(k)}_{sk}. \tag{8}$$

Proof. By (7) and Proposition 1, we get

$$\begin{aligned} GQ^{(k)}_{(s+1)k} - GQ^{(k)}_{sk} &= GQ^{k-k}_s GQ^k_{s+1} - GQ^k_s \\ &= GQ^k_{s+1} - GQ^k_s. \end{aligned}$$

□

Theorem 3. *For k and n, we have the relation*

$$GQ^{(k)}_{kn+1} = 2GQ^{(k)}_{kn} + GQ^{(k)}_{kn-1}.$$

Proof. By (1), (7) and Proposition 1, we obtain

$$\begin{aligned} 2GQ^{(k)}_{kn} + GQ^{(k)}_{kn-1} &= 2GQ^k_n + GQ_{n-1}GQ^{k-1}_n \\ &= GQ^{k-1}_n \left(2GQ_n + GQ_{n-1}\right) \\ &= GQ^{k-1}_n GQ_{n+1} \\ &= GQ^{(k)}_{kn+1}. \end{aligned}$$

□

Theorem 4. ***(Cassini's Identity)*** *Let $GQ^{(k)}_n$ be the generalized Gaussian k-Pell-Lucas numbers. For $n, k \geq 2$, the following gives the Cassini's identity for $GQ^{(k)}_n$:*

$$GQ^{(k)}_{kn+t}GQ^{(k)}_{kn+t-2} - \left(GQ^{(k)}_{kn+t-1}\right)^2 = \begin{cases} GQ^{2k-2}_n \left(-1\right)^{n+1} 16\left(1-i\right), & t = 1, \\ 0, & t \neq 1. \end{cases}$$

Proof. If $t = 1$, by (3), (7) and Proposition 1, then we have

$$\begin{aligned} GQ^{(k)}_{kn+1}GQ^{(k)}_{kn-1} - \left(GQ^{(k)}_{kn}\right)^2 &= \left(GQ^{k-1}_n GQ_{n+1}\right)\left(GQ_{n-1}GQ^{k-1}_n\right) - \left(GQ^k_n\right)^2 \\ &= GQ^{2k-2}_n \left(GQ_{n+1}GQ_{n-1} - GQ^2_n\right) \\ &= GQ^{2k-2}_n \left(-1\right)^{n+1} 16\left(1-i\right), \end{aligned}$$

and if $t \neq 1$, by (7), then we have

$$\begin{aligned} GQ^{(k)}_{kn+t}GQ^{(k)}_{kn+t-2} - \left(GQ^{(k)}_{kn+t-1}\right)^2 &= \left(GQ^{k-t}_n GQ^t_{n+1}\right)\left(GQ^{k-t+2}_n GQ^{t-2}_{n+1}\right) \\ &\quad - \left(GQ^{k-t+1}_n GQ^{t-1}_{n+1}\right)^2 \\ &= GQ^{2k-2t+2}_n \left(GQ^{2t-2}_{n+1} - GQ^{2t-2}_{n+1}\right) \\ &= 0. \end{aligned}$$

□

For $t = 0, 1, 2, \ldots, k-1$, we have the following relations:

$$GQ_{kn+t}^{(k)} = GQ_n^{k-t} GQ_{n+1}^t.$$

3.2. **The generalized Gaussian k-Pell-Lucas polynomials.**

Definition 4. *There are unique numbers m and r such that $n = mk + r$ and $0 \leq r < k$, for $n, k \in \mathbb{N}\,(k \neq 0)$. Then we define the generalized Gaussian k-Pell-Lucas numbers $GQ_n^{(k)}(x)$ by*

$$\begin{aligned} GQ_n^{(k)}(x) \;&:=\; \left[(\alpha^m(x) + \beta^m(x)) - i\,(\alpha(x)\beta^m(x) + \beta(x)\alpha^m(x))\right]^{k-r} \\ &\quad \times \left[\left(\alpha^{m+1}(x) + \beta^{m+1}(x)\right) - i\left(\alpha(x)\beta^{m+1}(x) + \beta(x)\alpha^{m+1}(x)\right)\right]^r, \end{aligned}$$

where $\alpha(x) = x + \sqrt{1+x^2}$ and $\beta(x) = x - \sqrt{1+x^2}$.

