Konuralp Journal of Mathematics, 12 (1) (2024) 80-85

Konuralp Journal of Mathematics

Research Paper
Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Operator Norm-Numerical Radius Gaps for Analytic Function of Hilbert Space Operators

Pembe Ipek Al[1], Rukiye Öztürk Mert[2*] and Zameddin I. Ismailov[1]

[1]*Department of Mathematics, Faculty of Science, Karadeniz Technical University, Trabzon, Türkiye*
[2]*Department of Mathematics, Faculty of Art and Sciences, Hitit University, Çorum, Türkiye*
[*]*Corresponding author*

Abstract

In this study, some estimates are obtained by means of the number of differences between the operator norm of the analytic functions of the linear bounded Hilbert space operator and the numerical radius, and the difference numbers of the powers of the corresponding Hilbert space operators. Firstly, these evaluations are made for the polynomial functions of the linear bounded Hilbert space operator. Later, this topic is generalized for the analytical functions of the linear bounded Hilbert space operator. In the end, two general results are proved.

Keywords: *Analytic functions of operator; Numerical radius; Operator norm.*
2010 Mathematics Subject Classification: *47A10; 47A30; 47A60*

1. Introduction

Let $B(H)$ be the class of linear bounded operators in Hilbert space H. For $A \in B(H)$, the numerical radius of A is defined by

$$\omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|,$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are the inner product and its corresponding norm on H, respectively. It is known that

$$\omega(A) = \sup_{t\in\mathbb{R}} \|Re(e^{it}A)\| = \sup_{t\in\mathbb{R}} \|Im(e^{it}A)\|$$

(see, e.g. [13]). It is well-known that $\omega(A)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, the following inequalities hold for every $A \in B(H)$:

$$\frac{\|A\|}{2} \leq \omega(A) \leq \|A\|. \tag{1.1}$$

The following are well-known facts on the numerical radius:

$$\text{If } A^2 = 0, \text{then } \omega(A) = \frac{\|A\|}{2}$$

and

$$\text{if } A \text{ is normal, then } \omega(A) = \|A\|.$$

We refer the reader to [8], [9] for other basic facts and results on the numerical radius. Furthermore, developments of the numerical radius inequalities (1.1) can be found in [2], [4], [5], [10], [11], [13] and references there in.
It is well known that for every two operators $T, S \in B(H)$ is valid

$$\omega(T+S) \leq \omega(T) + \omega(S) \tag{1.2}$$

(see, e.g. [[7], page 2]). Let $gap_\omega(A) := \|A\| - \omega(A)$, $A \in B(H)$ be the numerical type gap of the operator A.

Email addresses: **ipekpembe@gmail.com (P. Ipek Al), rukiyeozturkmert@hitit.edu.tr (R. Öztürk Mert), zameddin.ismailov@gmail.com (Z. I. Ismailov)**

In [1], some numerical radius inequalities for Hilbert space operators are introduced. In [12], some generalizations and refinements inequalities for the operator norm and numerical radius of the product and sum of Hilbert space operators are established.

The open problem posed by Demuth in 2015 and the works of Kittaneh and his researcher group in this area had an important effect on the formation of the subject discussed in this study (see, [6], [10] and [11]).

This work is organized as follow: In Section 2, some important evaluations for the numerical type gap for polynomial functions of linear bounded operator in Hilbert space via the numerical type gap of powers of given operator are given. In Section 3, this subject for analytic functions of linear bounded Hilbert space operator is been generalized. Also, the obtained results are supported by an application. In Section 4, two general results are proved.

2. Gaps between operator norm and numerical radius for polynomial of operator

In this section, some important evaluations for the numerical type gap for polynomial functions of linear bounded operator in Hilbert space via the numerical type gap of powers of given operator are given.

Let prove the following theorem.

