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On Γ-Paracompact Spaces

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Abstract

We introduce the class of Γ-paracompact spaces as a stronger form of paracompactness. A space X is said to be Γ-paracompact (Γ-P, for short) space if every open cover of X has a strongly locally finite (SLF) open refinement. We give some characterizations of Γ-P spaces. We also define some weaker forms of Γ-P spaces as Γ_σ-paracompact and feebly Γ-P spaces We later introduce Γ-expandable spaces and study the relationships between Γ-expandable and Γ-P spaces. We also investigate some of topological properties of Γ-P spaces.

Keywords: *paracompact; Γ-paracompact; regular open set; strongly locally finite collection*
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1. Introduction

A space will always denote a topological space and no separation axioms unless explicitly stated will be assumed. $\mathbb{N},\mathbb{R}$ will denote set of positive integers and set of real numbers, respectively. ω will denote the first infinite cardinal. A space is said to be paracompact if every open cover of the space admits a locally finite (LF) open refinement. Many generalizations and as stronger forms of paracompactness got their places in many papers [2, 3, 4, 5, 6, 9, 10, 11, 14]. The aim of this paper is to introduce and study Γ-paracompactness as a stronger form of paracompactness by using strongly locally finite collections. In Section 2, we give definition of Γ-paracompactness, its fewer characterizations and define Γ_σ-paracompact and feebly Γ-paracompact spaces as weaker forms of Γ-paracompactness. We investigate the relationships between these spaces and paracompactness. We construct some related counter examples that some of them does not imply the other ones. In Section 3, We introduce the class of Γ-expandable spaces and give some characterizations and the relations between Γ-paracompactness. We later study some topological properties of Γ-paracompactness by considering it under some certain type of mappings, subspaces and sum.

Let X be space and $A \subset X$. $Int(A)$ and $\overline{A}$ will denote the interior and closure of A, respectively. The subset A is said to be a regular open set [11] if $A = Int(\overline{A})$ and $RO(X)$ will denote the set of all regular open subsets of X. It is well known that every regular open set is an open set, so $RO(X)$ is contained by the topology on X. The intersection of finitely many regular open set is regular open while even the union of two regular open set need not be a regular open. For an open set A, the minimal regular open set containing A is $Int(\overline{A})$ which is called [7] regularly open envelope. If the collection $\mathscr{A} = \{A_\alpha : \alpha \in \Delta\}$ is a LF collection of a space X and U_x is the open set containing x and intersects at most finitely many members of $\mathscr{A}$, we sometimes write U_x has LFP of $\mathscr{A}$. Other notions and terminology will follow [15].

2. Strongly locally finiteness and Γ-paracompactness

Definition 2.1. *[10]* A collection of $\mathscr{A} = \{A_\alpha : a \in \Delta\} \subset \mathscr{P}(X)$ is called *strongly locally finite (SLF, for short)* if there is $V_x \in RO(X)$ for each $x \in X$ such that $x \in V_x$ and $|\{\alpha \in \Delta : V_x \cap A_\alpha \neq \emptyset\}| < \omega$.

Clearly every SLF family of a space X is LF, but statement converse is not, in general, true as the following example illustrates;

Example 2.2. *Let X be an uncountable set and $a \in X$ be fixed. Consider the particular point topology $\tau = \{U \subset X : a \in U\} \cup \{\emptyset\}$ on X. The family $\mathscr{B} = \{\{x\} : x \in X\}$ is clearly a LF collection of X which fails to be SLF since $RO(X) = \{X, \emptyset\}$.*

Theorem 2.3. *Let X be a regular space. Then every LF collection of X is SLF.*

Proof. If $\mathscr{D}$ is any LF collection of X, there is an open set O_x for each $x \in X$ such that $x \in O_x$ and O_x has LFP of $\mathscr{D}$. Then by the regularity of X, one can find and open set V_x containing x such that $Int(\overline{V_x}) \subset O_x$ where $Int(\overline{V_x}) \in RO(X)$. This shows that $\mathscr{D}$ is SLF. □

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Lemma 2.4. *Let X be any space and $\mathscr{A} = \{A_\alpha : a \in \Delta\} \subset \mathscr{P}(X)$*

1. *If $\mathscr{A}$ is SLF then $\overline{\mathscr{A}}$ is SLF where $\overline{\mathscr{A}} = \{\overline{A_\alpha} : \alpha \in \Delta\}$.*
2. *If $\mathscr{A}$ strongly localy finite and $B_\alpha \subset A_\alpha$ for every $\alpha \in \Delta$, then $\mathscr{B} = \{B_\alpha : \alpha \in \Delta\}$ is SLF.*

Proof. Straightforward. □

Definition 2.5. A space X is said to be Γ-*paracompact* (Γ-P) if every open cover of X has a SLF open refinement.

