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A Classification of Parallel Normalized Biconservative Submanifold in the Minkowski Space in Arbitrary Dimension

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ABSTRACT

In this paper, we examine PNMCV-MCGL biconservative submanifold in a Minkowski space $\mathbb{E}_1^{n+2}$ with nondiagonalizable shape operator, where PNMCV-MCGL submanifold denotes a submanifold with parallel normalized mean curvature vector and the mean curvature whose gradient is lightlike ($\langle \nabla H, \nabla H \rangle = 0$). We obtain some conditions about connection forms, principal curvatures and some results about them. Then we use them to obtain a classification of such submanifolds. Finally, we showed that there is no biconservative such submanifold in Minkowski space of arbitrary dimension.

Keywords: *Biconservative submanifolds, non-diagonalizable shape operator, Minkowski space, biconservative isometric immersions.*
AMS Subject Classification (2020): *Primary: 53C42 ; Secondary: 53B25.*

1. Introduction

In 1980's Chen B.Y. introduced a conjecture that "biharmonic submanifolds of Euclidean spaces are minimal" that would later be named after him. Then many geometers studied on this conjecture and obtained results that confirm this conjecture. Concerning to the pseudo-Euclidean spaces this conjecture is not satisfied everytime. In 1990's Chen B.Y. and Ishikawa S. gave some examples of non-minimal biharmonic submanifolds which are called proper biharmonic submanifold [5],[6]. Biconservative submanifolds are generalizations of biharmonic submanifolds such that every biharmonic submanifold is biconservative at the same time but inverse does not hold generally.

Let $x : M \to N$ be an isometric immersion with the mean curvature vector H. Then x is bihamonic if and only if the following equations

$$m\mathrm{grad}\,\|H\|^2 + 4\mathrm{tr}A_{\nabla^\perp_\cdot H}(\cdot) + 4\mathrm{tr}(\tilde{R}(\cdot, H)\cdot)^T = 0 \tag{1.1}$$

and

$$-\Delta^\perp H + \mathrm{tr}h(A_H(\cdot), \cdot) + \mathrm{tr}(\tilde{R}(\cdot, H)\cdot)^\perp = 0 \tag{1.2}$$

are satisfied, where m is the dimension of M and $\Delta^\perp$ is the Laplacian associated with $\nabla^\perp$. x is said to be biconservative map if (1.1) is satisfied [13]. A submanifold is said to be biconservative if it has biconservative map. So biconservative manifolds are much bigger family than bihamonic submanifolds and it has been studied in many geometers so far ([13],[14],[10],[11]). In [2], Chen gave the definition of parallel normalized mean curvature vector such that a submanifold is said to have parallel normalized mean curvature vector if the mean curvature vector is nonzero and the unit vector in the direction of the mean curvature vector is parallel, i.e. $\nabla^\perp(H/\|H\|) = 0$, and he showed that a surface which is isometrically immersed in a Euclidean m-space $\mathbb{E}^m$ then it is the minimal surface of $\mathbb{E}^m$ or minimal surface of a hypersphere of $\mathbb{E}^m$ or surfaces in an affine 4-space $\mathbb{E}^4$ of $\mathbb{E}^m$. Moreover, In [3], Chen obtained some results of such surfaces which is analytic and in 2019 he showed

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that "A biharmonic surface in $\mathbb{E}^m$ with a parallel normalized mean curvature vector does not exist.". This was an another solution to Chen's conjecture. Concerning to the pseudo-Euclidean case, Chen investigated to classify space-like and Lorentz surfaces with parallel mean curvature vector in Riemannian and indefinite space forms. Du L. and Zhang J. classified completely such surface by adding pseudo-umbilical property and they showed that such submanifolds have parallel mean curvature vector field under some geometric conditions [8]. Later Du L. completely classified such pseudo-umbilical submanifolds and $f-$biharmonic. A submanifold is said to $f-$biharmonic if the left hand side of (1.2) is equal to fH, where f is a function.

In [16] Şen R. and Turgay N. C. obtained some results on the biconservative submanifolds with parallel normalized mean curvature vector as well as biharmonic ones. In [15], Şen R. studied on biconservative m-dimensional submanifolds with parallel normalized mean curvature vector field in $\mathbb{E}^{m+2}$ and obtain canonical forms of the shape operators of such submanifolds. So, it is still open working area of such submanifolds in pseudo-Euclidean space. In [17], in 2016, Turgay N. C. studied such hypersurface in Minkowski space in arbitrary dimension for biharmonic ones. He showed that there is no such biharmonic hypersurface with at most 5 distinct principal curvatures. So biconservative part of the problem about submanifolds with parallel normalized mean curvature is still open.

In this paper we investigate biconservative $n-$dimensional submanifold with parallel normalized mean curvature vector and non-diagonalizable shape operator in arbitrary Minkowski space $\mathbb{E}^{n+2}_1$ such that it has mean curvature vector whose gradient is lightlike. We proved that there is no such submanifold.

2. Preliminaries

Let M be an n-dimensional submanifold of Minkowski space $\mathbb{E}^{n+2}_1$ and $x : M \to \mathbb{E}^{n+2}_1$ be an isometric immersion. Let $\nabla^\perp$ denote its normal connection then a normal vector field η is said to be parallel if $\nabla^\perp_X \eta = 0$ whenever X is tangent to submanifold.

We put ∇ and $\tilde{\nabla}$ for the Levi-Civita connection of M and $\mathbb{E}^{n+2}_1$, respectively. Then

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h(X,Y) && (2.1)\\ \langle h(X,Y), N\rangle &= \langle A_N X, Y\rangle && (2.2) \end{aligned}$$

respectively, for any vector fields X, Y tangent to M, where h is the second fundamental form and A is the shape operator. Moreover, the mean curvature vector H of M is defined by

$$H = \frac{1}{n} tr(h) \qquad (2.3)$$

and its norm $\|H\| = |\langle H, H\rangle|^{1/2}$ is called the mean curvature of M.