Theorem 2.1. *For any linear bounded operator A in any complex Hilbert space H and any $n \in \mathbb{N}$, the following inequalities hold*

$$n\left(gap_\omega(A) - \Delta_n(A)\right)\omega^{n-1}(A) \le gap_\omega(A^n) \le n\left(gap_\omega(A) + \delta_n(A)\right)\|A\|^{n-1},$$

where

$$\begin{aligned}\Delta_n(A) &= \|A^n\|^{\frac{1}{n}} - \|A\| \le 0,\\ \delta_n(A) &= \omega(A) - \omega^{\frac{1}{n}}(A^n) \ge 0,\ n \ge 1.\end{aligned}$$

Proof. For any $n \ge 1$, from [[7], page 3] it is obtained that

$$\begin{aligned}
gap_\omega(A^n) &= \|A^n\| - \omega(A^n)\\
&= (\|A^n\| - \omega^n(A)) + (\omega^n(A) - \omega(A^n))\\
&\le (\|A\|^n - \omega^n(A)) + \left(\omega^n(A) - \left(\omega^{\frac{1}{n}}(A^n)\right)^n\right)\\
&= (\|A\| - \omega(A))\left(\|A\|^{n-1} + \|A\|^{n-2}\omega(A) + \ldots + \omega^{n-1}(A)\right) + \left(\omega(A) - \omega^{\frac{1}{n}}(A^n)\right)\left(\omega^{n-1}(A) + \omega^{n-2}(A)\omega^{\frac{1}{n}}(A^n) + \ldots + \omega^{\frac{n-1}{n}}(A^n)\right)\\
&\le (\|A\| - \omega(A))\, n\|A\|^{n-1} + \left(\omega(A) - \omega^{\frac{1}{n}}(A^n)\right) n\omega^{n-1}(A)\\
&\le n\left(gap_\omega(A) + \delta_n(A)\right)\|A\|^{n-1},
\end{aligned}$$

where $\delta_n(A) = \omega(A) - \omega^{\frac{1}{n}}(A^n),\ n \ge 1$.
On the other hand, from [[7], page 3] it is implies that

$$\begin{aligned}
\|A^n\| - \omega(A^n) &\ge \|A^n\| - \omega^n(A)\\
&= (\|A\|^n - \omega^n(A)) + (\|A^n\| - \|A\|^n)\\
&= (\|A\| - \omega(A))\left(\|A\|^{n-1} + \|A\|^{n-2}\omega(A) + \ldots + \omega^{n-1}(A)\right) + \left(\|A^n\|^{\frac{1}{n}} - \|A\|\right)\left(\|A^n\|^{\frac{n-1}{n}} + \|A^n\|^{\frac{n-2}{n}}\|A\| + \ldots + \|A\|^{n-1}\right)\\
&\ge (\|A\| - \omega(A))\, n\omega^{n-1}(A) + \left(\|A^n\|^{\frac{1}{n}} - \|A\|\right) n\|A\|^{n-1}\\
&\ge n\left(gap_\omega(A) - \Delta_n(A)\right)\omega^{n-1}(A),
\end{aligned}$$

where, $\Delta_n(A) = \|A^n\|^{\frac{1}{n}} - \|A\|$ for each $n \ge 1$. □

Theorem 2.2. *For any $A \in B(H)$ and polynomial*

$$P(A) = \sum_{k=0}^{n} a_k A^k,\ a_k \in \mathbb{C},\ 0 \le k \le n,$$

the following inequality holds

$$\left| gap_\omega(P(A)) - \sum_{k=0}^{n} |a_k| gap_\omega\left(A^k\right)\right| \le 2 \max_{0 \le k \le n-1} |a_k| \sum_{k=0}^{n-1} \|A^k\|.$$

Proof. From [[7], page 2], we have

$$\begin{aligned}
gap_{\omega}(P(A)) &= \|P(A)\| - \omega(P(A)) \\
&\geq \|P(A)\| - \sum_{k=0}^{n} |a_k| \omega(A^k) \\
&\geq \left(|a_n| \|A^n\| - \sum_{k=0}^{n-1} |a_k| \|A^k\| \right) - \sum_{k=0}^{n} |a_k| \omega(A^k) \\
&= \sum_{k=0}^{n} |a_k| \|A^k\| - \sum_{k=0}^{n} |a_k| \omega(A^k) - 2 \sum_{k=0}^{n-1} |a_k| \|A^k\| \\
&= \left(\sum_{k=0}^{n} |a_k| \left(\|A^k\| - \omega(A^k) \right) \right) - 2 \max_{0 \leq k \leq n-1} |a_k| \left(\sum_{k=0}^{n-1} \|A^k\| \right) \\
&= \sum_{k=0}^{n} |a_k| gap_{\omega}(A^k) - 2 \max_{0 \leq k \leq n-1} |a_k| \left(\sum_{k=0}^{n-1} \|A^k\| \right).
\end{aligned}$$