It is clear that every compact space is Γ-P and every Γ-P space is paracompact. But the inverse implications is not true in general.

Example 2.6. *A paracompact space which is not Γ-P.*

Let $\mathbb{R}$ be set of real numbers, $(\mathbb{R},\tau_1)$ be discrete space and $(\mathbb{R},\tau_2)$ be finite complement space. Consider the product space (X,τ) where $X = \mathbb{R}\times\mathbb{R}$ and $\tau = \tau_1 \times \tau_2$. Since $(\mathbb{R},\tau_1)$ is paracompact and $(\mathbb{R},\tau_2)$ is compact, (X,τ) is paracompact. Consider the open cover $\mathscr{D}$ of X where $\mathscr{D} = \mathscr{D}_1 \times \mathscr{D}_2$ such that $\mathscr{D}_1 = \{(-n,n) : n \in \mathbb{N}\}$ and $\mathscr{D}_2 = \{X \setminus \{n\} : n \in \mathbb{N}\}$. None of LF refinement of $\mathscr{D}$ is finite. Otherwise it would have a finite subcover. So (X,τ) is not Γ-P space since every nonempty open subset of $(\mathbb{R},\tau_2)$ is dense.

Example 2.7. *A Γ-P space which is not compact.*

Let ξ is a nonprincipal ultrafilter on $\mathbb{N}$. Assume $X = \mathbb{N}\cup\{\xi\}$. Consider the topology on X that has open sets of the form $A\cup\{\xi\}$ where $A \in \xi$ and the points of $\mathbb{N}$. The set X with this topology is called single ulftrafilter topology. (See [13]).
Since X is countable and zero dimensional, it is a regular Lindelöf space. Then X is paracompact space. X is also Γ-P space because every open subsets of X is clopen (closed+open). But X is not a compact space. As ξ is an ultrafilter on $\mathbb{N}$, assume that the even integers set $\mathbb{Z}_e$ is a member of ξ. Then $\mathscr{D} = \{\{x\} : x \notin \mathbb{Z}_e\}\cup\{\mathbb{Z}_e\cup\{\xi\}\}$ is an open cover of X which fails to have a finite subcover.

Theorem 2.8. *Every countably compact Γ-P space is compact.*

Proof. Obvious consequence of every Γ-P space is paracompact. □

Theorem 2.9. *A regular space is paracompact if and only if it is Γ-P.*

Proof. It is a direct consequence of Theorem 2.3. □

Theorem 2.10. *If every open cover of a space X has a SLF closed refinement, then the space X is Γ-P.*