Let R denote the curvature tensor of M. Then, Gauss, Ricci and Codazzi equations are as follows

$$\begin{aligned} R(X,Y,Z,W) &= h(X,W)h(Y,Z) - h(X,Z)h(Y,W), && (2.4)\\ R^\perp(X,Y)N &= h(X, A_N Y) - h(A_N X, Y), && (2.5)\\ (\bar{\nabla}_Y h)(X,Z) &= (\bar{\nabla}_X h)(Y,Z). && (2.6) \end{aligned}$$

The covariant derivative $\bar{\nabla} h$ of h is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

We study on submanifold with nondiagonalizable shape operator. So we need to construct the cannonical form of the shape operator. To do this we give well-known the following lemma.

Lemma 2.1. *[12] Let M be a Lorentzian surface, $p \in M$ and A be a symmetric endomorphism of T_pM. Then, by choosing an appropriated base for T_pM, A can put into one of the following four canonical forms:*

Case (i). $A \sim \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$, *Case (ii).* $A \sim \begin{bmatrix} a_0 & & & & 0 \\ -1 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ 0 & & & & a_{n-2} \end{bmatrix}$

Case (iii). $A \sim \begin{bmatrix} a_0 & 0 & 0 & & & \\ 0 & a_0 & 1 & & & \\ -1 & 0 & a_0 & & & \\ & & & a_1 & & \\ & & & & \ddots & \\ & & & & & a_{n-3} \end{bmatrix}$ *Case (iv).* $A \sim \begin{bmatrix} a_0 & b_0 & & & \\ -b_0 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ & & & & a_{n-2}. \end{bmatrix}$

Note that the base field is pseudo-orthonormal in Case (ii) and Case(iii) while it is orthonormal in the other cases. Moreover, b_0 is nonzero.

Let M be a Lorentzian hypersurface. We can choose a pseudo-orthonormal frame $\{e_1, e_2, e_3 \cdots, e_n\}$ such that

$$\langle e_i, e_j \rangle = \delta_{ij} - 1 \qquad \langle e_i, e_\alpha \rangle = 0 \qquad \langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta}$$

where $i, j \in \{1, 2\}, \alpha, \beta \in \{3, 4, \cdots, n\}$. Then, Levi-Civita Connection ∇ related to M as the following

$$\nabla_{e_i} e_1 = \phi_i e_1 + \sum_{b=3}^{n} \omega_{1b}(e_i) e_b \tag{2.7}$$

$$\nabla_{e_i} e_2 = -\phi_i e_2 + \sum_{b=3}^{n} \omega_{2b}(e_i) e_b \tag{2.8}$$

$$\nabla_{e_i} e_\alpha = \omega_{2\alpha}(e_i) e_1 + \omega_{1\alpha}(e_i) e_2 + \sum_{b=3}^{n} \omega_{\alpha b}(e_i) e_b \tag{2.9}$$

where $\phi_i = \langle \nabla_{e_i} e_2, e_1 \rangle = -\omega_{12}(e_i)$.

Proposition 2.1. *Let $\psi : (\Omega, g) \to (\mathbb{E}_1^{n+2}, g)$ be an isometric immersion of Lorentzian manifold of dimension n into Minkowski space $\mathbb{E}_1^{n+2}$. Then ψ is biconservative if and only if*

$$A_{N_1}(\nabla f) = -\frac{nf}{2} \nabla f \tag{2.10}$$

is satisfied, where N_1 is the normalized mean curvature vector and $f = \|H\|$ and ∇f is gradient of f.

From now on we abbreviate submanifold with mean curvature whose gradient is lightlike as MCGL submanifold.

3. MCGL Submanifolds with Codimension 2 In $\mathbb{E}_1^{n+2}$

In this section we examine the submanifolds of codimension-2. It means M has two normal vectors and so two shape operators. we study on submanifolds with nondiagonalizable shape operator. So it is more difficult than hypersurface case. Definition of submanifolds with parallel normalized mean curvature vector has been given in introduction. So, from now on we abbreviate such submanifold as PNMCV submanifold.

Before we proceed we would like to emphasis that we choose the mean curvature vector as $H = fN_1$ then we have $\nabla^\perp N_2 = 0$ since H is parallel normalized mean curvature vector.

Remark 3.1. Note that ∇f is proportional to only e_1 or e_2 since if $\nabla f = ae_1 + be_2$ for non-zero funtions a, b then $\langle \nabla f, \nabla f \rangle = -2ab \neq 0$ and ∇f would not be lightlike.

Now, consider case(iv) in Lemma 2.1. If ∇f is proportional e_1 or e_2 by Remark 3.1 then b_0 would be zero. So, the rest of the problem is to examine the cases of the shape operators depending on N_1 which is equal the case(ii) and case(iii) in Lemma 2.1, seperately. we examine these matrices in the subsections named by Case 1 and Case 2, respectively.

3.1. Case1

Firstly we inqury the structure of the shape operators of M. By Lemma 2.1 we have two matrices as the following:

$$A_{N_1} = \begin{bmatrix} k_1 & 1 & & & \\ & k_1 & & & \\ & & k_3 & & \\ & & & \ddots & \\ & & & & k_n \end{bmatrix} \tag{3.1}$$

$$A_{N_2} = \begin{bmatrix} l_1 & a_1 & & & \\ a_2 & l_2 & & & \\ & & l_3 & & \\ & & & \ddots & \\ & & & & l_n \end{bmatrix}. \tag{3.2}$$

Then, one can get easily the second fundamental form as follows;

$$h(e_1, e_1) = -a_2 N_2, \qquad h(e_1, e_2) = -k_1 N_1 - l_1 N_2, \qquad h(e_2, e_2) = -N_1 - a_1 N_2. \tag{3.3}$$