On the other hand, using the result in [[7], page 2] we get

$$\begin{aligned}
gap_{\omega}(P(A)) &= \|P(A)\| - \omega(P(A)) \\
&\leq \sum_{k=0}^{n} |a_k| \|A^k\| - \left(\omega\left((a_n)(A^n)\right) - \omega\left(\sum_{k=0}^{n-1} a_k A^k \right) \right) \\
&\leq \sum_{k=0}^{n} |a_k| \|A^k\| - |a_n| \omega(A^n) + \sum_{k=0}^{n-1} |a_k| \omega(A^k) \\
&= \sum_{k=0}^{n} |a_k| \left(\|A^k\| - \omega(A^k) \right) + 2 \sum_{k=0}^{n-1} |a_k| \omega(A^k) \\
&= \sum_{k=0}^{n} |a_k| gap_{\omega}(A^k) + 2 \max_{0 \leq k \leq n-1} |a_k| \sum_{k=0}^{n-1} \|A^k\|.
\end{aligned}$$

□

3. Gaps between operator norm and numerical radius for analytic functions of operator

Firstly, note that one important property of the numerical radius.

Lemma 3.1. *For any $A, B \in B(H)$ the following inequality holds*

$$|\omega(A) - \omega(B)| \leq \omega(A \pm B).$$

Proof. From the subaddivity of the numerical radius function, we have

$$\omega(A) = \omega(A \mp B \pm B) \leq \omega(A \mp B) + \omega(B),$$

i.e.,

$$\omega(A) - \omega(B) \leq \omega(A \mp B).$$

Similarly, we get

$$\omega(B) = \omega(B \mp A \pm A) \leq \omega(A \mp B) + \omega(A),$$

that is

$$-(\omega(A) - \omega(B)) \leq \omega(A \mp B).$$

Consequently, it is established that

$$|\omega(A) - \omega(B)| \leq \omega(A \pm B).$$

□

Let A be a linear bounded operator in any complex Hilbert space H. The sets $\rho(A)$ and $\sigma(A)$ are resolvent and spectrum sets of the operator A, respectively.
Suppose that $f(\cdot)$ is an analytic function defined in a neighbourhood of the spectrum $\sigma(A)$ of the operator A. The family of all functions of this type will be denoted by $\mathscr{A} = \mathscr{A}(\sigma(A))$ [3].

Definition 3.2 ([3], page 375)**.** *For $f \in \mathscr{A}$ the operator $f(A) \in B(H)$ it will be denoted by*

$$f(A) = \frac{-1}{2\pi i} \oint_{\Gamma} f(z) R_z(A) dz,$$

where $\Gamma \subset dom(f)$ is a closed contour, composed of finitely many restifable Jordan curves oriented to the positive direction and $R_z(A)$ is a resolvent operator of A in the point $z \in \rho(A)$.

Theorem 3.3. *Let A be linear bounded operator in any complex Hilbert space H. Then, for $f \in \mathscr{A}$ the following inequality holds*

$$gap_{\omega}(f(A)) = \lim_{n\to\infty} gap_{\omega}(P_n(A,a))$$

where, $a \in \rho(A)$ and $P_n(A,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(A-aI)^k$, $n \geq 0$.

Proof. For fixed point $a \in \rho(A)$ and $f \in \mathscr{A}$,

$$f(A) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(A-aI)^k$$

is true [3]. Here, this series is converges on the norm of $B(H)$ [3]. Denote by for any $n = 0,1,2,\dots$

$$P_n = P_n(A,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(A-aI)^k,$$

$$Q_n = Q_n(A,a) = \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(A-aI)^k.$$

