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Approximation On Modular Spaces Via P-Statistical Relative $\mathcal{A}$-Summation Process

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Research Article

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Abstract

In this paper, we first present the notions of statistical relative modular and F-norm convergence concerning the power series method. Then, we also present theorems of Korovkin-type via statistical relative $\mathcal{A}$-summation process via power series method on modular spaces, including as particular cases weighted spaces, certain interpolation spaces, Orlicz and Musielak-Orlicz spaces, L_p spaces and many others. Later, we consider some applications to Kantorovich-type operators in Orlicz spaces. Moreover, we present some estimates of rates of convergence via modulus of continuity. We end the paper with giving some concluding remarks.

Keywords: Korovkin theorem, modular space, statistical convergence, matrix summability, power series method

Modüler Uzaylar Üzerinde P-İstatistiksel $\mathcal{A}$-Toplam Süreci Aracılığıyla Yaklaşım

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Öz

Bu çalışmada, ilk olarak, kuvvet serisi yöntemiyle ilgili istatistiksel relative modüler ve F-norm yakınsama kavramlarını sunuyoruz. Daha sonra, özel durumlar olarak ağırlıklı uzaylar, belirli enterpolasyon uzayları, Orlicz ve Musielak-Orlicz uzayları, L_p uzayları ve diğer bir çok uzayları içeren modüler uzaylar üzerinde kuvvet serisi yöntemiyle istatistiksel relative $\mathcal{A}$-toplam süreci aracılığıyla Korovkin-tipi teoremleri de sunuyoruz. Daha sonra, Orlicz uzaylarında Kantorovich tipi operatörlere bazı uygulamaları göz önüne alıyoruz. Dahası, süreklilik modülü aracılığıyla yakınsama oranlarının bazı tahminlerini sunuyoruz. Makaleyi bazı son sözler vererek bitiriyoruz.

Anahtar Kelimeler: Korovkin teoremi, modüler uzay, istatistiksel yakınsaklık, matris toplanabilme, kuvvet serisi yöntemi

Introduction and Preliminaries Notations

We begin this section with the notions of statistical convergence and the power series statistical convergence a sequence z :

First let $\mathbb{N}$ be the set of natural numbers. Next let, $E \subseteq \mathbb{N}$, then *the natural density of* E, indicated by the symbol $\delta(E)$, is determined by:

$$\delta(E) := \lim_n \frac{\sharp\{k \leq n \ : k \in E\}}{n}$$

whenever the limit exists, where $\sharp\{.\}$ denotes the cardinality of the set [1].
A sequence $z = \{z_k\}$ of numbers is *statistically convergent* to κ iff, for every $\varepsilon > 0$,

$$\lim_n \frac{\sharp\{k \leq n : |z_k - \kappa| \geq \varepsilon\}}{n} = 0$$

that is, $E_n(\varepsilon) := \{k \leq n : |z_k - \kappa| \geq \varepsilon\}$ has natural density zero. This is denoted by $st - \lim_k z_k = \kappa$ [2, 3]. It is evident from the definition that a convergent sequence (in the usual sense) is statistically convergent to the same value. For all that a statistically convergent sequence need not be convergent in light of the above.
Another interesting convergence method is P-statistical convergence, which recently introduced by Ünver and Orhan [4] and is a different type of statistical convergence given via power series methods. They gave striking examples to demonstrate the incompatibility between statistical convergence and P-statistical convergence. The Abel and Borel methods are two well-known power series methods that are more effective than ordinary convergence.
Let $\{p_k\}$ be a positive real sequence such that $p_0 > 0$ and the associated power series

$$p(t) := \sum_{k=0}^{\infty} p_k t^k$$

has radius of convergence $\mathcal{R}$ with $0 < \mathcal{R} \leq \infty$. A sequence $z = \{z_k\}$ is *convergent in the sense of power series method* provided that
$\lim_{0<t\to\mathcal{R}^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k z_k = \kappa$ ([5,6]).
Keep in mind that the method is regular iff $\lim_{0<t\to\mathcal{R}^-} \frac{p_k t^k}{p(t)} = 0$ for every k (see, e.g. [7]).

Remark 1. We should note the special cases of power series methods. In case of $\mathcal{R} = 1$, if $p_k = 1$ and $p_k = \frac{1}{k+1}$, the power series methods give us Abel method and logarithmic method, respectively. Also, if $\mathcal{R} = \infty$ and $p_k = \frac{1}{k!}$, then the power series method gives us Borel method.

The rest of the paper, we assume that the method of power series is regular.
Now, we recall the next definitions:

Definition 1. [4] Let $E \subset \mathbb{N}_0$. If the limit

$$\delta_P(E) := \lim_{0<t\to\mathcal{R}^-} \frac{1}{p(t)} \sum_{k\in E} p_k t^k$$

exists, then $\delta_P(E)$ is called the P-density of E.

It is evident from the definition of a power series method and P-density that $0 \leq \delta_P(E) \leq 1$ whenever it exists.

In a manner similar to the natural density, we can give the following properties for the P-density:
$i)$ $\delta_P(\mathbb{N}_0) = 1,$
$ii)$ if $E \subset F$ then $\delta_P(E) \leq \delta_P(F),$
$iii)$ if E has P-density then $\delta_P(\mathbb{N}_0 \backslash E) = 1 - \delta_P(E).$

Definition 2. [4] Let $z = \{z_k\}$ be a sequence. Then z is said to be P-statistically convergent, i.e., statistically convergent with respect to power series method to κ provided that for any $\varepsilon > 0$

$$\lim_{0<t\to\mathcal{R}^-} \frac{1}{p(t)} \sum_{k\in E_\varepsilon} p_k t^k = 0$$

that is $\delta_P(E_\varepsilon) = 0$ for any $\varepsilon > 0$ where $E_\varepsilon = \{k \in \mathbb{N}_0 : |z_k - \kappa| \geq \varepsilon\}$. This is denoted by $st_P - \lim z_k = \kappa$.

Definition 3. A sequence of real numbers $z = \{z_k\}$ is said to be P-statistically bounded if for some $H > 0$ such that $\delta_P(\{k : |z_k| > H\}) = 0$.