So, we use the Ricci equation (2.5) to find a_1, a_2.

$$\begin{aligned} R^{\perp}(e_1, e_2) N_2 &= h(e_1, A_{N_2} e_2) - h(A_{N_2} e_1, e_2) \\ 0 &= a_1 h(e_1, e_1) + (l_2 - l_1) h(e_1, e_2) - a_2 h(e_2, e_2). \end{aligned}$$

Direct calculations give

$$0 = (l_2 - l_1) l_1 N_2 + (k_1(l_2 - l_1) - a_2) N_1.$$

It is obvious that $a_2 = 0$ and if $l_1 \neq 0$ then $l_1 = l_2$. Then the matrix (3.2) becomes

$$A_{N_2} = \begin{bmatrix} l_1 & \varphi & & & \\ & l_1 & & & \\ & & l_3 & & \\ & & & \ddots & \\ & & & & l_n \end{bmatrix} \tag{3.4}$$

and the second fundamental form (3.3) becomes

$$h(e_1, e_1) = 0, \qquad h(e_1, e_2) = -k_1 N_1 - l_1 N_2, \qquad h(e_2, e_2) = -N_1 - \varphi N_2.$$

So, we can choose ∇f is proportional to e_1 by Remark 3.1. So we have

$$e_2(f) \neq 0, \qquad e_i(f) = 0, \qquad \nabla f = -e_2(f) e_1 \tag{3.5}$$

where $i = 1, 3, \cdots, n$. Moreover, by (2.10) we have

$$-2k_1 = nf. \tag{3.6}$$

Consider distinct princial curvatures $K_1, K_2, \cdots, K_p$ with its multiplicities $v_1, v_2, \cdots, v_p$ and $L_1, L_2, \cdots, L_q$ with its multiplicities $u_1, u_2, \cdots, u_q$ as in the previous section then we have

$$v_2 K_2 + v_3 K_3 + \cdots + v_p K_p = -(2 + v_1) K_1 \tag{3.7}$$

$$u_2 L_2 + u_3 L_3 + \cdots + u_q L_q = -u_1 L_1 \tag{3.8}$$

Now we use the Codazzi equation (2.6)
• The triplet $(e_1, e_\alpha, e_\alpha)$ gives

$$e_1(k_\alpha) = \psi_\alpha(k_1 - k_\alpha) \tag{3.9}$$

$$e_1(l_\alpha) = \psi_\alpha(l_1 - l_\alpha) \tag{3.10}$$

• The triplet $(e_2, e_\alpha, e_\alpha)$ gives

$$e_2(k_\alpha) = \Phi_\alpha(k_1 - k_\alpha) + \psi_\alpha \tag{3.11}$$
$$e_2(l_\alpha) = \Phi_\alpha(l_1 - l_\alpha) + \psi_\alpha\varphi \tag{3.12}$$

• The triplet (e_1, e_α, e_β) gives

$$\omega_{\alpha\beta}(e_1)(k_\alpha - k_\beta) = \omega_{1\beta}(e_\alpha)(k_1 - k_\beta) \tag{3.13}$$
$$\omega_{\alpha\beta}(e_1)(l_\alpha - l_\beta) = \omega_{1\beta}(e_\alpha)(l_1 - {}_\beta) \tag{3.14}$$

• The triplet (e_α, e_β, e_1) gives

$$\omega_{1\beta}(e_\alpha)(k_1 - k_\beta) = \omega_{1\alpha}(e_\beta)(k_1 - k_\alpha) \tag{3.15}$$
$$\omega_{1\beta}(e_\alpha)(l_1 - l_\beta) = \omega_{1\alpha}(e_\beta)(l_1 - {}_\alpha) \tag{3.16}$$

Moreover; by (3.5) we have $[e_\alpha, e_\beta](k_1) = 0$. So

$$\omega_{1\alpha}(e_\beta) = \omega_{1\beta}(e_\alpha) \tag{3.17}$$

by subsituiting (3.17) into (3.15) and (3.13), we get

$$w_{1\alpha}(e_\beta) = \omega_{1\beta}(e_\alpha) = 0 \tag{3.18}$$

for $k_\alpha \neq k_\beta$. Further if $k_1 \neq k_\alpha = k_\beta$ then $\omega_{\alpha\beta}(e_1) = 0$.
• The triplet (e_2, e_α, e_β) implies

$$(k_\alpha - k_\beta)\omega_{\alpha\beta}(e_2) = (k_1 - k_\beta)\omega_{2\beta}(e_\alpha) \tag{3.19}$$
$$(l_\alpha - l_\beta)\omega_{\alpha\beta}(e_2) = (l_1 - l_\beta)\omega_{2\beta}(e_\alpha). \tag{3.20}$$

• The triplet (e_α, e_β, e_2) implies

$$(k_1 - k_\beta)\omega_{2\beta}(e_\alpha) = (k_1 - k_\alpha)\omega_{2\alpha}(e_\beta). \tag{3.21}$$

By combining (3.19) and (3.21), suppose that $k_1 \neq k_\alpha = k_\beta$, we see that

$$\omega_{2\alpha}(e_\beta) = \omega_{2\beta}(e_\alpha) = 0. \tag{3.22}$$

• The triplet (e_α, e_1, e_1) implies

$$\omega_{1\alpha}(e_1) = 0 \tag{3.23}$$

• The triplet (e_2, e_1, e_1) gives

$$e_1(l_1) = 0 \tag{3.24}$$

• The triplet (e_α, e_1, e_2) gives

$$\omega_{2\alpha}(e_1)(k_1 - k_\alpha) = 0$$

and by $k_1 \neq k_\alpha$ then we have

$$\omega_{2\alpha}(e_1) = 0 \tag{3.25}$$

Now we use Gauss equations (2.4)

$$R(e_\alpha, e_1, e_2, e_\alpha) = -k_1 k_\alpha - l_1 l_\alpha. \tag{3.26}$$

we make some calculation on (3.26).

