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On Rough $\mathscr{I}$-Convergence and $\mathscr{I}$-Cauchy Sequence for Functions Defined on Amenable Semigroups

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Abstract

In this paper, firstly we introduced the concepts of rough $\mathscr{I}$-convergence, rough $\mathscr{I}^*$-convergence, rough $\mathscr{I}$-Cauchy sequence, and rough $\mathscr{I}^*$-Cauchy sequence of a function defined on discrete countable amenable semigroups. Then, we investigated the relations between them.

1. Introduction

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [1] as a generalization of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of $\mathbb{N}$.
Phu [2] introduced, firstly, the notion of rough convergence in finite-dimensional normed spaces. In [2], he investigated some properties of $\text{LIM}^r x$ such as boundedness, closedness and convexity, and also he defined the notion of rough Cauchy sequence. Then, Phu [3] studied on rough convergence and some important properties of this concept. Furthermore, recently some authors [4–8] investigated the rough convergence types in some normed spaces.
In [9], Day studied on the concept of amenable semigroups (or briefly ASG). Then, some authors [10–12] studied the notions of summability in ASG. Douglas [13] extended the notion of arithmetic mean to ASG and obtained a characterization for almost convergence in ASG. In [14], Nuray and Rhoades presented the concepts of convergence and statistical convergence in ASG. Dündar et al. [15] and Dündar, Ulusu [16] introduced rough convergence and investigated some properties of rough convergence in ASG. Dündar, Ulusu [17] studied rough statistical convergence in ASG. Also, Dündar et al. [18] defined rough ideal convergence and some properties in ASG. Recently, some authors studied on the new concepts in ASG (see [19–22]).
First of all, we remember the basic definitions and concepts that we will use in our study such as amenable semigroups, rough convergence, rough ideal convergence, etc. (see [2, 3, 8–16, 18–24, 26, 27]).
Let a real number $r \geq 0$ and $\mathbb{R}^n$ (the real n-dimensional space) with the norm $\|.\|$, and a sequence $x = (x_k)_{k=0}^{n} \subset \mathbb{R}^n$.
A sequence (x_k) is said to be r-convergent to L, denoted by $x_k \xrightarrow{r} L$, provided that

$$\forall \varepsilon > 0 \; \exists k_\varepsilon \in \mathbb{N} : \; k \geq k_\varepsilon \Rightarrow \|x_k - L\| < r + \varepsilon.$$

The rough limit set of the sequence $x = (x_k)$ is showed by $\text{LIM}^r x = \{L \in \mathbb{R}^n : x_k \xrightarrow{r} L\}$.
A sequence $x = (x_k)$ is said to be r-convergent if $\text{LIM}^r x \neq \emptyset$ and r is called the convergence degree of the sequence (x_k). For $r = 0$, we get the ordinary convergence.
Let G be a discrete countable amenable semigroups (or briefly DCASG) with identity in which both left and right cancelation laws hold, and $w(G)$ denotes the space of all real valued functions on G.
If G is a countable amenable group, there exists a sequence $\{S_n\}$ of finite subsets of G such that

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(i) $G=\bigcup_{n=1}^{\infty} S_n$,
(ii) $S_n \subset S_{n+1}$ $(n=1,2,...)$,
(iii) $\lim\limits_{n\to\infty} \frac{|S_n g \cap S_n|}{|S_n|}=1$, $\lim\limits_{n\to\infty} \frac{|g S_n \cap S_n|}{|S_n|}=1$, for all $g\in G$.

If a sequence of finite subsets of G satisfy (i)-(iii), then it is called a Folner sequence (or briefly FS) of G.
Throughout the paper, we take G be a DCASG with identity in which both left and right cancelation laws hold.
For any FS $\{S_n\}$ of G, a function $f\in w(G)$ is said to be convergent to t if for every $\varepsilon>0$ there exists a $k_0=k_0(\varepsilon)\in\mathbb{N}$ such that $|f(g)-t|<\varepsilon$, for all $m>k_0$ and $g\in G\setminus S_m$.
Let $X\neq\emptyset$. A class $\mathscr{I}$ of subsets of X is said to be an ideal in X provided:

i) $\emptyset\in\mathscr{I}$,
ii) $A,B\in\mathscr{I}$ implies $A\cup B\in\mathscr{I}$,
iii) $A\in\mathscr{I}$, $B\subset A$ implies $B\in\mathscr{I}$.

$\mathscr{I}$ is called a nontrivial ideal if $X\notin\mathscr{I}$. A nontrivial ideal $\mathscr{I}$ in X is called admissible if $\{x\}\in\mathscr{I}$, for each $x\in X$.
Throughout the paper, we take $\mathscr{I}$ as an admissible ideal in $\mathbb{N}$.
Let $X\neq\emptyset$. A class $\emptyset\neq\mathscr{F}$ of subsets of X is said to be a filter in X provided:

i) $\emptyset\notin\mathscr{F}$,
ii) $A,B\in\mathscr{F}$ implies $A\cap B\in\mathscr{F}$,
iii) $A\in\mathscr{F}$, $A\subset B$ implies $B\in\mathscr{F}$.