The terms statistical limit superior and statistical limit inferior have been given by Fridy and Orhan [8]. Then, Demirci [9] has generalized these ideas to ideal limit superior and limit inferior. We can now introduce the ideas of P-statistical limit superior and P-statistical limit inferior in light of these investigations. The *P-statistical limit superior* of a number sequence $z = \{z_k\}$, we write $st_P - \limsup z_k$, is defined by

$$st_P - \limsup z_k = \begin{cases} \sup B_z, & \text{if } B_z \neq \phi, \\ -\infty, & \text{if } B_z = \phi, \end{cases}$$

where $B_z := \{b \in \mathbb{R} : \delta_P(\{k : z_k > b\}) > 0 \text{ or does not exist in } \mathbb{R}\}$ and the symbol ϕ represents the empty set. Similarly, the *P-statistical limit inferior* of $\{z_k\}$, we write $st_P - \liminf z_k$, is defined by

$$st_P - \liminf z_k = \begin{cases} \inf C_z, & \text{if } C_z \neq \phi, \\ +\infty, & \text{if } C_z = \phi, \end{cases}$$

where $C_z := \{c \in \mathbb{R} : \delta_P(\{k : z_k < c\}) > 0 \text{ or does not exist in } \mathbb{R}\}$. It is discovered that, similar to the statistical superior or inferior limit $st_P - \liminf z_k \leq st_P - \limsup z_k$ and notice that, for any sequence $z = \{z_k\}$ satisfying $\delta_P(\{k : |z_k| > H\}) = 0$ for some $H > 0$, $st_P - \lim z_k = \kappa$ iff $st_P - \liminf z_k = st_P - \limsup z_k = \kappa$.
Let us briefly recall some simple, basic facts about modular spaces which will be used in our main section.
Let $I = [c, d]$ be the Lebesgue-measured bounded interval of the real line $\mathbb{R}$. The space of all real-valued measurable functions on I that are guaranteed to have equality a.e. will then be denoted by the symbol $X(I)$. A functional $\rho : X(I) \to [0, +\infty]$ is said to be a modular on $X(I)$ if $\rho(f) = 0$ if and only if $f = 0$ *a.e.* in I, $\rho(-f) = \rho(f)$ for every $f \in X(I)$, $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $f, g \in X(I)$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.
If there is a constant $\mathcal{N} \geq 1$ such that $\rho(af) \leq \mathcal{N} a \rho(\mathcal{N} f)$ holds for any $f \in X(I)$ and $a \in (0, 1]$, then a modular ρ is said to be $\mathcal{N}$-quasi semiconvex. A modular ρ is *$\mathcal{N}$-quasi convex* if there exists a constant

$\mathcal{N} \geq 1$ such that $\rho(\alpha f+\beta g) \leq \mathcal{N}\alpha\rho(\mathcal{N}f)+\mathcal{N}\beta\rho(\mathcal{N}g)$ holds for every $f, g \in X(I)$, $\alpha, \beta \geq 0$ with $\alpha+\beta=1$. If $\mathcal{N}=1$, then ρ is called *convex*.

An $\mathcal{N}$-quasi convex modular is obviously an $\mathcal{N}$-quasi semiconvex modular. It is worth noting that Bardaro et. al. [10] introduced and detailed the above two ideas.

By means of a modular ρ, we recall the vector subspace of $X(I)$, denoted by $L^{\rho}(I)$ and it is called the modular space:

$$L^{\rho}(I) := \left\{ f \in X(I) : \lim_{\lambda \to 0^{+}} \rho(\lambda f) = 0 \right\}$$

and the subspace of $L^{\rho}(I)$ defined by

$$E^{\rho}(I) := \{ f \in L^{\rho}(I) : \rho(\lambda f) < +\infty \text{ for all } \lambda > 0 \}.$$

It is easy to verify that if ρ is $\mathcal{N}$-quasi semiconvex, then we have the following characterization of the modular space $L^{\rho}(I)$:

$$L^{\rho}(I) = \{ f \in X(I) : \rho(\lambda f) < +\infty \text{ for some } \lambda > 0 \}.$$

For the notions about modulars, we refer the readers to [10, 11] (Also, see [12, 13]).

Now we deal with the convergence notions in modular spaces.

Let a function sequence $\{f_k\}$ be in $L^{\rho}(I)$. Then, $\{f_k\}$ is said to be *modularly convergent* to a function $f \in L^{\rho}(I)$ iff

$$\lim_{k} \rho(\lambda_0(f_k - f)) = 0, \text{ for some } \lambda_0 > 0. \tag{1}$$

Also, if (1) is provided for all $\lambda > 0$, then it is called *F-norm convergent* (or, *strongly convergent*), i.e.,

$$\lim_{k} \rho(\lambda(f_k - f)) = 0, \text{ for every } \lambda > 0. \tag{2}$$

It is well known from Musielak [13] that (1) and (2) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, that is to say, $\exists H > 0$ such that $\rho(2f) \leq H\rho(f)$ for every $f \in X(I)$.

We also recall the definitions of relative modular and strong convergence given by Yılmaz et al. [14]:

Let a function sequence $\{f_k\}$ be in $L^{\rho}(I)$. Then, $\{f_k\}$ is said to be *relatively modularly convergent* to a function $f \in L^{\rho}(I)$ if there exists a function $\sigma(z)$, called a scale function $\sigma \in X(I)$, $|\sigma(z)| \neq 0$ such that

$$\lim_{k} \rho\left(\lambda_0\left(\frac{f_k - f}{\sigma}\right)\right) = 0, \text{ for some } \lambda_0 > 0.$$

Also, $\{f_k\}$ is F-norm convergent (or, strongly convergent) to f iff

$$\lim_{k} \rho\left(\lambda\left(\frac{f_k - f}{\sigma}\right)\right) = 0, \text{ for every } \lambda > 0.$$

Notice that modular convergence is a particular case of relative modular convergence where the scale

function is a constant that is not zero.
The majority of classical operators have a tendency to approximate the function's value, although at discontinuous points they typically converge to the average of the function's left and right limits. As previously mentioned, the fundamental tools for resolving the lack of convergence are the matrix summability methods or, more generally, $\mathcal{A}$-summation methods.
Let $\mathcal{A} := \left\{A^{(j)}\right\} = \left\{\left(a_{nk}^{(j)}\right)\right\}$ be a sequence of infinite non-negative real matrices. For a sequence of real numbers, $z = \{z_k\}$, the double sequence $\mathcal{A}z := \left\{(Az)_n^j : n, j \in \mathbb{N}\right\}$ defined by $(Az)_n^j := \sum_{k=1}^{\infty} a_{nk}^{(j)} z_k$ is referred to as the $\mathcal{A}$-transform of z whenever the series converges for all n and j. A sequence z is said to be $\mathcal{A}$-summable to κ if

$$\lim_n \sum_{k=1}^{\infty} a_{nk}^{(j)} z_k = \kappa$$

uniformly in j [15, 16]. If $A^{(j)} = \mathcal{B}$ for some matrix $\mathcal{B}$ then, we get the ordinary matrix summability.
Orhan and Demirci [17] presented $\mathcal{A}$-summation process and then Kolay et al. [18] presented relative $\mathcal{A}$-summation process on a modular space as follows:
First let ρ be a monotone and finite modular on $X(I)$. Then, assume that D is a set satisfying $C^{\infty}(I) \subset D \subset L^{\rho}(I)$. Next let $\mathbb{T} := \{T_j\}$ be a sequence of positive linear operators from D into $X(I)$, more precisely there holds, for all $n, j \in \mathbb{N}$, $f \in D$ the series

$$A_{n,j}^{\mathbb{T}}(f) := \sum_{k=1}^{\infty} a_{nk}^{(j)} T_k f \tag{3}$$

is absolutely convergent almost everywhere with respect to Lebesgue measure.
A sequence $\mathbb{T} := \{T_k\}$ of positive linear operators of D into $X(I)$ constitutes a relative $\mathcal{A}$-summation process on D if $\{T_k f\}$ is relatively $\mathcal{A}$-summable to f (with respect to modular ρ) for every $f \in D$, i.e.,

$$\lim_n \rho\left[\lambda\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right] = 0, \text{ uniformly in } j. \tag{4}$$