$$\begin{aligned}
\langle \nabla_{e_\alpha} \nabla_{e_1} e_2 \rangle &= \langle \nabla_{e_\alpha}(-\phi_1 e_2 + \sum_{b=3}^{n} \omega_{2b}(e_1) e_b, e_\alpha) \rangle && \text{(3.27a)}\\
&= -\phi_1 \Phi_\alpha + e_\alpha(\omega_{2\alpha}(e_1)) + \sum_{\alpha\neq b=3}^{n} \omega_{2b}(e_1)\omega_{b\alpha}(e_\alpha) && \text{(3.27b)}\\
\langle \nabla_{e_1} \nabla_{e_\alpha} e_2 \rangle &= \langle e_1(-\phi_\alpha e_2 + \sum_{b=3}^{n} \omega_{2b}(e_\alpha) e_b), e_\alpha \rangle && \text{(3.27c)}\\
&= -\phi_\alpha \omega_{2\alpha}(e_1) + e_1(\Phi_\alpha) + \sum_{\alpha\neq b=3}^{n} \omega_{2b}(e_\alpha)\omega_{b\alpha}(e_1) && \text{(3.27d)}\\
\langle \nabla_{[e_\alpha, e_1]} e_2, e_\alpha \rangle &= (\phi_\alpha - \omega_{2\alpha}(e_1))\omega_{2\alpha}(e_1) && \text{(3.27e)}
\end{aligned}$$

Under the consideration (3.18) and (3.22) with its contractions, one can obtain

$$\sum_{\alpha\neq b=3}^{n} \omega_{2b}(e_\alpha)\omega_{b\alpha}(e_1) = 0.$$

In addition to this result, $k_1 \neq k_\alpha$ turns (3.26) into

$$e_1(\Phi_\alpha) + \Phi_\alpha(\phi_1 + \psi_\alpha) = k_1 k_\alpha + l_1 l_\alpha \tag{3.28}$$

since (3.25). Moreover, $R(e_\alpha, e_1, e_1, e_\alpha) = 0$ gives

$$e_1(\psi_\alpha) = \psi_\alpha(\phi_1 - \psi_\alpha). \tag{3.29}$$

Now consider equalities (3.7) and (3.8). By using (3.5), (3.6), (3.9), (3.10), (3.24) and (3.29), if we deriviate (3.7) p times and (3.8) q times we have following two matrices, respectively

$$\begin{bmatrix} \psi_2 & \psi_3 & \cdots & \psi_p \\ \psi_2^2 & \psi_3^2 & \cdots & \psi_p^2 \\ \vdots & \vdots & \ddots & \vdots \\ \psi_2^p & \psi_3^p & \cdots & \psi_p^p \end{bmatrix} \begin{bmatrix} v_2(K_1 - K_2) \\ v_3(K_1 - K_3) \\ \vdots \\ v_p(K_1 - K_p) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{3.30}$$

$$\begin{bmatrix} \psi_2 & \psi_3 & \cdots & \psi_q \\ \psi_2^2 & \psi_3^2 & \cdots & \psi_q^2 \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1^q & \psi_2^q & \cdots & \psi_q^q \end{bmatrix} \begin{bmatrix} u_2(L_1 - L_2) \\ u_3(L_1 - L_3) \\ \vdots \\ u_q(L_1 - L_q) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.31}$$

Before proceed, we would like to notice that we examine what kind of contradiction gives us the fact that all ψ_α's are zero. Thus we shall give the following Lemma.

Lemma 3.1. *For $k_1 \neq k_\alpha$, if $\psi_\alpha = 0$ then $k_1 k_\alpha = -l_1 l_\alpha$, where $\alpha = 3, 4, \cdots, n$.*

Proof. By taking a deriviation of (3.11) along e_1, we have

$$[e_1, e_2]k_\alpha = e_1(\Phi_\alpha)(k_1 - k_\alpha)$$

Notice that we can use the Lie bracket i.e. $[e_1, e_2](k_\alpha) = e_1 e_2(k_\alpha)$ since $e_1(k_\alpha) = 0$. So,

$$\begin{aligned} -\phi_1 e_2(k_\alpha) &= e_1(\Phi_\alpha)(k_1 - k_\alpha) \\ -\phi_1 \Phi_\alpha(k_1 - k_\alpha) &= e_1(\Phi_\alpha)(k_1 - k_\alpha) \end{aligned}$$

by simplifying with $(k_1 - k_\alpha)$, we have

$$-\phi_1 \Phi_\alpha = e_1(\Phi_\alpha) \tag{3.32}$$

By subsituiting (3.32) into the Gauss equation (3.28) then we get the result that we want to show.

Remark 3.2. The indice α begins from 2 for capital K_α while beginning from 3 for lowercase k_α due to the matrix (3.1). Because some principal curvatures may be the same for some $\alpha's$ for k_α, but it is not valid for K_α. Because k_α's are just principal curvatures, but K_α's are distinct principal curvatures.

Lemma 3.2. *There is no biconservative PVMCV-MCGL submanifold in $\mathbb{E}_1^{n+2}$ with the shape operator given in (3.1) with two distinct principal curvatures.*

Proof. Assume that A_{N_1} has two distinct principal curvatures K_1, K_2 then (3.7) becomes

$$v_2 K_2 = -(2 + v_1)K_1. \tag{3.33}$$

Before we proceed, we would like to notice that taking a derivative (3.33) along e_1 gives $\psi_2 = 0$. Because all ψ_α are equal to ψ_2, Lemma 3.1 gives $K_1 K_2 = -l_1 l_\alpha$ and $K_1 K_2 = -l_1 l_\beta$. So, we have $l_1 l_\alpha = l_1 l_\beta$. Notice that if $l_1 = 0$ then $K_1 K_2 = 0$ and this means $K_1 = 0$ since (3.87) which yields a contradiction. It follows $l_\alpha = l_\beta$. So, A_{N_2} has two distinct principal curvatures as well. Then (3.8) becomes

$$u_2 L_2 = -u_1 L_1. \tag{3.34}$$

In addititon, we have $K_1K_2 = -L_1L_2$ by Lemma 3.1. Multiplying (3.33) and (3.34) by K_1 and L_1 seperately and some direct calculations give

$$0 = \frac{2+v_1}{v_2}K_1^2 + \frac{u_1}{u_2}L_1^2 \tag{3.35}$$

which yields a contradiction since then K_1 would be zero and it follows H is zero.