Proof. Let $\mathscr{D}$ be an open cover of X and $\mathscr{C} = \{V_\beta : \beta \in B\}$ be a SLF closed refinement of $\mathscr{D}$. Let consider the open cover $\mathscr{R} = \{R_x : x \in X\}$ where $R_x \in RO(X)$ which contains x and has LFP of $\mathscr{C}$. If $\mathscr{K} = \{K_\alpha : \alpha \in \Delta\}$ is SLF closed refinement of $\mathscr{R}$, then $V'_\beta = X \setminus \bigcup\{K_\alpha \in \mathscr{K} : K_\alpha \cap V_\beta = \emptyset\}$ is an open subset of X for each $V_\beta \in \mathscr{C}$ since every SLF collection is LF. Since $\mathscr{C}$ is a cover of X, there is a $V_{\beta(x)} \in \mathscr{C}$ such that $x \in V_{\beta(x)}$ for each $x \in X$. Then $x \in V'_{\beta(x)}$ and $\mathscr{C}' = \{V'_{\beta(x)} : V_{\beta(x)} \in \mathscr{C}\}$ is a SLF open cover of X. For seeing this $\mathscr{C}'$ is SLF, let $x_0 \in X$ be arbitrary and $W_{x_0} \in RO(X)$ containing x_0 and meeting at most finitely many members $K_{\alpha_1}, K_{\alpha_2}, ..., K_{\alpha_{n(x_0)}}$ of $\mathscr{K}$. Since $\mathscr{K}$ is a cover, $W_{x_0} \subset \bigcup_{i=1}^{n(x_0)} K_{\alpha_i}$ holds. Hence if $W_{x_0} \cap V'_{\beta(x)} \neq \emptyset$ then $K_m \cap V'_{\beta(x)} \neq \emptyset$ holds at least for one $m \in \{\alpha_1, \alpha_2, ..., \alpha_{n(x_0)}\}$ and witnesses that $K_m \cap V_{\beta(x)} \neq \emptyset$. On the other hand, we can find a $R_x^m \in \mathscr{R}$ for each K_m such that $|\{\beta \in B : R_x^m \cap V_\beta \neq \emptyset\}| < \omega$. Then every K_m meets at most finitely many V_β of $\mathscr{C}$ and so $W_{x_0} \cap V'_{\beta(x)} \neq \emptyset$ is true for only at most finitely many $V'_{\beta(x)} \in \mathscr{C}'$. Now let choose a $U_{V_\beta} \in \mathscr{D}$ for each $V_\beta \in \mathscr{C}$ such that $V_\beta \subset U_{V_\beta}$ and say $\mathscr{D}' = \{U_{V_{\beta(x)}} \cap V'_{\beta(x)} : V_{\beta(x)} \in \mathscr{C}\}$. By Lemma 2.4(2), $\mathscr{D}'$ is SLF and clearly is an open refinement of $\mathscr{D}$ which completes the proof. □

An example witnessing that the inverse implication of the theorem above does not always hold would be finite complement topology on any set X such that $|X| \geq \omega$. Since X is compact, then it is Γ-P. But none of open cover $\mathscr{D}$ of X such that $X \notin \mathscr{D}$ can have SLF closed refinement since every closed subset of X is finite or X itself and $RO(X) = \{\emptyset, X\}$.

Definition 2.11. *[10]* Let X be a space and $\mathscr{A} \subset \mathscr{P}(X)$. If $\mathscr{A}$ can be represented as countably union of SLF collection, $\mathscr{A}$ is said to be σ-SLF.

Definition 2.12. A space X is called;

1. Γ_σ-*paracompact* if every open cover of X has a σ-SLF open refinement.
2. *Feebly* Γ-*paracompact* if every open cover has a SLF refinement.

Proposition 2.13. *Every Γ-P space is Γ_σ-paracompact.*

Proof. Since every SLF collection is σ-SLF, it follows that the space is Γ_σ-paracompact. □

Theorem 2.14. *Every Γ_σ-paracompact space is feebly Γ-P.*

Proof. Let space X be Γ_σ-paracompact and $\mathscr{O}$ be an open cover of X. Let $\mathscr{G} = \bigcup_{n\in\mathbb{N}}\mathscr{G}_n$ be σ-SLF open refinement of $\mathscr{O}$ where each $\mathscr{G}_n$ is SLF. Let $U_n = \bigcup\{G : G \in \mathscr{G}_n\}$ for each $n \in \mathbb{N}$. Then we obtain an open cover $\mathscr{D} = \{U_n : n \in \mathbb{N}\}$ of X. Now put $U_1 = V_1$ and $V_n = U_n \setminus \bigcup_{m=1}^{n-1} U_m$ for each $n \in \mathbb{N}\setminus\{1\}$. Let $\mathscr{B} = \{V_n : n \in \mathbb{N}\}$. We firstly shall show that $\mathscr{B}$ is LF. For seeing this, one can find $n_x \in \mathbb{N}$ for each $x \in X$ such that $n_x = min\{n \in \mathbb{N} : x \in U_n\}$. Then $U_{n_x} \cap V_k = \emptyset$ for each each $k > n_x$. Hence $\mathscr{B}$ is LF. Now let $\mathscr{W}_n = \{G\cap V_n : G \in \mathscr{G}_n\}$ for each $n \in \mathbb{N}$ and $\mathscr{W} = \bigcup_{n\in\mathbb{N}}\mathscr{W}_n$. We claim that $\mathscr{W}$ is a SLF refinement of $\mathscr{O}$. Since $x \in U_{n_x}$, there is $G_x \in \mathscr{G}_{n_x}$ such that $x \in G_x \cap V_{n_x} \in \mathscr{W}_{n_x} \subset \mathscr{W}$ and thus $\mathscr{W}$ is a cover X. Now let $W \in \mathscr{W}$. Then there is $n_0 \in \mathbb{N}$ such that $W = G_W \cap V_{n_0}$ where $G_W \in \mathscr{G}_{n_0}$. Also one can find an $O_W \in \mathscr{O}$ such that $G_W \subset O_W$, hence $W \subset O_W$ witnessing that $\mathscr{W}$ refines $\mathscr{O}$. Now let $x \in X$ be arbitrary. Since $\mathscr{B}$ is LF, there is an open set T_x such that $x \in T_x$ and has LFP $V_{n_1}, V_{n_2}, ..., V_{n_m}$ of $\mathscr{B}$. On the other hand, by Lemma 2.4, each $\mathscr{W}_n$ is SLF, so $\mathscr{W}_{n_i}$ is SLF for each $i = 1,2,...,m$. Hence, there is an regular open set T_i containing x and has LFP of $\mathscr{W}_{n_i}$. Then $T = \cap_{i=1}^m T_i \subset T_x \cap (\cap_{i=1}^m \cap T_i)$ is the desired regular open set which contains x and has LFP of $\mathscr{W}$. This completes the proof. □