In addition, using the matrix methods, we can derive the generalized Gaussian k-Pell-Lucas polynomials. Indeed, it is obvious that

$$GQ_n^{k-1}(x)\, gq_n(x) = \begin{bmatrix} GQ_{kn+1}^{(k)}(x) & GQ_{kn}^{(k)}(x) \\ GQ_{kn}^{(k)}(x) & GQ_{kn-1}^{(k)}(x) \end{bmatrix},$$

where $n > 0$ and

$$gq_n(x) = \begin{bmatrix} GQ_{n+1}(x) & GQ_n(x) \\ GQ_n(x) & GQ_{n-1}(x) \end{bmatrix}.$$

Various values for the generalized Gaussian k-Pell-Lucas polynomials are given in Table 2. From (5) and Definition 4, we have the following relationship between the generalized Gaussian k-Pell-Lucas polynomials and the Gaussian Pell-Lucas polynomials.

$$GQ_n^{(k)}(x) := (GQ_m(x))^{k-r}\,(GQ_{m+1}(x))^r,\ n = mk + r \tag{9}$$

If we take $k = 1$ in (9), then we have that $m = n$ and $r = 0$ so $GQ_n^{(1)}(x) = GQ_n(x)$.

For $k = 2, 3, 4$ and n, we have the following relations between the generalized Gaussian k-Pell-Lucas polynomials and the Gaussian Pell-Lucas polynomials by (4) and (9):

(1) $GQ_{2n}^{(2)}(x) = GQ_n^2(x)$,
(2) $GQ_{2n+1}^{(2)}(x) = GQ_n(x)\, GQ_{n+1}(x)$,
(3) $GQ_{2n+1}^{(2)}(x) = 2xGQ_{2n}^{(2)}(x) + GQ_{2n-1}^{(2)}(x)$,
(4) $GQ_{3n}^{(2)}(x) = GQ_n^3(x)$,
(5) $GQ_{3n+1}^{(3)}(x) = GQ_n^2(x)\, GQ_{n+1}(x)$,
(6) $GQ_{3n+1}^{(3)}(x) = 2xGQ_{3n}^{(3)}(x) + GQ_{3n-1}^{(3)}(x)$,
(7) $GQ_{3n+2}^{(3)}(x) = GQ_n(x)\, GQ_{n+1}^2(x)$,
(8) $GQ_{4n}^{(4)}(x) = GQ_n^4(x)$,
(9) $GQ_{4n+1}^{(4)}(x) = GQ_n^3(x)\, GQ_{n+1}(x)$,

TABLE 2. The generalized Gaussian k-Pell-Lucas polynomials $GQ_n^{(k)}(x)$ for some k and n.