Lemma 3.3. *For $\alpha = 2, 3, \ldots, n$, if all $\psi_\alpha = 0$ then A_{N_1} and A_{N_2} has the same number of principle curvatures which is equal to zero, i.e. $p = q$.*

Proof. Firstly notice that $k_1 \neq 0$. Assume that all $\psi_\alpha = 0$ then we have $k_1k_\alpha = -l_1l_\alpha$ and $k_1k_\beta = -l_1l_\beta$ for distinct α, β and $3 \leq \alpha, \beta \leq n$. If $l_1 = 0$ then $k_\alpha = k_\beta$ and $p = 2$. Applying Lemma 3.2 gives a contradiction. If $l_1 \neq 0$ then it is obviously obtained that

$$k_\alpha = k_\beta \Leftrightarrow l_\alpha = l_\beta.$$

This implies $p = q$.

Now one can give the following lemma

Lemma 3.4. *If (3.30) is satisfied then all $\psi_\alpha = 0$, where $\alpha = 2, \ldots, p$.*

Proof. For $p = 2$ case has been shown in Lemma 3.2.
Assume that for $p = r$ if the equation

$$\sum_{\alpha=2}^{r} \psi_\alpha v_\alpha (K_1 - K_\alpha) = 0 \tag{3.36}$$

is satisfied then $\psi_2 = \cdots = \psi_r = 0$. Now we shall show for $p = r+1$. Then one can say

$$\sum_{\alpha=2}^{r+1} \psi_\alpha v_\alpha (K_1 - K_\alpha) = 0 \tag{3.37}$$

So

$$\sum_{\alpha=2}^{r} \psi_\alpha v_\alpha (K_1 - K_\alpha) + \psi_{r+1} v_{r+1} (K_1 - K_{r+1}) = 0 \tag{3.38}$$

The first term in the left hand side of (3.38) is equal to zero by acceptance. So this gives $\psi_{r+1} = 0$ since K_1 and K_{r+1} are distinct principle curvatures and v_{r+1} is multiplicity.

Theorem 3.1. *There is no PNMCV-MCGL submanifold in Minkowski space $\mathbb{E}_1^{n+2}$ with the shape operator given in (3.1) and (3.2).*

Proof. Firstly we prove that multiplicities of distinct principal curvatures K_α and L_α are equal, i.e. $u_\alpha = v_\alpha$ for every $\alpha = 2, \ldots, p$. By Lemma 3.4 and 3.1 one can say $K_1K_\alpha = -L_1L_\alpha$. Now assume that $u_\alpha \neq v_\alpha$ then there is some integers a, r such that

$$\begin{aligned} k_a &= k_{a+1} = \cdots = k_{a+r-1} = K_\alpha, \qquad k_{a+r} = K_{\alpha+1} \qquad (3.39)\\ l_a &= l_{a+1} = \cdots = l_{a+r-1} = l_{a+r} = L_\alpha \qquad (3.40)\end{aligned}$$

So, by Lemma 3.1, we have

$$\begin{aligned} k_1k_{a+r} &= -l_1l_{a+r} \qquad (3.41)\\ K_1K_{\alpha+1} &= -L_1L_\alpha. \qquad (3.42)\end{aligned}$$

By Lemma 3.4 and 3.1 the equation (3.42) gives

$$-K_1K_{\alpha+1} = -K_1K_\alpha. \tag{3.43}$$

The equation (3.43) is satisfied than either $K_1 = 0$ or $K_\alpha = K_{\alpha+1}$. Now (3.7) and (3.8) becomes

$$\begin{aligned} \sum_{i=2}^{p} v_iK_i &= -(2+v_1)K_1 \qquad (3.44)\\ \sum_{i=2}^{p} v_iL_i &= -u_1L_1 \qquad (3.45)\end{aligned}$$

By multiplying (3.44) and (3.45) by K_1 and L_1, seperately and using Lemma 3.4, we have the following equality

$$0 = -(2+v_1)K_1^2 - u_1L_1^2. \tag{3.46}$$

It gives $K_1 = 0$ which yields a contradiction.

3.2. Case 2:

We inqury the structure of the shape operators of M. By Lemma 2.1 we have two matrices as the following:

$$A_{N_1} = \begin{bmatrix} k_1 & & & & & \\ & k_1 & 1 & & & \\ -1 & & k_1 & & & \\ & & & k_4 & & \\ & & & & \ddots & \\ & & & & & k_n \end{bmatrix} \tag{3.47}$$

$$A_{N_2} = \begin{bmatrix} l_1 & a & b & & & \\ d & l_2 & c & & & \\ e & \tilde{a} & l_3 & & & \\ & & & l_4 & & \\ & & & & \ddots & \\ & & & & & l_n \end{bmatrix}. \tag{3.48}$$

Then, one can get easily then second fundamental form as follows;

$$h(e_1,e_1) = -dN_2 \,, h(e_1,e_2) = -k_1N_1 - l_1N_2 \,, h(e_1,e_3) = -N_1 + eN_2 \tag{3.49a}$$
$$h(e_2,e_2) = -aN_2 \,, h(e_2,e_3) = \tilde{a}N_2 \,, h(e_3,e_3) = k_1N_1 + l_3N_2 \tag{3.49b}$$