Then, from Lemma 3.1 for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
0 \leq gap_{\omega}(f(A)) &= \|f(A)\| - \omega(f(A)) \\
&= \|P_n + Q_n\| - \omega(P_n + Q_n) \\
&\leq \|P_n\| + \|Q_n\| - \omega(P_n + Q_n) \\
&= \|P_n\| + \|Q_n\| - \omega(P_n) + (\omega(P_n) - \omega(P_n + Q_n)) \\
&\leq (\|P_n\| - \omega(P_n)) + \|Q_n\| + (\omega(Q_n) \\
&= gap_{\omega}(P_n) + \|Q_n\| + \omega(Q_n).
\end{aligned}$$

Since the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(A-aI)^k$ is converges in norm of $B(H)$ [3], then the sequences $\|Q_n\| = \|f(A) - P_n\|$ converges to zero as $n \to \infty$. Hence from (1.1) it is implies that $\lim_{n\to\infty} \omega(Q_n) = 0$. From this and the last relation, it is obtained that

$$gap_{\omega}(f(A)) \leq \lim_{n\to\infty} \inf gap_{\omega}(P_n). \tag{3.1}$$

On the other hand, by using the inequality (1.2) for any $A \in B(H)$ we have

$$\begin{aligned}
gap_{\omega}(f(A)) &= \|f(A)\| - \omega(f(A)) \\
&= \|P_n + Q_n\| - \omega(P_n + Q_n) \\
&\geq |\|P_n\| - \|Q_n\|| - \omega(P_n + Q_n) \\
&\geq |\|P_n\| - \|Q_n\|| - (\omega(P_n) + \omega(Q_n)) \\
&\geq (\|P_n\| - \|Q_n\|) - (\omega(P_n) + \omega(Q_n)) \\
&= (\|P_n\| - \omega(P_n)) - \|Q_n\| - \omega(Q_n) \\
&= gap_{\omega}(P_n) - \|Q_n\| - \omega(Q_n).
\end{aligned}$$

Since $\lim_{n\to\infty} \omega(Q_n) = 0$ and (1.1), then from the last relation it is obtained that

$$\lim_{n\to\infty} \sup gap_{\omega}(P_n) \leq gap_{\omega}(f(A)). \tag{3.2}$$

Therefore, from (3.1) and (3.2) we have

$$gap_{\omega}(f(A)) = \lim_{n\to\infty} gap_{\omega}(P_n).$$

□

4. Some general results on operator norm and numerical radius gaps

In this section, we prove two general results.

Theorem 4.1. *For the operator $A \in B(H)$, the following inequality holds*

$$gap_{\omega}(A) \leq 2 \inf_{t\in\mathbb{R}} \|sint Im(e^{it}A)\| + \|ImA\|$$

or

$$gap_{\omega}(A) \leq \inf_{t\in\mathbb{R}} \|Re(((1-sint)+icost)A) + iImA\|.$$

Proof. For the operator $A \in B(H)$, we get

$$\begin{aligned}
gap_{\omega}(A) &= \|A\| - \omega(A) \\
&= \|A\| - \sup_{t\in\mathbb{R}} \|Re(e^{it}A)\| \\
&= \inf_{t\in\mathbb{R}} \left(\|A\| - \|Re(e^{it}A)\|\right) \\
&\leq \inf_{t\in\mathbb{R}} \|A - Re(e^{it}A)\| \\
&= \inf_{t\in\mathbb{R}} \|ReA + iImA - Re\left((cost + isint)(ReA + iImA)\right)\| \\
&= \inf_{t\in\mathbb{R}} \|ReA + iImA - (costReA - sintImA)\| \\
&= \inf_{t\in\mathbb{R}} \|\left((1-cost)ReA + sintImA\right) + iImA\| \\
&= \inf_{t\in\mathbb{R}} \|2sin(t/2)\left(sin(t/2)ReA + cos(t/2)ImA\right) + iImA\| \\
&= \inf_{t\in\mathbb{R}} \|2sin(t/2)Re\left(-ie^{it/2}A\right) + iImA\| \\
&= \inf_{t\in\mathbb{R}} \|2sin(t/2)Im\left(e^{it/2}A\right) + iImA\|.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
gap_{\omega}(A) &= \|A\| - \omega(A) \\
&= \|A\| - \sup_{t\in\mathbb{R}} \|Im(e^{it}A)\| \\
&= \inf_{t\in\mathbb{R}} \left(\|A\| - \|Im(e^{it}A)\|\right) \\
&\leq \inf_{t\in\mathbb{R}} \|A - Im(e^{it}A)\| \\
&= \inf_{t\in\mathbb{R}} \|ReA + iImA - Im\left((cost + isint)(ReA + iImA)\right)\| \\
&= \inf_{t\in\mathbb{R}} \|ReA + iImA - costImA - sintReA\| \\
&= \inf_{t\in\mathbb{R}} \|(1-sint)ReA - costImA + iImA\| \\
&= \inf_{t\in\mathbb{R}} \|Re\left(((1-sint) + icost)A\right) + iImA\|.
\end{aligned}$$