Example 2.15. *There is a feebly Γ-paracompact space which is not Γ_σ paracompact. Namely, let X be an uncountable discrete space and $P = \{x,y\}$ with $P \cap X = \emptyset$. Consider the topology on $Y = X \cup P$ that has open sets are the form of $G \cup A$ where G is open in X and $A \subset P$. The space Y is feebly Γ-paracompact space while it is not Γ_σ-paracompact. It has at least an open cover which does not admit a LF open refinement. Say $\mathscr{D} = \{\{a\}\cup\{x,y\} : a \in X\}$. It obviously is an open cover of Y. Assume that $\mathscr{D}$ has a LF open refinement $\mathscr{B} = \{V_\alpha : \alpha \in \Delta\}$. Then there is an open set G_x containing x and satisfying $V_\alpha \cap G_x = \emptyset$ for each $\alpha \in \Delta$ with $\alpha \neq \alpha_1, \alpha_2, ..., \alpha_{n_x}$. Then for all $\alpha \neq \alpha_1, \alpha_2, ..., \alpha_{n_x}$, V_α is of the form $G_\alpha \cup \{y\}$ for some open set G_α in X. But this contradicts that $\mathscr{B}$ is LF since every open set containig y meets infinitely many members of $\mathscr{B}$. Then since Y is regular, it is impossible to find an σ-LF open refinement of $\mathscr{D}$ and thus a σ-SLF open refinement of $\mathscr{D}$. But Y is feebly Γ-paracompact. The collection $\mathscr{B} = \{\{a\} : a \in Y\}$ is clearly a refinement for every open cover $\mathscr{D}$ of X. Now choose $\{a,x\}$ and $\{x\} \in RO(Y)$ for each $a \in Y\setminus\{x\}$ and $x \in Y$, respectively and conclude that $\mathscr{B}$ is SLF.*

On the other hand, every Γ_σ-paracompact space need not be Γ-P as follows;

Example 2.16. *Take the topology $\tau = \{\mathbb{N},\emptyset\}\cup\{\{1,2,3,...,n\} : n \in \mathbb{N}\}$ on $\mathbb{N}$. So the space $(\mathbb{N},\tau)$ is a Γ_σ-paracompact space which fails to be Γ-P. Let $\mathscr{D}$ be an open cover of $\mathbb{N}$. Let $\mathscr{B}_n = \{\{1,2,3,...,n\}\}$ for each $n \in \mathbb{N}$ and $\mathscr{B} = \bigcup_{n\in\mathbb{N}}\mathscr{B}_n$. Since every $\mathscr{B}_n$ is SLF, $\mathscr{B}$ is σ-SLF. It can easily be seen that $\mathscr{B}$ is σ-SLF open refinement of $\mathscr{D}$. But the space is not Γ-P since it is not even paracompact.*

3. Γ-expandibility and preservation of Γ-paracompactness

Definition 3.1. *[8]* A space X is is said to be *expandable* if for each LF collection $\mathscr{B} = \{V_\alpha : \alpha \in \Delta\}$ of X, there is a LF collection $\mathscr{D} = \{U_\alpha : \alpha \in \Delta\}$ such that $V_\alpha \subset U_\alpha$ and U_α is open set for each $\alpha \in \Delta$.