Consider (2.2),

- $\langle A_{N_2}e_3, e_2\rangle = \langle N_2, h(e_3,e_2)\rangle$ gives
$$-b = \tilde{a} \tag{3.50}$$
- $\langle A_{N_2}e_1, e_3\rangle = \langle N_2, h(e_1,e_3)\rangle$ gives
$$-c = e \tag{3.51}$$
- $\langle A_{N_2}e_2, e_1\rangle = \langle N_2, h(e_2,e_1)\rangle$ gives
$$l_1 = l_2 \tag{3.52}$$

Now, we choose the mean curvature vector as in the Case 1. Then we have $\nabla^\perp N_2 = 0$ again and $f = \|H\|$. So, we use the Ricci equation 2.5 to find $a, b, c, d, \tilde{a}$.

$$\begin{aligned} R^\perp(e_1,e_2)N_2 &= h(e_1, A_{N_2}e_2) - h(A_{N_2}e_1, e_2) \\ 0 &= \tilde{a}h(e_1,e_3) + (l_2 - l_1)h(e_1,e_2) + ah(e_1,e_1) - dh(e_2,e_2) - eh(e_3,e_2). \end{aligned}$$

Direct calculations give

$$0 = (-\tilde{a} + (l_1 - l_2)k_1)N_1 + (l_1 - l_2)l_1N_2$$

and so

$$\tilde{a} = 0. \tag{3.53}$$

Now

$$\begin{aligned} R^\perp(e_1,e_3)N_2 &= h(e_1, A_{N_2}e_3) - h(A_{N_2}e_1, e_3) \\ 0 &= ch(e_1,e_2) + (l_3 - l_1)h(e_1,e_3) - dh(e_2,e_3) - eh(e_3,e_3). \end{aligned}$$

Direct calculations and (3.51) give

$$l_1 = l_3 \tag{3.54}$$

Now

$$\begin{aligned} R^\perp(e_2,e_3)N_2 &= h(e_2, A_{N_2}e_3) - h(A_{N_2}e_2, e_3) \\ 0 &= (l_3 - l_2)h(e_2,e_3) + ch(e_2,e_2) - ah(e_1,e_3). \end{aligned}$$

Direct calculations give

$$a = 0 \tag{3.55}$$

Now the second fundamental forms (3.49) becomes

$$h(e_1,e_1) = -dN_2\ , h(e_1,e_2) = -k_1N_1 - l_1N_2 = -h(e_3,e_3), \tag{3.56a}$$
$$h(e_1,e_3) = -N_1 - cN_2,\ h(e_2,e_2) = 0 = h(e_2,e_3), \tag{3.56b}$$
$$h(e_\alpha,e_\alpha) = k_\alpha N_1 + l_\alpha N_2, \tag{3.56c}$$
$$h(e_\alpha,e_\beta) = 0 \text{ for } \alpha \neq \beta. \tag{3.56d}$$

Subsituiting (3.50), (3.51), (3.52), (3.53), (3.54) and (3.55) into (3.48), we have

$$A_{N_2} = \begin{bmatrix} l_1 & 0 & 0 & & & \\ d & l_1 & c & & & \\ -c & 0 & l_1 & & & \\ & & & l_4 & & \\ & & & & \ddots & \\ & & & & & l_n \end{bmatrix}. \tag{3.57}$$

Note that ∇f is proportional to e_2. So we have

$$e_1(f) \neq 0, \qquad e_i(f) = 0, \qquad \nabla f = -e_1(f)e_2 \tag{3.58}$$

where $i = 2, 3, \cdots, n$. Moreover, by (2.10) we have

$$-2k_1 = nf. \tag{3.59}$$

Note that $[e_2, e_\alpha](k_1) = e_2e_\alpha(k_1) - e_\alpha e_2(k_1)$. By (3.59) one can say

$$[e_2, e_\alpha](k_1) = 0 \tag{3.60}$$

So, the same process gives

$$[e_2, e_3](k_1) = [e_\alpha, e_\beta](k_1) = [e_3, e_\alpha](k_1) = 0. \tag{3.61}$$

Equations (3.60) and (3.61) give

$$\omega_{2\alpha}(e_2) = \omega_{23}(e_2) = 0, \tag{3.62}$$
$$\omega_{2\beta}(e_\alpha) = \omega_{2\alpha}(e_\beta), \tag{3.63}$$
$$\omega_{2\alpha}(e_3) = \omega_{23}(e_\alpha), \tag{3.64}$$

where $\alpha, \beta = 4, 5, \ldots, n$.

Now we use the Codazzi equation (2.6)

• The triplet $X = e_2, Y = Z = e_\alpha$ gives

$$e_2(k_\alpha) = \Phi_\alpha(k_1 - k_\alpha) \tag{3.65}$$
$$e_2(l_\alpha) = \Phi_\alpha(l_1 - l_\alpha) \tag{3.66}$$

,where $\Phi_\alpha = \omega_{2\alpha}(e_\alpha)$.

• The triplet $X = e_1, Y = e_\alpha, Z = e_\alpha$ gives

$$e_1(k_\alpha) = \psi_\alpha(k_1 - k_\alpha) + \omega_{\alpha 3}(e_\alpha) \tag{3.67}$$

• For $\alpha \neq \beta$, the triplet $X = e_\beta, Y = e_\alpha, Z = e_\alpha$ gives

$$e_\beta(k_\alpha) = \omega_{\beta\alpha}(e_\alpha)(k_\beta - k_\alpha). \tag{3.68}$$