Note that the scale function in an $\mathcal{A}$-summation process is a non-zero constant, making it a special case of a relative $\mathcal{A}$-summation process [18].
Statistical type convergences in the sense of the power series method in modular spaces are now introduced:

Definition 4. Let $\{f_k\}$ be a function sequence whose terms belong to $L^{\rho}(I)$. Then, $\{f_k\}$ is said to be *P-statistically relatively modularly convergent* to a function $f \in L^{\rho}(I)$ iff there exists a scale function $\sigma \in X(I)$, $|\sigma(z)| \neq 0$ such that

$$st_P - \lim \rho\left(\lambda_0\left(\frac{f_k - f}{\sigma}\right)\right) = 0, \text{ for some } \lambda_0 > 0. \tag{5}$$

In addition, if (5) is provided for all $\lambda > 0$, then it is called *P-statistical relative F-norm convergence* (or, P-*statistical relative strong convergence*), i.e.,

$$st_P - \lim \rho\left(\lambda\left(\frac{f_k - f}{\sigma}\right)\right) = 0, \quad \text{for every } \lambda > 0. \tag{6}$$

It can be immediately seen that (5) and (6) are equivalent if and only if the modular ρ satisfies the Δ_2-condition. Indeed, P-statistical relative strong convergence of the sequence $\{f_k\}$ to f is equivalent to the condition $st_P - \lim \rho\left(2^N\lambda\left(\frac{f_k-f}{\sigma}\right)\right) = 0$, for all $N = 1, 2, ...$ and some $\lambda > 0$. Let $\{f_k\}$ be P-statistically relatively modularly convergent to f, hence, there exists a $\lambda > 0$ such that $st_P - \lim \rho\left(\lambda\left(\frac{f_k-f}{\sigma}\right)\right) = 0$. Also, Δ_2-condition implies by induction that $\rho\left(2^N\lambda\left(\frac{f_k-f}{\sigma}\right)\right) \leq H^N\rho\left(\lambda\left(\frac{f_{k,}-f}{\sigma}\right)\right)$. Then, we get

$$st_P - \lim \rho\left(2^N\lambda\left(\frac{f_k - f}{\sigma}\right)\right) = 0.$$

It is also important to keep in mind that P-statistical relative modular convergence is a special case of P-statistical modular convergence where the scale function is a non-zero constant.
Here, we provide a striking example of how our new definitions are more precise.

Example 1. Take $I = [0, 1]$ and with the following requirements, let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function:

- φ is convex,
- $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$.

Therefore, take into consideration the functional ρ^φ on $X(I)$ defined by

$$\rho^\varphi(f) := \int_0^1 \varphi(|f(z)|)\, dz \quad \text{for } f \in X(I).$$

In this case, all of the assumptions listed in this section are satisfied since ρ^φ is a convex modular on $X(I)$.Take into account the Orlicz space generated by φ by the following:

$$L^\rho_\varphi(I) := \{f \in X(I) : \rho^\varphi(\lambda f) < +\infty \quad \text{for some } \lambda > 0\}.$$

For each $k \in \mathbb{N}$ $(k \geq 2)$, define $h_k : I \to \mathbb{R}$ by

$$h_k(z) = \begin{cases} k, & k = 2n, \\ k^2 z, & 0 \leq z \leq \frac{1}{k},\ k = 2n+1, \\ 2k - k^2 z, & \frac{1}{k} < z \leq \frac{2}{k},\ k = 2n+1, \\ 0, & \frac{2}{k} < z \leq 1,\ k = 2n+1, \end{cases} \quad n = 1, 2, ... \tag{7}$$

If $\varphi(z) = z^p$ for $1 \leq p < \infty$, $z \geq 0$, then $L^\rho_\varphi(I) = L_p(I)$. Moreover, we have for any function $f \in L^\rho_\varphi(I)$ that $\rho^\varphi(f) = \|f\|^p_{L_p}$. Additionally, let's say that the power series method is provided with

$$p_k = \begin{cases} 0, & k = 2n, \\ 1, & \text{otherwise}, \end{cases} \quad n = 1, 2, ...$$

It is evident that $\{h_k\}$ does not converge either statistically modularly or modularly. Additionally,$\{h_k\}$ converges P-statistically modularly relative to a scale function σ rather than P -statistically modularly to $h = 0$ where

$$\sigma(z) = \begin{cases} 1, & z = 0 \\ \frac{1}{z^2}, & z \in (0,1] \end{cases}$$

on $L_1(I)$. Indeed, for some $\lambda_0 > 0$, with the choice of $p = 1$ we have $\rho^{\varphi}(.) = \|.\|_{L_1}$,

$$\rho(\lambda_0(h_k - h)) = \|\lambda_0(h_k - h)\|_{L_1} = \lambda_0 \begin{cases} k, & k = 2n, \\ 1, & k = 2n+1, \end{cases} \quad n = 1, 2, ... \tag{8}$$

As a result, it is simple to prove that $\{h_k\}$ does not converge statistically or P-statistically. Using the scale function σ, let's now

$$\rho\left(\lambda_0\left(\frac{h_k - h}{\sigma}\right)\right) = \left\|\lambda_0\left(\frac{h_k - h}{\sigma}\right)\right\|_{L_1} = \lambda_0 \begin{cases} \frac{k}{3}, & k = 2n, \\ \\ \frac{7}{6k^2}, & k = 2n+1, \end{cases} \quad n = 1, 2, ...$$

then, we get

$$st_P - \lim \rho\left(\lambda_0\left(\frac{h_k - h}{\sigma}\right)\right) = 0. \tag{9}$$

On the other hand, referring to (8), we see that the sequence $\{h_k\}$ neither converges to $h = 0$ statistically modularly nor P-statistically modularly on $L_1(I)$.

We conclude this section with the following assumptions:

- A modular ρ is said to be *finite* if $\chi_A \in L^\rho(I)$ when A is measurable subset of I such that $\mu(A) < \infty$,
- If a modular ρ is finite and, for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho(\lambda\chi_B) < \varepsilon$ for any measurable subset $B \subset I$ with $\mu(B) < \delta$, then ρ is *absolutely finite* and if $\chi_I \in E^\rho(I)$, then ρ is *strongly finite,*
- A modular ρ is *monotone* if $\rho(f) \leq \rho(g)$ for $|f| \leq |g|$,
- A modular ρ is *absolutely continuous* provided that there exists an $\alpha > 0$ such that, for every $f \in X(I)$ with $\rho(f) < +\infty$, the following condition holds: for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f \chi_B) < \varepsilon$ whenever B is any measurable subset of I with $\mu(B) < \delta$.