Definition 3.2. A space X is said to be Γ-*expandable* if for each LF collection $\mathscr{B} = \{V_\alpha : \alpha \in \Delta\}$ of X, there is a SLF collection $\mathscr{D} = \{U_\alpha : \alpha \in \Delta\}$ such that $V_\alpha \subset U_\alpha$ and U_α is open set for each $\alpha \in \Delta$. If $|\Delta| \leq \omega$, it is said X is ω-Γ-expandable.

Theorem 3.3. *Followings are equivalent for a space X;*

1. *X is Γ-expandable*
2. *For every LF collection $\mathscr{C} = \{K_\alpha : \alpha \in \Delta\}$ of X where K_α closed subset for each $\alpha \in \Delta$, there is a SLF collection $\mathscr{D} = \{U_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $K_\alpha \subset U_\alpha$ for each $\alpha \in \Delta$.*

Proof. It is clear from Definition 3.2 and Lemma 2.4(1). □

Theorem 3.4. *Every Γ-P space is Γ-expandable.*

Proof. Let the space X be Γ-P and $\mathscr{C} = \{K_\alpha : \alpha \in \Delta\}$ be a LF collection of closed subsets of X. Let $\mathscr{F} \subset \Delta$ be the family of all finite subsets of Δ. Define $V_F = X \setminus \bigcup\{K_\alpha : \alpha \notin F\}$ for every $F \in \mathscr{F}$. It is clear that each V_F is open and has LFP of $\mathscr{C}$. Since $\mathscr{C}$ is LF, there is an open set O_x for each $x \in X$ such that O_x contains x and $O_x \cap K_\alpha = \emptyset$ for each $\alpha \neq \alpha_1, \alpha_2, ..., \alpha_{n_x}$. Then for $F = \{\alpha_1, \alpha_2, ..., \alpha_{n_x}\}$, $x \in V_F$. So we obtain $\mathscr{B} = \{V_F : F \in \mathscr{F}\}$ such that $X = \bigcup_{F\in\mathscr{F}} V_F$. Since X is Γ-P, $\mathscr{B}$ has a SLF open refinement. Say $\mathscr{D} = \{U_\beta : \beta \in \Delta\}$ and define $G_\alpha = \bigcup\{U_\beta \in \mathscr{D} : U_\beta \cap K_\alpha \neq \emptyset\}$ for each $\alpha \in \Delta$. Then G_α is clearly open and $K_\alpha \subset G_\alpha$ for each $\alpha \in \Delta$. Moreover, $\mathscr{G} = \{G_\alpha : \alpha \in \Delta\}$ is SLF. For seeing this, let $x \in X$ be arbitrary and $U_x \in RO(X)$ which contains x and LFP of $\mathscr{D}$. Then by the construction of G_α, $U_x \cap G_\alpha \neq \emptyset$ if and only if $U_x \cap U_\beta \neq \emptyset$ and $U_\beta \cap K_\alpha$ for some $\beta \in \Delta$. Since there is F_β such that $U_\beta \subset V_{F_\beta}$ for each $\beta \in \Delta$, then $|\{\alpha \in \Delta : U_\beta \cap K_\alpha \neq \emptyset\}| < \omega$, witnessing that U_x meets at most finitely many G_α. Hence $\mathscr{G}$ is SLF. □

Theorem 3.5. *Every countably compact space is Γ-expandable.*

Proof. Recall from [8] that every LF collection of subsets of a countably paracompact space is finite and every countably compact space is expandable. Thus, every countably compact space is clearly Γ-expandable. □

Not eveery Γ-expandable spaces need to be Γ-P space. The long line L (see [13]) is a nice example which is countably compact but not even paracompact space.

Theorem 3.6. *Following statements are equivalent for any space X;*

1. *X is ω-Γ-expandable*
2. *Each countable open cover of X admits a SLF open refinement.*
3. *Each countable open cover $\mathscr{D} = \{U_n : n \in \mathbb{N}\}$ of X, there is a SLF open cover $\mathscr{B} = \{V_n : n \in \mathbb{N}\}$ such that $V_n \subset U_n$ for each $n \in \mathbb{N}$.*