Notice that for some α with $k_1 = k_\alpha \neq k_\beta$ we have

$$\omega_{\beta\alpha}(e_\alpha) = 0 \tag{3.69}$$

since (3.58).
• For $\alpha \neq \beta$, the triplet $X = e_2, Y = e_\beta, Z = e_\alpha$ gives

$$\omega_{\alpha\beta}(e_2)(k_\alpha - k_\beta) = -\omega_{2\alpha}(e_\beta)(k_\alpha - k_1) \tag{3.70}$$
$$\omega_{\alpha\beta}(e_2)(l_\alpha - l_\beta) = -\omega_{2\alpha}(e_\beta)(l_\alpha - l_1) \tag{3.71}$$

• For $\alpha \neq \beta$, the triplet $X = e_\alpha, Y = e_2, Z = e_\beta$ gives

$$\omega_{2\beta}(e_\alpha)(k_1 - k_\beta) = \omega_{\alpha\beta}(e_2)(k_\alpha - k_\beta) \tag{3.72}$$
$$\omega_{2\beta}(e_\alpha)(l_1 - l_\beta) = \omega_{\alpha\beta}(e_2)(l_\alpha - l_\beta) \tag{3.73}$$

By combining (3.72) and (3.70) we get

$$\omega_{2\beta}(e_\alpha)(k_1 - k_\beta) = \omega_{2\alpha}(e_\beta)(k_1 - k_\alpha) \tag{3.74}$$

Subsituiting (3.63) into (3.74) gives $\omega_{2\alpha}(e_\beta) = 0$. So we have

$$k_1 \neq k_\alpha = k_\beta \quad \Rightarrow \quad \omega_{2\beta}(e_\alpha) = \omega_{2\alpha}(e_\beta) = 0 \tag{3.75a}$$
$$k_\alpha \neq k_\beta \quad \Rightarrow \quad \omega_{2\beta}(e_\alpha) = \omega_{2\alpha}(e_\beta) = \omega_{\alpha\beta}(e_2) = 0 \tag{3.75b}$$

• Under the result of (3.75), the triplet $X = e_3, Y = e_\alpha, Z = e_\beta$ and $X = e_3, Y = e_\beta, Z = e_\alpha$ give

$$\omega_{\alpha\beta}(e_3)(k_\alpha - k_\beta) = \omega_{3\beta}(e_\alpha)(k_1 - k_\beta) = \omega_{3\alpha}(e_\beta)(k_1 - k_\alpha) \tag{3.76}$$

• The triplet $X = e_1, Y = e_\alpha, Z = e_\beta$ and $X = e_1, Y = e_\beta, Z = e_\alpha$ give

$$\omega_{\alpha\beta}(e_1)(k_\alpha - k_\beta) = \omega_{1\beta}(e_\alpha)(k_1 - k_\beta) + \omega_{\beta 3}(e_\alpha) = \omega_{1\alpha}(e_\beta)(k_1 - k_\alpha) + \omega_{\alpha 3}(e_\beta) \tag{3.77}$$

By combining (3.76) and (3.77), one can say easily, for $k_1 \neq k_\alpha = k_\beta$,

$$\omega_{3\beta}(e_\alpha) = \omega_{1\beta}(e_\alpha) = 0 \tag{3.78}$$

• The triplet $X = e_2, Y = e_1, Z = e_2$ gives

$$e_2(l_1) = 0 \tag{3.79}$$

• The triplet $X = e_2, Y = e_1, Z = e_3$ gives

$$\phi_2 = 0 \tag{3.80}$$

• Triplets $(e_\alpha, e_2, e_3), (e_2, e_3, e_\alpha)$ and $(e_\alpha, e_1, e_2), (e_\alpha, e_2, e_1)$ give

$$\begin{aligned} \omega_{3\alpha}(e_2)(k_1 - k_\alpha) &= 0 = \omega_{2\alpha}(e_3)(k_1 - k_\alpha) \\ \omega_{23}(e_\alpha) = \omega_{2\alpha}(e_1)(k_1 - k_\alpha) &= \omega_{\alpha 3}(e_2) - \omega_{1\alpha}(e_2)(k_1 - k_\alpha), \end{aligned}$$

respectively, So, for $k_1 \neq k_\alpha$, we have

$$\omega_{3\alpha}(e_2) = \omega_{2\alpha}(e_3) = \omega_{23}(e_\alpha) = 0 \tag{3.81}$$
$$\omega_{2\alpha}(e_1) = \omega_{\alpha 2}(e_1) = \omega_{1\alpha}(e_2) = 0, \tag{3.82}$$

respectively. Further, notice that (3.81) also holds for $k_1 = k_\alpha$. Now we use Gauss equations (2.4) by considering (3.70),(3.72), (3.80) and (3.82).

$$R(e_\alpha, e_1, e_2, e_\alpha) = -(k_1 k_\alpha + l_1 l_\alpha) \tag{3.83}$$

One can obtain, for $k_1 \neq k_\alpha$,

$$e_1(\Phi_\alpha) + \Phi_\alpha(\phi_1 + \psi_\alpha) = k_1 k_\alpha + l_1 l_\alpha \tag{3.84}$$

Lemma 3.5. *For $k_1 \neq k_\alpha$, if $\Phi_\alpha = 0$ then $k_1 k_\alpha = -l_1 l_\alpha$, where $\alpha = 4, , 5 \cdots , n$.*

Proof. Putting $\Phi_\alpha = 0$ into (3.84) gives the result that we want to show.