Let $C(I)$ stands for the space of all continuous real-valued functions, as usual. We have $C(I) \subset L^\rho(I)$ if a modular ρ is monotone and finite (see [21]). Also, if ρ is monotone and strongly finite, then $C(I) \subset E^\rho(I)$. Let $C^\infty(I)$ denote the space of all infinitely differentiable functions on I. If ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^\infty(I)} = L^\rho(I)$ with respect to the modular convergence in the ordinary sense (see [10, 11, 22, 23]).

In this study, we analyze the statistical relative $\mathcal{A}$-summation process's approximation of the Korovkin-type with respect to the power series method on modular spaces. Then, in Orlicz spaces, we apply some Kantorovich-type operators. We also provide some estimates of convergence rates obtained from moduli of continuity. We provide some final remarks at the end of the study.

P-Statistical Korovkin Theorems in Modular Spaces

The Korovkin-type approximation [24,25], which is undoubtably one of the most elegant and fascinating result, has emerged as a wide field of study in recent years. There is a wide variety of Korovkin type results in the approximation (see e.g. [26–41]). Korovkin-type approximation first obtained by Bardaro and Mantellini [21] in the abstract setting of the modular spaces, a class of function spaces which includes weighted spaces, certain interpolation spaces, Orlicz and Musielak–Orlicz spaces, L_p spaces. Especially, using the concept of statistical convergence, Karakuş, Demirci and Duman [42] have introduced the Korovkin theorems on modular space and also, using matrix summability methods Karakuş and Demirci proved Korovkin theorems in [43]. In the absence of convergence, it is also become a powerfull tool to use the matrix summability methods or more generally, summation methods. It is the reason why Nishishiraho introduced and studied the notion of $\mathcal{A}$-summation process on a compact Hausdorff space [44, 45]. Recently, Kolay et al. have studied an approximation theorem of statistical type via relative $\mathcal{A}$-summation process in modular function space in [18].
The key findings of a P-statistical relative $\mathcal{A}$-summation process on a modular space are presented in this part in order to provide the Korovkin type approximation theorem for a sequence of positive linear operators. The major difference from the classical line is that convergence method which is used.
Let ρ be a monotone and finite modular on $X(I)$. Assume that D is a set satisfying $C^{\infty}(I) \subset D \subset L^{\rho}(I)$. Assume further that $\mathbb{T} := \{T_j\}$ is a sequence of positive linear operators from D into $X(I)$ for which there exists a subset $X_{\mathbb{T}} \subset D$ with $C^{\infty}(I) \subset X_{\mathbb{T}}$ and an unbounded function $\sigma \in X(I)$ satisfying $\sigma(z) \neq 0$ such that

$$st_P - \limsup \rho\left(\lambda\left(\frac{A_{n,j}^{\mathbb{T}}(h)}{\sigma}\right)\right) \leq R\rho(\lambda h), \text{ uniformly in } j, \tag{10}$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and with a constant $R > 0$ where $A_{n,j}^{\mathbb{T}}(h) = \sum_{k=1}^{\infty} a_{nk}^{(j)} T_k h$ as in (3).
The test functions e_i are used throughout the entire paper defined by $e_i(z) = z^i \quad (i = 0, 1, 2)$.

Theorem 1. Let $\mathcal{A} = \{A^{(j)}\}$ be a sequence of infinite non-negative real matrices and let ρ be an absolutely continuous, monotone, $\mathcal{N}$-quasi semiconvex and strongly finite modular on $X(I)$. Let $\mathbb{T} := \{T_k\}$ be a sequence of positive linear operators from D into $X(I)$ satisfying (10) holds for each $f \in D$. Moreover, suppose that σ_i is an unbounded function satisfying $|\sigma_i(z)| \geq b_i > 0$ $(i = 0, 1, 2)$. Suppose that

$$st_P - \lim \rho\left(\lambda\left(\frac{A_{n,j}^{\mathbb{T}}(e_i) - e_i}{\sigma_i}\right)\right) = 0, \text{ uniformly in } j, \tag{11}$$

for every $\lambda > 0$ and $i = 0, 1, 2$. Now let f be any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$

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for every $g \in C^{\infty}(I)$. Then we get

$$st_P - \lim \rho\left(\lambda_0\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) = 0, \text{ uniformly in } j, \tag{12}$$

for some $\lambda_0 > 0$ where $\sigma(z) = \max\{|\sigma_i(z)|; i = 0, 1, 2\}$.

Proof. As an auxiliary result, we first prove that

$$st_P - \lim \rho\left(\eta\left(\frac{A_{n,j}^{\mathbb{T}}(g) - g}{\sigma}\right)\right) = 0, \text{ uniformly in } j, \tag{13}$$

for every $g \in C(I) \cap D$ and $\eta > 0$. Assume that g is a member of $C(I) \cap D$ and η is any positive number. Thanks to continuity of g on I and from the linearity and positivity of the operators T_k, it is readily seen that (see, for instance [26, 46]), for a given $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|g(u) - g(z)| < \varepsilon + \frac{2H}{\delta^2}(u - z)^2$ where $H = \sup_{z \in I} |f(z)|$. Assume that $0 < \varepsilon \leq 1$. As in the proof of Theorem 1 in [18], because of ρ is $\mathcal{N}$-quasi semiconvex and strongly finite, we can see that

$$\begin{aligned}\rho\left(\eta\left(\frac{A_{n,j}^{\mathbb{T}}(g) - g}{\sigma}\right)\right) \leq \mathcal{N}\varepsilon\rho\left(\frac{4\eta\mathcal{N}}{\sigma}\right) &+ \rho\left(4\eta K\left(\frac{A_{n,j}^{\mathbb{T}}(e_0) - e_0}{\sigma_0}\right)\right)\\ &+ \rho\left(4\eta K\left(\frac{A_{n,j}^{\mathbb{T}}(e_1) - e_1}{\sigma_1}\right)\right)\\ &+ \rho\left(4\eta K\left(\frac{A_{n,j}^{\mathbb{T}}(e_2) - e_2}{\sigma_2}\right)\right)\end{aligned}$$

where $K := \max\left\{\varepsilon + H + \frac{2Hs^2}{\delta^2}, \frac{4Hs}{\delta^2}, \frac{2H}{\delta^2}\right\}$, $s := \max|z|$. For a given $r > 0$, choose an $\varepsilon \in (0, 1]$ such that $\mathcal{N}\varepsilon\rho\left(\frac{4\eta\mathcal{N}}{\sigma}\right) < r$. Now we define the following sets:

$$G_\eta := \left\{n : \rho\left(\eta\left(\frac{A_{n,j}^{\mathbb{T}}(g) - g}{\sigma}\right)\right) \geq r\right\},$$

$$G_{\eta,i} := \left\{n : \rho\left(4\eta K\left(\frac{A_{n,j}^{\mathbb{T}}(e_i) - e_i}{\sigma_i}\right)\right) \geq \frac{r - N\varepsilon\rho\left(\frac{4\eta\mathcal{N}}{\sigma}\right)}{3}\right\}, i = 0, 1, 2.$$