Proof. $1 \Rightarrow 2$ Let $\mathscr{D} = \{U_n : n \in \mathbb{N}\}$ be any countable open cover of X. Define $V_n = \bigcup_{m \leq n} U_m$ for each $n \in \mathbb{N}$ and put $W_1 = V_1$ and $W_n = V_n \setminus V_{n-1}$ for each $n \in \mathbb{N} \setminus \{1\}$. Observe that $W_n \subset U_n$ for each $n \in \mathbb{N}$. Let $x \in X$ and n_x be the minimum natural number such that $x \in U_{n_x}$. Then $x \in W_{n_x}$ and thus $\mathscr{W} = \{W_n : n \in \mathbb{N}\}$ is refinement of $\mathscr{D}$. Since $U_n \cap W_m = \emptyset$ for each $m > n$ and X is ω-Γ-expandable, there is a SLF collection $\mathscr{G} = \{G_n : n \in \mathbb{N}\}$ of X such that G_n is open and $W_n \subset G_n$ for each $n \in \mathbb{N}$. Now let $O_n = U_n \cap G_n$ for each $n \in \mathbb{N}$ and $\mathscr{O} = \{O_n : n \in \mathbb{N}\}$. Then $\mathscr{O}$ is SLF. On the other hand, there is some $n \in \mathbb{N}$ for each $x \in X$ such that $x \in W_n$ and since $W_n \subset U_n$, then $x \in O_n = U_n \cap G_n$ and thus, $\mathscr{D}$ admits $\mathscr{O}$ as SLF open refinement.

$2 \Rightarrow 3$ Let $\mathscr{D} = \{U_n : n \in \mathbb{N}\}$ be any countable open cover of X and $\mathscr{O}$ be a SLF open refinement of $\mathscr{D}$. Then there is a $n(V) \in \mathbb{N}$ for each $V \in \mathscr{O}$ such that $V \subset U_{n(V)}$. Now let define $V_n = \bigcup\{V : n(V) = n\}$. Then $\mathscr{B} = \{V_n : n \in \mathbb{N}\}$ is the desired cover. $\mathscr{B}$ is clearly an open cover of X satisfying $V_n \subset U_n$ for each $n \in \mathbb{N}$. To show that $\mathscr{B}$ is SLF, let $x \in X$ and $U_x \in RO(X)$ containing x and $\eta(U_x, \mathscr{O}) = \{V \in \mathscr{O} : U_x \cap V \neq \emptyset\}$ be finite. If $U_x \cap V_n \neq \emptyset$, then one can find $V \in \mathscr{O}$ such that $n(V) = n$ and $U_x \cap V \neq \emptyset$. Then $\{V_n : U_x \cap V \neq \emptyset\}$ is finite since $\eta(U_x, \mathscr{O})$ is finite.

The implication $2 \Rightarrow 1$ can be done similar to the Theorem 3.4 and the implication $3 \Rightarrow 2$ is an obvious fact. Thus, that completes the proof. □

Theorem 3.7. *Every ω-Γ-expandable Γ_σ-paracompact space is Γ-P.*

Proof. Let X be ω-Γ-expandable and Γ_σ-paracompact space and $\mathscr{O}$ be any open cover of X. Since X Γ_σ-paracompact, let $\mathscr{D} = \{U_n : n \in \mathbb{N}\}$ be obtained as in Theorem 2.14. Since X is ω-Γ-expandable, then by Theorem 3.5, there is an SLF open cover $\mathscr{B} = \{V_n : n \in \mathbb{N}\}$ such that $V_n \subset U_n$ for each $n \in \mathbb{N}$. Now let $\mathscr{W} = \{G \cap V_n : G \in \mathscr{G}_n, n \in \mathbb{N}\}$. We shall show that $\mathscr{W}$ is SLF open refinement of $\mathscr{O}$. For any $W \in \mathscr{W}$, one can find $n(W) \in \mathbb{N}$ such that $W = G \cap V_{n(W)}$ where $G \in \mathscr{G}_{n(W)}$. Since $\mathscr{G}$ refines $\mathscr{O}$, there is O_G such that $G \subset O_G$ which is witnessing that $\mathscr{W}$ refines $\mathscr{O}$. On the other hand, there is n_x for each $x \in X$ such that $x \in V_{n_x}$ and $T_x \in RO(X)$ which contains x and has LFP of $\mathscr{B}$. Hence, there is $k(x) \geq n_x$ such that $V_n \cap T_x = \emptyset$ for each $n > k(x)$. Also there is a T_x^n open set containing x and LFP of $\mathscr{G}_n$ for each $n \leq k(x)$. Then, $T = T_x \cap (\bigcap_{m=1}^n T_x^n) \in RO(X)$ has LFP of $\mathscr{W}$. □

In what follows, we look at the behaviour of Γ-P spaces under certain types of mappings, subspaces and sum.