$GQ_n^{(k)}(x)$	$k=1$	$k=2$	$k=3$	$k=4$
$GQ_0^{(k)}(x)$	$2-2xi$	$-4x^2+4-8xi$	$-24x^2+8+\left(8x^3-24x\right)i$	$16x^4-96x^2+16+\left(64x^3-64x\right)i$
$GQ_1^{(k)}(x)$	$2x+2i$	$8x+\left(-4x^2+4\right)i$	$-8x^3+24x+\left(-24x^2+8\right)i$	$-64x^3+64x+\left(16x^4-96x^2+16\right)i$
$GQ_2^{(k)}(x)$	$4x^2+2+2xi$	$4x^2-4+8xi$	$24x^2-8+\left(-8x^3+24x\right)i$	$-16x^4+96x^2-16+\left(-64x^3+64x\right)i$
$GQ_3^{(k)}(x)$	$8x^3+6x+\left(4x^2+2\right)i$	$8x^3+\left(12x^2+4\right)i$	$8x^3-24x+\left(24x^2-8\right)i$	$64x^3-64x+\left(-16x^4+96x^2-16\right)i$
$GQ_4^{(k)}(x)$	$16x^4+16x^2+2+\left(8x^3+6x\right)i$	$16x^4+12x^2+4+\left(16x^3+8x\right)i$	$16x^4-24x^2-8+\left(40x^3+8x\right)i$	$16x^4-96x^2+16+\left(64x^3-64x\right)i$
$GQ_5^{(k)}(x)$	$32x^5+40x^3+10x+\left(16x^4+16x^2+2\right)i$	$32x^5+32x^3+8x+\left(32x^4+28x^2+4\right)i$	$32x^5-8x^3-8x+\left(64x^4+40x^2+8\right)i$	$32x^5-128x^3-32x+\left(112x^4-32x^2-16\right)i$
$GQ_6^{(k)}(x)$	$64x^6+96x^4+36x^2+2+(32x^5+40x^3+10x)i$	$64x^6+80x^4+20x^2-4+(64x^5+80x^3+24x)i$	$64x^6+48x^4+24x^2+8+(96x^5+88x^3+24x)i$	$64x^6-144x^4-96x^2-16+\left(192x^5+64x^3\right)i$
$GQ_7^{(k)}(x)$	$128x^7+224x^5+112x^3+14x+(64x^6+96x^4+36x^2+2)i$	$128x^7+192x^5+72x^3+(128x^6+192x^4+76x^2+4)i$	$128x^7+128x^5+40x^3+8x+(192x^6+240x^4+88x^2+8)i$	$128x^7-96x^5-128x^3-32x+(320x^6)+270x^4+96x^2+16)i$

(10) $GQ_{4n+1}^{(4)}(x)=2xGQ_{4n}^{(4)}(x)+GQ_{4n-1}^{(4)}(x)$,

(11) $GQ_{4n+2}^{(4)}(x)=GQ_n^2(x)\,GQ_{n+1}^2(x)$,

(12) $GQ_{4n+3}^{(4)}(x)=GQ_n(x)\,GQ_{n+1}^3(x)$.

Proposition 2. *For k and n, we have $GQ_{kn}^{(k)}(x)=GQ_n^k(x)$.*

Proof. By (9), we get $GQ_{kn}^{(k)}(x)=GQ_n^k(x)\,GQ_{n+1}^0(x)=GQ_n^k(x)$. □

Theorem 5. *For n and s such that $n+s\geq 2$, we have*

$$GQ_{n+s}(x)\,GQ_{n+s-2}(x)-GQ_{2(n+s-1)}^{(2)}(x)=8(-1)^{n+s}\left(1+x^2\right)(1-xi).$$

Proof. By Proposition 2 and (6), we get

$$\begin{aligned}GQ_{n+s}(x)\,GQ_{n+s-2}(x)-GQ_{2(n+s-1)}^{(2)}(x) &= GQ_{n+s}(x)\,GQ_{n+s-2}(x)-GQ_{n+s-1}^2(x)\\ &= 8(-1)^{n+s}\left(1+x^2\right)(1-xi).\end{aligned}$$

□

Theorem 6. *For k and s, we have*

$$GQ_{s+1}^k(x)-GQ_s^k(x)=GQ_{(s+1)k}^{(k)}(x)-GQ_{sk}^{(k)}(x). \tag{10}$$

Proof. By (9) and Proposition 2, we get

$$\begin{aligned}GQ_{(s+1)k}^{(k)}(x)-GQ_{sk}^{(k)}(x) &= GQ_s^{k-k}(x)\,GQ_{s+1}^k(x)-GQ_s^k(x)\\ &= GQ_{s+1}^k(x)-GQ_s^k(x).\end{aligned}$$