Then, notice that $G_\eta \subseteq \bigcup_{i=0}^{2} G_{\eta,i}$. Hence, this allowed us to write

$$\delta_P(G_\eta) \leq \sum_{i=0}^{2} \delta_P(G_{\eta,i}).$$

It is evident from hypothesis (11) that

$$\delta_P(G_\eta) = 0,$$

which gives our claim (13). Observe that (13) also holds for every $g \in C^{\infty}(I)$ since $C^{\infty}(I) \subset C(I) \cap D$. Now let $f \in L^{\rho}(I)$ satisfying $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$. Since $\mu(I) < \infty$ and ρ is strongly finite and absolutely continuous, it is a simple matter to see that ρ is also absolutely finite on $X(I)$ (see [22]). Using these properties of the modular ρ, it is known from [10, 23] that the space $C^{\infty}(I)$ is modularly dense in $L^{\rho}(I)$, i.e., there exists a sequence $\{g_n\} \subset C^{\infty}(I)$ such that $\lim_n \rho(3\lambda_0^*(g_n - f)) = 0$ for some $\lambda_0^* > 0$. This means that, for every $\varepsilon > 0$, there is a positive number $n_0 = n_0(\varepsilon)$ so that

$$\rho(3\lambda_0^*(g_n - f)) < \varepsilon \quad \text{for every } n \geq n_0. \tag{14}$$

However, utilizing a well-known process and the positivity and linearity of the operators T_k, we may write that,

$$\begin{aligned}\lambda_0^* \left| \sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(f; z) - f(z) \right| \leq & \lambda_0^* \left| \sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(f - g_{n_0}; z) \right| \\ & + \lambda_0^* \left| \sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(g_{n_0}; z) - g_{n_0}(z) \right| + \lambda_0^* \left| g_{n_0}(z) - f(z) \right|,\end{aligned}$$

holds for every $z \in I$ and $j \in \mathbb{N}$. Now, by multiplying both sides of the above inequality by $\frac{1}{|\sigma(z)|}$, using the modular ρ, and moreover considering the monotonicity of ρ, we get

$$\begin{aligned}\rho\left(\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) \leq & \rho\left(3\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f - g_{n_0})}{\sigma}\right)\right) \\ & + \rho\left(3\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(g_{n_0}) - g_{n_0}}{\sigma}\right)\right) + \rho\left(3\lambda_0^*\left(\frac{g_{n_0} - f}{\sigma}\right)\right).\end{aligned}$$

Hence observing that $|\sigma| \geq b > 0$ $(b = \max\{b_i : i = 0, 1, 2\})$ and taking into consideration that ρ is monotone, we can write that

$$\begin{aligned}\rho\left(\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) \leq & \rho\left(3\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f - g_{n_0})}{\sigma}\right)\right) \\ & + \rho\left(3\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(g_{n_0}) - g_{n_0}}{\sigma}\right)\right) + \rho\left(\frac{3\lambda_0^*}{b}(g_{n_0} - f)\right).\end{aligned} \tag{15}$$

Then, in view of (14) and (15), we have

$$\rho\left(\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) \leq \varepsilon + \rho\left(3\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f - g_{n_0})}{\sigma}\right)\right) + \rho\left(3\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(g_{n_0}) - g_{n_0}}{\sigma}\right)\right). \tag{16}$$

Consequently, assuming that the P-statistical limit superior on both sides of the (16) and noting that

$g_{n_0} \in C^{\infty}(I)$ and $f - g_{n_0} \in X_{\mathbb{T}}$, we obtained from (10) that

$$st_P - \limsup \rho\left(\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) \leq \varepsilon + R\rho\left(3\lambda_0^*(f - g_{n_0})\right) + st_P - \limsup \rho\left(3\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(g_{n_0}) - g_{n_0}}{\sigma}\right)\right).$$

Referring to (14) in above inequality, we find that

$$st_P - \limsup \rho\left(\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) \leq \varepsilon(R+1) + st_P - \limsup \rho\left(3\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(g_{n_0}) - g_{n_0}}{\sigma}\right)\right). \quad (17)$$

Thanks to (13), since $st_P - \lim \rho\left(3\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(g_{n_0}) - g_{n_0}}{\sigma}\right)\right) = 0$, uniformly in j, we get

$$st_P - \limsup \rho\left(3\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(g_{n_0}) - g_{n_0}}{\sigma}\right)\right) = 0, \text{ uniformly in } j. \quad (18)$$

When (17) and (18) are combined, we see that

$$st_P - \limsup \rho\left(\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) \leq \varepsilon(R+1).$$

From arbitrariety of $\varepsilon > 0$, we find $st_P - \limsup \rho\left(\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) = 0$ uniformly in j. Furthermore, since $\rho\left(\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right)$ is non-negative for all $n, j \in \mathbb{N}$, we can easily show that

$$st_P - \lim \rho\left(\lambda_0^*\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) = 0, \text{ uniformly in } j,$$

this completes the proof.

Remark 2. We can establish an identical result using a similar method in the space $C_{2\pi}(\mathbb{R})$ of all continuous real-valued functions on $\mathbb{R}$ of period 2π.

The following conclusion can be derived from Theorem 1 in a direct manner if the modular ρ satisfies the Δ_2-condition.

Theorem 2. Let $\mathcal{A} = \left\{A^{(j)}\right\}$ be a sequence of infinite non-negative real matrices and $\mathbb{T} := \{T_k\}$, ρ and σ be the same as in Theorem 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:

(a) $st_P - \lim \rho\left(\lambda\left(\frac{A_{n,j}^{\mathbb{T}}(e_i) - e_i}{\sigma_i}\right)\right) = 0$ uniformly in j, for every $\lambda > 0$ and $i = 0, 1, 2,$

(b) $st_P - \lim \rho\left(\lambda\left(\frac{A^{\mathbb{T}}_{n,j}(f)-f}{\sigma}\right)\right) = 0$ uniformly in j, for every $\lambda > 0$ provided that f is any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$.

The condition (10) reduces to the following if the identity matrix is used in place of the matrices $A^{(j)}$ and the scale function is assumed to be a non-zero constant

$$st_P - \limsup \rho\left(\lambda\left(T_k h\right)\right) \leq R\rho\left(\lambda h\right) \tag{19}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R. In this case, we obtain the next results from our main theorems, Theorem 1 and Theorem 2.

Corollary 1. Let ρ be an absolutely continuous, monotone, $\mathcal{N}$-quasi semiconvex and strongly finite modular on $X(I)$. Let $\mathbb{T} := \{T_k\}$ be a sequence of positive linear operators satisfying (19) from D into $X(I)$. If $\{T_k e_i\}$ is P-statistically strongly convergent to e_i for each $i = 0, 1, 2$, then $\{T_k f\}$ is P-statistically modularly convergent to f provided that f is any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$.

Corollary 2. $\mathbb{T} := \{T_k\}$ and ρ be the same as in Corollary 1. The following statements are equivalent if ρ satisfies the Δ_2-condition:

(a) $\{T_k e_i\}$ is P-statistically strongly convergent to e_i for each $i = 0, 1, 2$,

(b) $\{T_k f\}$ is P-statistically strongly convergent to f provided that f is any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$.