Definition 3.8. *[1]* Let X and Y be spaces. A map $f : X \to Y$ is said to be *completely continuous* if $f^{-1}(U) \in RO(X)$ for every open subset U of Y.

Theorem 3.9. *Let f be completely continuous, closed surjection from the space X to Y such that fibers $f^{-1}(y)$ is compact for each $y \in Y$. If Y is Γ-P, then X is Γ-P.*

Proof. Let $\mathscr{D} = \{U_\alpha : \alpha \in \Delta\}$ be any open cover X. Firstly, there is a finite subfamily Δ_y of Δ for each $y \in Y$ such that $U^y = \bigcup_{\alpha \in \Delta_y} U_\alpha \supset f^{-1}(y)$. Since f is closed and U^y is open, there is an open set W_y of Y such that $f^{-1}(W_y) \subset U^y$. Then $\mathscr{W} = \{W_y : y \in Y\}$ is an open cover of Y and thus, it has a SLF open refinement. Say, $\mathscr{B} = \{V_\gamma : \gamma \in \Phi\}$. Now if $x \in X$, then there is $V \in RO(Y)$ such that $f(x) \in V$ and $V \cap V_\gamma = \emptyset$ for each $\gamma \neq \gamma_1, \gamma_2, ..., \gamma_n$. Then $f^{-1}(V) \cap f^{-1}(V_\gamma) = \emptyset$ for each $\gamma \neq \gamma_1, \gamma_2, ..., \gamma_n$ and $f^{-1}(V) \in RO(X)$. Hence, $f^{-1}(\mathscr{B}) = \{f^{-1}(V_\gamma) : \gamma \in \Phi\}$ is SLF and an open cover of X since f is also a continuous map. Since $\mathscr{B}$ refines $\mathscr{W}$, there is $y(\gamma)$ for each $\gamma \in \Phi$ such that $V_\gamma \subset W_{y(\gamma)}$. And thus, $f^{-1}(V_\gamma) \subset \bigcup_{\alpha \in \Delta_{y(\gamma)}} U_\alpha = U^{y(\gamma)}$. Now let $\mathscr{O} = \{f^{-1}(V_\gamma) \wedge U^{y(\gamma)} : \gamma \in \Phi, y(\gamma) \in Y\}$ where $f^{-1}(V_\gamma) \wedge U^{y(\gamma)} = \{f^{-1}(V_\gamma) \cap U_\alpha : \gamma \in \Phi, \alpha \in \Delta_{y(\gamma)}\}$. Then $\mathscr{O}$ is clearly SLF and since $f^{-1}(\mathscr{B})$ is an open cover of X, $\mathscr{O}$ is also an open cover of X and obviously refines $\mathscr{D}$. Hence, X is Γ-P. □

Remark 3.10. *Being Γ-P is not inverse invariant under perfect mappings. (See Example* 2.6*).*

Definition 3.11. *[12]* A mapping $f : X \to Y$ is said to be *cl-supercontinuous* if for each $x \in X$ and each open set U containing $f(x)$, there is a clopen subset V of X such that $f(V) \subset U$.

Say f is cl-perfect if it is cl-supercontinuous, closed and every fiber $f^{-1}(y)$ is compact.

Theorem 3.12. *Let f be cl perfect and open surjection from X to Y. If X is Γ-P, then Y is Γ-P.*