Now consider distinct principal curvatures $K_1, K_2, \cdots, K_p$ with its multiplicities $v_1, v_2, \cdots, v_p$ and $L_1, L_2, \cdots, L_q$ with its multiplicities $u_1, u_2, \cdots, u_q$ as in the previous section then we have (3.7) and (3.8) again. When taking a derivative of (3.7) and (3.8) alon e_2, p times and q times, respectively, one can get the following matrices;

$$\begin{bmatrix} \Phi_2 & \Phi_3 & \cdots & \Phi_p \\ \Phi_2^2 & \Phi_3^2 & \cdots & \Phi_p^2 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_2^p & \Phi_3^p & \cdots & \Phi_p^p \end{bmatrix} \begin{bmatrix} v_2(K_1 - K_2) \\ v_3(K_1 - K_3) \\ \vdots \\ v_p(K_1 - K_p) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{3.85}$$

$$\begin{bmatrix} \Phi_2 & \Phi_3 & \cdots & \Phi_q \\ \Phi_2^2 & \Phi_3^2 & \cdots & \Phi_q^2 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_1^q & \Phi_2^q & \cdots & \Phi_q^q \end{bmatrix} \begin{bmatrix} u_2(L_1 - L_2) \\ u_3(L_1 - L_3) \\ \vdots \\ u_q(L_1 - L_q) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.86}$$

Remark 3.3. The indice α starts at 2 for capital K_α while starting at 4 for lowercase k_α due to the matrix (3.47). The rest of this remark is the same with Remark 3.2.

Remark 3.4. Note that the using the method of Gauss equation for which we use to find a contradiction has not changed for the shape operator (3.47) either. So we obtain the same of Lemma 3.3. With the same logic, it is enough to prove that absence condition for $p = 2$ to obtain the same of Lemma 3.2, Lemma 3.4 and Theorem 3.1.

So we can give the following Lemma.

Lemma 3.6. *There is no biconservative PVMCV-MCGL submanifold in $\mathbb{E}_1^{n+2}$ with two distinct principal curvatures .*

Proof. Assume that A_{N_1} given in (3.47) has two distinct principal curvatures K_1, K_2 then A_{N_2} given in (3.48) has two distinct principal curvatures L_1, L_2 by Lemma 3.3. Consider (3.7) and (3.8), we have

$$v_2K_2 = -(2 + v_1)K_1 \tag{3.87}$$
$$u_2L_2 = -u_1L_1 \tag{3.88}$$

Taking a derivative (3.87) along e_2 give $\Phi_2 = 0$. It follows $K_1K_2 = -L_1L_2$ by Lemma 3.1. Multiplying (3.87) and (3.88) by K_1 and L_1, respectively and some direct calculations give

$$0 = \frac{2 + v_1}{v_2}K_1^2 + \frac{u_1}{u_2}L_1^2 \tag{3.89}$$

which yields a contradiction since then K_1 would be zero and it follows H is zero.

So one can prove easily that every elements Φ_α of the matrix (3.85) being zero with the help of induction and then get easily the same of Lemma 3.4 for Φ_α. Moreover, one can give the following final theorem whose proof is the same of Theorem 3.1 since it depends on the condition $K_1K_\alpha + L_1L_\alpha$ is zero which is satisfied for the shape operator given in (3.47) either.

Theorem 3.2. *There is no PNMCV-MCGL submanifold in Minkowski space $\mathbb{E}_1^{n+2}$ with nondiagonalizable shape operator.*

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References

[1] Caddeo, R., Montaldo, S., Oniciuc, C., Piu, P.: *Surfaces in the three-dimensional space forms with divergence-free stress-bienergy tensor*. Annali di Matematica Pura ed Applicata **193**, 529–550 (2014).
[2] Chen, B. Y. : *On the surface with parallel mean curvature vector*. Indiana University Mathematics Journal. **22**, 655-666 (1973).
[3] Chen, B.Y.: *Surfaces with parallel normalized mean curvature vector*. Monatshefte für Mathematik. **90**, 185-194 (1980).
[4] Chen, B.Y.: *Some open problems and conjectures on submanifold of finite type* . Soochow Journal of Mathematics. **17**, 169-188 (1991).
[5] Chen, B.Y., Ishikawa, S.: *Biharmonic surfaces in pseudo-Euclidean spaces*. Kyushu Journal of Mathematics. **2** (45), 323-347 (1991).
[6] Chen, B.Y., Ishikawa, S.: *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*. Kyushu Journal of Mathematics. **52**, 167-185 (1998).
[7] Chen, B.Y.: *Chen's biharmonic conjecture and submanifolds with parallel normalized mean curvature vector*. Mathematics. **7**, 710 (2019).
[8] Du L., Zhang, J.: *Biharmonic Submanifolds with Parallel Normalized Mean Curvature Vector Field in Pseudo-Riemannian Space Forms*. Bulletin of the Malaysian Mathematical Sciences Society. **42**, 1469-1484 (2019).
[9] Du, L.: *Classification of f-biharmonic submanifolds in Lorentz space forms*. Open Mathematics. **19**, 1299-1314 (2021).
[10] Fu, Y.: *Explicit classification of biconservative surfaces in Lorentz 3-space forms*. Annali di Matematica Pura ed Applicata. **194**, 805-822 (2015).
[11] Fu, Y.: *On bi-conservative surfaces in Minkowski 3-space*. Journal of Geometry and Physics. **66**, 71-79 (2013).
[12] Magid, M. A.: *Lorentzian Isoparametric Hypersurfaces*. Pacific Journal of Mathematics. **118**, 165-197 (1995).
[13] Montaldo, S., Oniciuc C., Ratto, A. :, *Biconservative surfaces*. Journal of Geometric Analysis. **26**, 313-329 (2016).
[14] Montaldo, S., Oniciuc C., Ratto, A.: *Proper biconservative immersions into the Euclidean space*. Annali di Matematica Pura ed Applicata. **195**, 403-422 (2016).
[15] Şen, R.: *Biconservative Submanifolds with Parallel Normalized Mean Curvature Vector Field in Euclidean Space*. Bulletin of the Iranian Mathematical Society. **48**, 3185-3194 (2022).
[16] Şen, R., Turgay, N.C.: *Biharmonic PNMCV submanifolds in Euclidean 5-space*, Turkish Journal of Mathematics. **47** , 296-316 (2023).
[17] Turgay, N.C.: *A classifcation of biharmonic hypersurfaces in the Minkowski spaces of arbitrary dimension*. Hacettepe Journal of Mathematics and Statistics. **45**, 1125-1134 (2016).

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