An Application

The example in this section demonstrates how our approximation of Korovkin-type results in modular spaces are more powerful than those previously examined.

Example 2. Take $I = [0, 1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function such that φ is a convex function and $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u \to \infty} \varphi(u) = \infty$. Hence, consider the functional ρ^{φ} *on* $X(I)$ defined by

$$\rho^{\varphi}(f) := \int_0^1 \varphi\left(\left|f(z)\right|\right) dz \quad \text{for } f \in X(I). \tag{20}$$

In this case ρ^{φ} is a convex modular on $X(I)$. Consider the Orlicz space created by φ as follows:

$$L^{\rho}_{\varphi}(I) := \{f \in X(I) : \rho^{\varphi}(\lambda f) < +\infty \quad \text{for some } \lambda > 0\}.$$

Then take into account the following classic Bernstein-Kantorovich operator $\mathbf{U} := \{U_k\}$ on the space $L^{\rho}_{\varphi}(I)$ (see [21]) which is defined by:

$$U_k(f; z) := \sum_{n=0}^{k} \binom{k}{n} z^n (1-z)^{k-n} (k+1) \int_{n/(k+1)}^{(n+1)/(k+1)} f(u)\, du \ \text{ for } z \in I.$$

Keep in mind that the operators U_k map the Orlicz space $L_\varphi^\rho(I)$ into itself. Moreover, the property (19) is satisfied with the choice of $X_{\mathbf{U}} := L_\varphi^\rho(I)$. Then, it can be easily seen that, for any function $f \in L_\varphi^\rho(I)$ such that $f - g \in X_{\mathbf{U}}$ for every $g \in C^\infty(I)$, $\{U_k f\}$ is modularly convergent to f. Due to the regularity of the method, $\{U_k f\}$ is P-statistically modularly convergent to f.

If $\varphi(z) = z^p$ for $1 \le p < \infty$, $z \ge 0$, then $L_\varphi^\rho(I) = L_p(I)$. Moreover we have $\rho^\varphi(.) = \|.\|_{L_p}^p$. Also assume that $\mathcal{A} := \{A^{(j)}\} = \left\{\left(a_{nk}^{(j)}\right)\right\}$ is a sequence of infinite matrices defined by $a_{nk}^{(j)} = \frac{1}{n+1}$, for $j \le k \le n+j$, $(j = 1, 2, ...)$, and $a_{nk}^{(j)} = 0$ otherwise. It is well-known that, $\mathcal{A}$-summability reduces to almost convergence ([20]). Then, using the operators U_k, we define the sequence of positive linear operators $\mathbb{V} := \{V_k\}$ *on* $L_1([0,1])$ as follows:

$$V_k(f;z) = (1 + h_k(z))\, U_k(f;z) \quad \text{for } f \in L_1(I),\ z \in [0,1] \text{ and } k \in \mathbb{N}, \tag{21}$$

where $\{h_k\}$ is the same as in (7) and we choose $\sigma_i(z) = \sigma(z)$ $(i = 0, 1, 2)$,

where $\sigma(z) = \begin{cases} 1, & z = 0, \\ \frac{1}{z^2}, & 0 < z \le 1. \end{cases}$ For positive constant C, taking into account that $\|U_k(.)\|_{L_p} \le C\|.\|_{L_p}$([19]), we can easily see that

$$st_P - \limsup \left\| \lambda \frac{A_{n,j}^{\mathbb{V}}(f)}{\sigma} \right\|_{L_1} \le R\|\lambda f\|_{L_1}, \text{uniformly in } j, \tag{22}$$

where $A_{n,j}^{\mathbb{V}}(f) = \sum_{k=1}^{\infty} a_{nk}^{(j)} V_k f$ as in (3). We now claim that

$$st_P - \lim \left\| \lambda \frac{A_{n,j}^{\mathbb{V}}(e_i) - e_i}{\sigma} \right\|_{L_1} = 0, \text{ uniformly in } j,\ i = 0, 1, 2, \tag{23}$$

where $\sigma_i(z) = \sigma(z)$ for $i = 0, 1, 2$. Observe that $U_k(e_0; z) = e_0(z)$, $U_k(e_1; z) = \frac{kz}{k+1} + \frac{1}{2(k+1)}$ and $U_k(e_2; z) = \frac{k(k-1)z^2}{(k+1)^2} + \frac{2kz}{(k+1)^2} + \frac{1}{3(k+1)^2}$. So, we can see,

$$\left\| \lambda \frac{A_{n,j}^{\mathbb{V}}(e_0) - e_0}{\sigma} \right\|_{L_1} = \left\| \lambda \frac{\sum_{k=j}^{j+n} \frac{1}{n+1}(1 + h_k) - 1}{\sigma} \right\|_{L_1} \le \frac{1}{n+1} \sum_{k=j}^{j+n} \left\| \lambda \frac{h_k}{\sigma} \right\|_{L_1}.$$

It is well known that if a sequence is convergent, its arithmetic mean will also converge to the same value. Thus, by virtue of P-statistical convergence and thanks to (9) it is clear that

$$st_P - \lim \left(\sup_j \frac{1}{n+1} \sum_{k=j}^{j+n} \left\| \lambda \frac{h_k}{\sigma} \right\|_{L_1} \right) = 0, \tag{24}$$

and we get

$$st_P - \lim \left\| \lambda \frac{A_{n,j}^{\mathbb{V}}(e_0) - e_0}{\sigma} \right\|_{L_1} = 0, \text{ uniformly in } j,$$

which guarantees that (23) holds true for $i = 0$. Also, we can easily get that

$$\begin{aligned}
\left\| \lambda \frac{A_{n,j}^{\mathbb{V}}(e_1) - e_1}{\sigma} \right\|_{L_1} &= \left\| \lambda \left(\sum_{k=j}^{j+n} \frac{1}{n+1} \frac{(1+h_k)}{\sigma} U_k(e_1) - \frac{e_1}{\sigma} \right) \right\|_{L_1} \\
&< \left(\frac{1}{n+1} \sum_{k=j}^{j+n} \frac{k}{k+1} - 1 \right) \int_0^1 \left| \lambda z^3 \right| dz \\
&+ \left(\frac{1}{n+1} \sum_{k=j}^{j+n} \frac{1}{2(k+1)} \right) \int_0^1 \left| \lambda z^2 \right| dz + 2 \frac{1}{n+1} \sum_{k=j}^{j+n} \left\| \lambda \frac{h_k}{\sigma} \right\|_{L_1}.
\end{aligned}$$