∎

Theorem 7. *For k and n, we have the relation*

$$GQ^{(k)}_{kn+1}(x) = 2xGQ^{(k)}_{kn}(x) + GQ^{(k)}_{kn-1}(x).$$

Proof. By (4), (9) and Proposition 2, we obtain

$$\begin{aligned} 2xGQ^{(k)}_{kn}(x) + GQ^{(k)}_{kn-1}(x) &= 2xGQ^{k}_{n}(x) + GQ_{n-1}(x)\,GQ^{k-1}_{n}(x) \\ &= GQ^{k-1}_{n}(x)\left(2xGQ_{n}(x) + GQ_{n-1}(x)\right) \\ &= GQ^{k-1}_{n}(x)\,GQ_{n+1}(x) \\ &= GQ^{(k)}_{kn+1}(x). \end{aligned}$$

∎

Theorem 8. ***(Cassini's Identity)*** *Let $GQ^{(k)}_{n}(x)$ be the generalized Gaussian k-Pell-Lucas polynomials. For $n, k \geq 2$, the following gives the Cassini's identity for $GQ^{(k)}_{n}(x)$:*

$$\begin{aligned} & GQ^{(k)}_{kn+t}(x)\,GQ^{(k)}_{kn+t-2}(x) - \left(GQ^{(k)}_{kn+t-1}(x)\right)^2 \\ = & \begin{cases} GQ^{2k-2}_{n}(x)\,8\,(-1)^{n-1}\left(1+x^2\right)(1-xi), & t=1, \\ 0, & t \neq 1. \end{cases} \end{aligned}$$

Proof. If $t = 1$, by (6), (9) and Proosition 2, then we have

$$\begin{aligned} & GQ^{(k)}_{kn+1}(x)\,GQ^{(k)}_{kn-1}(x) - \left(GQ^{(k)}_{kn}(x)\right)^2 \\ = & \left(GQ^{k-1}_{n}(x)\,GQ_{n+1}(x)\right)\left(GQ_{n-1}(x)\,GQ^{k-1}_{n}(x)\right) - \left(GQ^{k}_{n}(x)\right)^2 \\ = & GQ^{2k-2}_{n}(x)\left(GQ_{n+1}(x)\,GQ_{n-1}(x) - GQ^{2}_{n}(x)\right) \\ = & GQ^{2k-2}_{n}(x)\,8\,(-1)^{n-1}\left(1+x^2\right)(1-xi), \end{aligned}$$

and if $t \neq 1$, by (9), then we have

$$\begin{aligned} & GQ^{(k)}_{kn+t}(x)\,GQ^{(k)}_{kn+t-2}(x) - \left(GQ^{(k)}_{kn+t-1}\right)^2(x) \\ = & \left(GQ^{k-t}_{n}(x)\,GQ^{t}_{n+1}(x)\right)\left(GQ^{k-t+2}_{n}(x)\,GQ^{t-2}_{n+1}(x)\right) \\ & - \left(GQ^{k-t+1}_{n}(x)\,GQ^{t-1}_{n+1}(x)\right)^2 \\ = & GQ^{2k-2t+2}_{n}(x)\left(GQ^{2t-2}_{n+1}(x) - GQ^{2t-2}_{n+1}(x)\right) \\ = & 0. \end{aligned}$$

∎

For $t = 0, 1, 2, \ldots, k-1$, we have the following relations:

$$GQ^{(k)}_{kn+t}(x) = GQ^{k-t}_{n}(x)\,GQ^{t}_{n+1}(x).$$

4. Conclusions

Halıcı and Öz defined Gaussian Pell-Lucas numbers in [6]. We introduced a generalization of these numbers as the generalized Gaussian k-Pell-Lucas numbers. Also, Yağmur defined Gaussian Pell-Lucas polynomials in [15]. We introduced a generalization of these polynomials as the generalized Gaussian k-Pell-Lucas polynomials. Some relations between the family of the generalized Gaussian k-Pell-Lucas numbers and the known Gaussian Pell-Lucas numbers are presented. Some relations between the family of the generalized Gaussian k-Pell-Lucas polynomials and the known Gaussian Pell-Lucas polynomials are presented. Then identities for these numbers and polynomials are proved.

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