Proof. Let $\mathscr{D} = \{U_\alpha : \alpha \in \Delta\}$ be any open cover of Y. The collection $f^{-1}(\mathscr{D})$ is an open cover of X. Then it has a SLF open refinement $\mathscr{B} = \{V_\gamma : \gamma \in \Phi\}$. Since f is open $f(\mathscr{B})$ is an OR of $\mathscr{D}$ where $f(\mathscr{B}) = \{f(V_\gamma) : \gamma \in \Phi\}$. Also $f(\mathscr{B})$ is SLF. For seeing this, let $y \in Y$ be arbitrary. Since $\mathscr{B}$ is SLF, there is a $U_x \in RO(X)$ for every $x \in f^{-1}(y)$ which contains x and has LFP of $\mathscr{B}$. So $\mathscr{W} = \{U_x : x \in f^{-1}(y)\}$ is an open cover of $f^{-1}(y)$ and thus, there is a finite subcover of $\mathscr{W}$ such that $f^{-1}(y) \subset \bigcup_{x \in F} U_x$ where $F \subset f^{-1}(y)$ is finite. Now find an open set T_y such that $y \in T_y$ and $f^{-1}(T_y) \subset \bigcup_{x \in F} U_x$. Then $f^{-1}(T_y)$ has LFP of $\mathscr{B}$. On the other hand, since f is cl-supercontinuous, there is a clopen subset and thus a regular open subset U of X such that $y \in f(U) \subset T_y$. Since $f^{-1}(T_y) \cap V_\gamma = \emptyset$ is true for all but finitely many $\gamma \in \Phi$, then $T_y \cap f(V_\gamma) = \emptyset$ is true for all but finitely many $\gamma \in \Phi$. And since f is both open and closed, then $f(U)$ is clopen and thus a regular open set which contains y and has LFP of $f(\mathscr{B})$ since $f(U) \subset T_y$. □

Theorem 3.13. *A clopen subspace of Γ-P space is Γ-P.*

Proof. Let X be a space and $R \subset X$ clopen subspace of X. Let $\mathscr{D} = \{U_\alpha : \alpha \in \Delta\}$ be any open cover of R. Find an open subset V_{U_α} of X such that $U_\alpha = R \cap V_{U_\alpha}$. Since R is a closed subset, then $\mathscr{B} = \{X \setminus R\} \cup \{V_{U_\alpha} : \alpha \in \Delta\}$ is an open cover of X. Then it admits a SLF open refinement $\mathscr{O} = \{O_\beta : \beta \in \Phi\}$. Let $\mathscr{O}^* = \{R \cap O_\beta : \beta \in \Phi\}$. $\mathscr{O}^*$ is a SLF refinement of $\mathscr{D}$. Firstly, $\mathscr{O}^*$ contains a non-empty open set. Because if $x \in R$ and since $\mathscr{O}$ is a cover of X, there is an $O_{\beta(x)}$ such that $x \in O_{\beta(x)}$ and thus, $x \in R \cap O_{\beta(x)}$. Let $R \cap O_\beta$ be arbitrary member of $\mathscr{O}^*$. Then O_β must be contained in V_{U_α} for some $\alpha \in \Delta$. So $O_\beta \cap R \subset V_{U_\alpha} \cap R = U_\alpha$. Then $\mathscr{O}^*$ refines $\mathscr{D}$. Now let $x \in R$. We can find $U_x \in RO(X)$ such that U_x contains x and has LFP of $\mathscr{O}$. But then $U_x \cap R$ has LFP of $\mathscr{O}^*$. On the other hand, since R is clopen, $U_x \cap R \in RO(A)$ which completes the proof. □

Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of spaces such that $X_\alpha \cap X_{\alpha'} = \emptyset$ for each $\alpha \neq \alpha'$. Let $X = \bigcup_{\alpha\in\Delta} X_\alpha$ and topologize X by $\tau = \{U \subset X : G \cap X_\alpha$ is open inX_α for each $\alpha \in \Delta\}$. Then (X, τ) is said to be sum of the spaces $\{X_\alpha : \alpha \in \Delta\}$. And it is written as $X = \bigoplus_{\alpha\in\Delta} X_\alpha$.

Theorem 3.14. *Topological sum $X = \bigoplus_{\alpha\in\Delta} X_\alpha$ is Γ-P if and only if each space X_α is Γ-P.*

Proof. If $X = \bigoplus_{\alpha\in\Delta} X_\alpha$ is Γ-P, since every X_α is clopen, it follows from Theorem 3.12. So X_α is Γ-P. For proving sufficiency, Let $\mathscr{G}$ be any open cover of X. Then $\mathscr{G}_\alpha = \{G \cap X_\alpha : G \in \mathscr{G}\}$ is an open cover of X_α for each $\alpha \in \Delta$. Thus, we can find a SLF open refinement $\mathscr{O}_\alpha$ of $\mathscr{G}_\alpha$ for each $\alpha \in \Delta$. Now let $\mathscr{O} = \bigcup_{\alpha\in\Delta} \mathscr{O}_\alpha$. It clearly is that $\mathscr{O}$ is SLF open refinement of $\mathscr{G}$ which completes the proof. □

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