Referring to $st_P - \lim \left(\sup_j \left(\frac{1}{n+1} \sum_{k=j}^{j+n} \frac{k}{k+1} - 1 \right) \right) = 0$, $st_P - \lim \left(\sup_j \frac{1}{n+1} \sum_{k=j}^{j+n} \frac{1}{2(k+1)} \right) = 0$ and from (24) we have,

$$st_P - \lim \left(\sup_j \left\| \lambda \frac{A_{n,j}^{\mathbb{V}}(e_1) - e_1}{\sigma} \right\|_{L_1} \right) = 0$$

which gives $st_P - \lim \left\| \lambda \frac{A_{n,j}^{\mathbb{V}}(e_1) - e_1}{\sigma} \right\|_{L_1} = 0$, uniformly in j. So (23) holds true for $i = 1$. Finally, since

$$\begin{aligned}
\left\| \lambda \frac{A_{n,j}^{\mathbb{V}}(e_2) - e_2}{\sigma} \right\|_{L_1} &= \left\| \lambda \left(\sum_{k=j}^{j+n} \frac{1}{n+1} \frac{(1+h_k)}{\sigma} U_k(e_2) - \frac{e_2}{\sigma} \right) \right\|_{L_1} \\
&< \left(\frac{1}{n+1} \sum_{k=j}^{j+n} \frac{k(k-1)}{(k+1)^2} - 1 \right) \int_0^1 \left| \lambda z^4 \right| dz + \left(\frac{1}{n+1} \sum_{k=j}^{j+n} \frac{2k}{(k+1)^2} \right) \int_0^1 \left| \lambda z^3 \right| dz \\
&+ \left(\frac{1}{n+1} \sum_{k=j}^{j+n} \frac{1}{3(k+1)^2} \right) \int_0^1 \left| \lambda z^2 \right| dz + 3 \frac{1}{n+1} \sum_{k=j}^{j+n} \left\| \lambda \frac{h_k}{\sigma} \right\|_{L_1}.
\end{aligned}$$

Thanks to $st_P - \lim \left(\sup_j \frac{1}{n+1} \sum_{k=j}^{j+n} \frac{k(k-1)}{(k+1)^2} \right) = 0$, $st_P - \lim \left(\sup_j \frac{1}{n+1} \sum_{k=j}^{j+n} \frac{2k}{(k+1)^2} \right) = 0$ and $st_P - \lim \left(\sup_j \frac{1}{n+1} \sum_{k=j}^{j+n} \frac{1}{3(k+1)^2} \right) = 0$ and also from (24) we have,

$$st_P - \lim \sup_j \left\| \lambda \frac{A_{n,j}^{\mathbb{V}}(e_2) - e_2}{\sigma} \right\|_{L_1} = 0$$

which gives $st_P - \lim \left\| \lambda \frac{A^{\mathbb{V}}_{n,j}(e_2)-e_2}{\sigma} \right\|_{L_1} = 0$, uniformly in j. So, our claim (23) holds true for each $i = 0, 1, 2$ and for any $\lambda > 0$. Now, from (22) and (23), we can say that our sequence $\mathbb{V} := \{V_k\}$ defined by (21) satisfy all assumptions of Theorem 1 and we conclude that

$$st_P - \lim \left\| \lambda_0 \frac{A^{\mathbb{V}}_{n,j}(f) - f}{\sigma} \right\|_{L_1} = 0, \text{ uniformly in } j, \text{ for some } \lambda_0 > 0,$$

holds for any $f \in L_1(I)$ such that $f - g \in X_{\mathbf{V}}$ for every $g \in C^{\infty}(I)$.
However, it can be easily seen that $\{V_k f\}$ is not P-statistically modularly convergent to f. Hence, Corollary 1 does not work for the sequence $\{V_k f\}$. In addition, in the view of (8), it can be shown that $\{V_k f\}$ is not statistically modularly convergent to f and we can say that the statistical Korovkin type theorem in [42] does not work for the sequence $\{V_k f\}$.

P-Statistical Convergence Rates

We present some estimates of P-statistical convergence rates for theorems of the Korovkin type in the case of modular convergence. We'll start with the definitions listed below:

Definition 5. Let $\{\alpha_k\}$ be a nonincreasing sequence of positive real numbers. A sequence $z = \{z_k\}$ is P-statistically convergent to a number κ with the rate of $o(\alpha_k)$ if, for every $\varepsilon > 0$,

$$\lim_{0<t\to\mathcal{R}^-} \frac{1}{p(t)} \sum_{k \in E_\varepsilon} p_k t^k = 0$$

where $E_\varepsilon = \{k \in \mathbb{N}_0 : |z_k - \kappa| \geq \varepsilon\alpha_k\}$. In this case, we write

$$z_k - \kappa = st_P - o(\alpha_k).$$

Definition 6. Let $\{\alpha_k\}$ be a nonincreasing sequence of positive real numbers. A sequence $z = \{z_k\}$ is P-statistically bounded with the rate of $O(\alpha_k)$ if there is an $B > 0$ with

$$\lim_{0<t\to\mathcal{R}^-} \frac{1}{p(t)} \sum_{k \in F_\varepsilon} p_k t^k = 0$$

where $F_\varepsilon = \{k \in \mathbb{N}_0 : |z_k| \geq B\alpha_k\}$. In this case, we write

$$z_k - \kappa = st_P - O(\alpha_k).$$

Using these definitions, let us give the following lemma:

Lemma 1. Let $\{z_k\}$ and $\{y_k\}$ be sequences. Assume that $\{\alpha_k\}$ and $\{\beta_k\}$ be positive non-increasing sequences. If $z_k - \kappa_1 = st_P - o(\alpha_k)$ and $y_k - \kappa_2 = st_P - o(\beta_k)$, then we have

(i) $(z_k - \kappa_1) \mp (y_k - \kappa_2) = st_P - o(\gamma_k)$, where $\gamma_k := \max\{\alpha_k, \beta_k\}$ for each $k \in \mathbb{N}_0$,

(ii) $\lambda(z_k - \kappa_1) = st_P - o(\alpha_k)$ for any real number λ.

Proof. (i) Assume that $z_k - \kappa_1 = st_P - o(\alpha_k)$ and $y_k - \kappa_2 = st_P - o(\beta_k)$. Also, for $\varepsilon > 0$, define

$$K_\varepsilon := \{\, k \in \mathbb{N}_0 : \ |(z_k - \kappa_1) \mp (y_k - \kappa_2)| \geq \varepsilon \gamma_k \} ,$$
$$K_\varepsilon^1 := \left\{ k \in \mathbb{N}_0 : \ |z_k - \kappa_1| \geq \frac{\varepsilon}{2} \alpha_k \right\},$$
$$K_\varepsilon^2 := \left\{ k \in \mathbb{N}_0 : \ |y_k - \kappa_2| \geq \frac{\varepsilon}{2} \beta_k \right\}.$$

Since $\gamma_k = \max\{\alpha_k, \beta_k\}$, then observe that

$$K_\varepsilon \subset K_\varepsilon^1 \cup K_\varepsilon^2,$$

which gives,

$$\delta_P(K_\varepsilon) \leq \sum_{i=1}^{2} \delta_P\left(K_\varepsilon^i\right)$$

Under the hypothesis, we conclude that

$$\delta_P(K_\varepsilon) = 0,$$

this conclusively demonstrates (i). We omit the proof of (ii) because it is similar.