□

Theorem 4.2. *For any $A \in B(H)$, the following inequality holds*

$$gap_{\omega}(A) \leq \frac{1}{2} \inf_{t\in\mathbb{R}} \|\left(2 - e^{it}\right)A\|.$$

Proof. Using the results by Yamazaki [[13], page 84], we have

$$\omega(A) = \sup_{t\in\mathbb{R}} \|Re(e^{it}A)\| = \sup_{t\in\mathbb{R}} \|Im(e^{it}A)\|.$$

Then, we get

$$\begin{aligned}
2\omega(A) &= \sup_{t\in\mathbb{R}} \|Re(e^{it}A)\| + \sup_{t\in\mathbb{R}} \|iIm(e^{it}A)\| \\
&\geq \sup_{t\in\mathbb{R}} \|Re(e^{it}A) + iIm(e^{it}A)\| \\
&= \sup_{t\in\mathbb{R}} \|e^{it}A\|.
\end{aligned}$$

Hence, one can easily check that

$$\begin{aligned}
2gap_{\omega}(A) &= 2\|A\| - 2\omega(A) \\
&\leq 2\|A\| - \sup_{t\in\mathbb{R}} \|e^{it}A\| \\
&= \inf_{t\in\mathbb{R}} \left(2\|A\| - \|e^{it}A\|\right) \\
&\leq \inf_{t\in\mathbb{R}} \|2A - e^{it}A\| \\
&= \inf_{t\in\mathbb{R}} \|\left(2 - e^{it}\right)A\|.
\end{aligned}$$

This completes the proof of theorem. □

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable

References

[1] T. M. W. Alomari, S. Sahoo and M. Bakherad, Further numerical radius inequalities, J. Math. Inequal., 16, 1 (2022), 307-326.
[2] W. Bani-Domi and F. Kittaneh, Refined and generalized numerical radius inequalities for 2×2 operator matrices, Linear Algebra Appl., 624 (2021), 364-386.
[3] Y. M. Berezansky, Z. G. Sheftel and G. H. Us, Functional Analysis, Birkhauser, 1th printing, Berlin, 1996.
[4] P. Bhunia and K. Paul, New upper bounds for the numerical radius of Hilbert space operators, Bull. Sci. Math., 167 (2021), 1-11.
[5] P. Bhunia, K. Paul and R. K. Nayak, Sharp inequalities for the numerical radius of Hilbert space operators and operator matrices, Math. Inequal. Appl., 24 (2021), 167-183.
[6] M. Demuth, Mathematical aspect of physics with non-selfadjoint operators, List of open problem, American Institute of Mathematics Workshop Germany, (8-12 June 2015).
[7] S. S. Dragomir, Inequalities for the Numerical Radius of Linear Operators in Hilbert Space, Springer, 1th printing, Chem, 2013.
[8] K. E. Gustafson and D. K. M. Rao, Numerical Range: The Field Of Values Of Linear Operators And Matrices, Springer, 1th printing, New York, 1997.
[9] P. R. Halmos, A Hilbert Space Problem Book, Van Nostrand, 1th printing, New York, 1967.
[10] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, Studia Math., 158 (2003), 11-17.
[11] F. Kittaneh,Numerical radius inequalities for Hilbert space operators, Studia Math., 168 (2005), 73-80.
[12] M. H. M Rashid and N. H. Altaweel, Some generalized numerical radius inequalities for Hilbert space operators, J. Math. Inequal., 16, 2 (2022), 541-560.
[13] T. Yamazaki, On upper and lower bounds of the numerical radius and equality condition, Studia Math., 178 (2007), 83-89.