Additionally, the same results apply when the symbol "o" is changed with "O".
We utilize the ordinary modulus of continuity, denoted by $\omega(f;\delta)$ defined as follows:

$$\omega(f;\delta) := \sup\{|f(u) - f(z)| : u, z \in I, |u - z| \leq \delta\}.$$

where $f \in C_b(I)$ and $\delta > 0$. Observe that $\omega(f;\delta)$ is an increasing function of δ, $|f(u) - f(z)| \leq \omega(f; |u - z|)$ for each $u, z \in I$ and $\omega(f;\delta) \leq 2H$ for every δ, where $H = \sup_{u \in I} |f(u)|$ and $\omega(f;\gamma\delta) \leq (1+\gamma)\,\omega(f;\delta)$ for every $\gamma, \delta > 0$ (see also [47]).
The following theorem gives appropriate and very simple sufficient conditions for the rate of P-statistical $\mathcal{A}$-summation process.

Theorem 3. Let $\mathcal{A} = \left\{A^{(j)}\right\}$, $\mathbb{T} = \{T_k\}$ and e_0 be as above, let ρ be a monotone and strongly finite modular. Let $\{\alpha_k\}$ and $\{\beta_k\}$ be two nonincreasing sequences of strictly positive real numbers, and put $\gamma_k := \max\{\alpha_k, \beta_k\}$ for each $k \in \mathbb{N}_0$. Let $\delta_k := \left\| A_{n,j}^{\mathbb{T}}(\varphi) \right\|$ with $\varphi(u) = |u - z|$ $(u, z \in I)$ where the symbol $\|.\|$ denotes the sup-norm, the supremum is taken with respect to the support of f and $A_{n,j}^{\mathbb{T}}(\varphi) = \sum_{k=1}^{\infty} a_{nk}^{(j)} T_k \varphi$ as in *(3)*. Furthermore, let there exists $\gamma > 0$ with
(i) $\rho\left(\gamma\left(\frac{A_{n,j}^{\mathbb{T}}(e_0) - e_0}{\sigma_0}\right)\right) = st_P - o(\alpha_k)$,
(ii) $\rho\left(\gamma \frac{\omega(f;\delta_k)}{\sigma_1}\right) = st_P - o(\beta_k)$. Then, for every $f \in C_c(I)$ we get

$$\rho\left(\gamma\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) = st_P - o(\,\gamma_k)$$

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where $\sigma(z) = \max\{|\sigma_0(z)|, |\sigma_1(z)|\}$, $|\sigma_i(z)| > 0$ and $\sigma_i(z)$ is unbounded, $i = 0, 1$.

Proof. Let $f \in C_c(I)$ and $H = \sup_{u \in I} |f(u)|$. Using the properties of the modulus of continuity, we get

$$|f(u) - f(z)| \leq \omega(f; \varphi(u)) \leq \left(1 + \frac{\varphi(u)}{\delta}\right) \omega(f; \delta) \tag{25}$$

for every $\delta > 0$ and $u, z \in I$. Let now $\delta = \delta_k$. By applying T_k, keeping fixed z and letting u in I, and from (25), we obtain while accounting for linearity and monotonicity

$$\begin{aligned}
\left|\sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(f; z) - f(z)\right| &\leq \sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(|f(u) - f(z)|; z) + |f(z)| \left|\sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(e_0; z) - e_0(z)\right| \\
&\leq \sum_{k=1}^{\infty} a_{nk}^{(j)} T_k\left(\left(1 + \frac{\varphi(u)}{\delta}\right) \omega(f; \delta); z\right) + H \left|\sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(e_0; z) - e_0(z)\right| \\
&\leq \omega(f; \delta) \left|\sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(e_0; z) - e_0(z)\right| + \frac{\omega(f; \delta)}{\delta} \sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(\varphi(u); z) \\
&+ \omega(f; \delta) e_0(z) + H \left|\sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(e_0; z) - e_0(z)\right| \\
&\leq 4H \left(\left|\sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(e_0; z) - e_0(z)\right| + \omega(f; \delta)\right)
\end{aligned}$$

for each $z \in I$. Now, by multiplying both sides of the above inequality by $\frac{1}{|\sigma(z)|}$ and

$$\left|\frac{\sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(f; z) - f(z)}{\sigma(z)}\right| \leq 4H \left(\left|\frac{\sum_{k=1}^{\infty} a_{nk}^{(j)} T_k(e_0; z) - e_0(z)}{\sigma_0(z)}\right| + \frac{\omega(f; \delta)}{|\sigma_1(z)|}\right).$$

Let now $\gamma > 0$. By applying the modular ρ, from the above inequality, we obtain

$$\rho\left(\gamma\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) \leq \rho\left(8H\gamma\left(\frac{A_{n,j}^{\mathbb{T}}(e_0) - e_0}{\sigma_0}\right)\right) + \rho\left(8H\gamma \frac{\omega(f; \delta)}{\sigma_1}\right). \tag{26}$$

For each $k \in \mathbb{N}_0$ and a given $r > 0$, define the following sets:

$$\begin{aligned}
U_k &:= \left\{k \in \mathbb{N}_0 : \rho\left(\gamma\left(\frac{A_{n,j}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) \geq r\gamma_k\right\}, \\
U_k^1 &:= \left\{k \in \mathbb{N}_0 : \rho\left(8H\gamma\left(\frac{A_{n,j}^{\mathbb{T}}(e_0) - e_0}{\sigma_0}\right)\right) \geq \frac{r\alpha_k}{2}\right\}, \\
U_k^2 &:= \left\{k \in \mathbb{N}_0 : \rho\left(8H\gamma \frac{\omega(f; \delta)}{\sigma_1}\right) \geq \frac{r\beta_k}{2}\right\}.
\end{aligned}$$

Referring to (26), we see that

$$U_k \subset U_k^1 \cup U_k^2.$$

Then, it follows from this inclusion that

$$\delta_P(U_k) \leq \delta_P\left(U_k^1\right) + \delta_P\left(U_k^2\right)$$

and using (i) and (ii), we obtain

$$\delta_P(U_k) = 0.$$

The theorem's proof is now complete.

Additionally, the same results apply when the symbol "o" is changed with "O".

Concluding Remarks

To conclude this paper, in order to highlight the significance of Theorem 1 and Theorem 2 in approximation theory, we present some condensed results:

- We can derive the Korovkin type theorems of the P-statistical $\mathcal{A}$-summation process from our Theorem 1 and Theorem 2 by substituting a non-zero constant for the scale function.
- The P-statistical relative Korovkin type theorems are directly obtained from our Theorem 1 and Theorem 2 if we take $A^{(j)}$ by the identity matrix. We also obtain P-statistical Korovkin type theorems if one replaces a non-zero constant for the scale function.
- It is a well-known conclusion that $(X, \|.\|)$ is a normed space then, $\rho(.) = \|.\|$ is a convex modular in X. As a result, evey theorem in this paper are considered valid on normed spaces as well.

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