If $\mathscr{I}$ is a nontrivial ideal in X, $X\neq\emptyset$, then the class

$$\mathscr{F}(\mathscr{I})=\{M\subset X:(\exists A\in\mathscr{I})(M=X\backslash A)\}$$

is a filter on X, called the filter associated with $\mathscr{I}$.
An admissible ideal $\mathscr{I}\subset 2^{\mathbb{N}}$ satisfies the property (AP), if for every countable family of mutually disjoint sets $\{A_1,A_2,\ldots\}$ belonging to $\mathscr{I}$, there exists a countable family of sets $\{B_1,B_2,\ldots\}$ such that $A_j\Delta B_j$ is a finite set for $j\in\mathbb{N}$ and $B=\bigcup_{j=1}^{\infty}B_j\in\mathscr{I}$ (hence $B_j\in\mathscr{I}$ for each $j\in\mathbb{N}$).
After then, we let $\mathscr{I}\subseteq 2^G$ be an admissible ideal for amenable semigroup G.
A function $f\in w(G)$ is said to be $\mathscr{I}$-convergent to s for any FS $\{S_n\}$ for G, if for every $\varepsilon>0$

$$\left\{g\in G:|f(g)-s|\geq\varepsilon\right\}\in\mathscr{I}.$$

In this case, we write $\mathscr{I}-\lim f(g)=s$.
A function $f\in w(G)$ is said to be $\mathscr{I}^*$-convergent to s, for any FS $\{S_n\}$ for G if there exists $M\subset G$, $M\in\mathscr{F}(\mathscr{I})$ (i.e., $G\setminus M\in\mathscr{I}$) and a $k_0=k_0(\varepsilon)\in\mathbb{N}$ such that for every $\varepsilon>0$ $|f(g)-s|<\varepsilon$, for all $n>k_0$ and all $g\in M\setminus S_n$. In this case, we write $\mathscr{I}^*-\lim f(g)=s$.
A function $f\in w(G)$ is said to be $\mathscr{I}$-Cauchy sequence, for any FS $\{S_n\}$ for G if for every $\varepsilon>0$, there exists an $h=h(\varepsilon)\in G$ such that

$$\left\{g\in G:|f(g)-f(h)|\geq\varepsilon\right\}\in\mathscr{I}.$$

A function $f\in w(G)$ is said to be $\mathscr{I}^*$-Cauchy sequence, for any FS $\{S_n\}$ for G if there exists $M\subset G$, $M\in\mathscr{F}(\mathscr{I})$ (i.e., $G\setminus M\in\mathscr{I}$) and a $k_0=k_0(\varepsilon)\in\mathbb{N}$ such that for every $\varepsilon>0$ $|f(g)-f(h)|<\varepsilon$, for all $n>k_0$ and $g,h\in M\setminus S_n$.
For any FS $\{S_n\}$ of G, a function $f\in w(G)$ is said to be rough convergent (r-convergent) to t if

$$\forall\varepsilon>0\ \exists k_\varepsilon\in\mathbb{N}:\ m\geq k_\varepsilon\Rightarrow|f(g)-t|<r+\varepsilon, \tag{1.1}$$

for all $g\in G\setminus S_m$ or equivalently if $\limsup|f(g)-t|\leq r$, for all $g\in G\setminus S_m$. In this instance, we write $r-\lim f(g)=t$ or $f(g)\xrightarrow{r}t$.
If (1.1) holds, then t is an r-limit point of the function $f\in w(G)$, which is usually no longer unique (for $r>0$). Hence, we have to think the so-called rough limit set (r-limit set) of the function $f\in w(G)$ defined by $\mathrm{LIM}^r f:=\{t:f(g)\xrightarrow{r}t\}$.
For any FS $\{S_n\}$ for G, the function $f\in w(G)$ is said to be r-convergent if $\mathrm{LIM}^r f\neq\emptyset$. In this instance, r is called the convergence degree of the $f\in w(G)$.
For any FS $\{S_n\}$ of G, a function $f\in w(G)$ is said to be a rough Cauchy sequence with roughness degree $\wp$, if $\forall\varepsilon>0\ \exists k_\varepsilon:\ m\geq k_\varepsilon\Rightarrow$ $|f(g)-f(h)|\leq\wp+\varepsilon$ is hold for $\wp>0$ and all $g,h\in G\setminus S_m$. $\wp$ is also said to be Cauchy degree of $f\in w(G)$.

2. Main Results

In this section, we introduced the concepts of rough $\mathscr{I}$-convergence, rough $\mathscr{I}^*$-convergence, rough $\mathscr{I}$-Cauchy sequence and rough $\mathscr{I}^*$-Cauchy sequence of a function defined on discrete countable amenable semigroups. Then, we investigated relations between them.

Definition 2.1. *For any FS $\{S_n\}$ of G, a function $f\in w(G)$ is said to be rough $\mathscr{I}$-convergent (r-$\mathscr{I}$-convergent) to s if for every $\varepsilon>0$*

$$\{g\in G:|f(g)-s|\geq r+\varepsilon\}\in\mathscr{I} \tag{2.1}$$

or equivalently if

$$\mathscr{I}-\limsup|f(g)-s|\leq r$$

is satisfied. In this instance, we write

$$r-\mathscr{I}-\lim f(g)=s \text{ or } f(g)\xrightarrow{r-\mathscr{I}}s.$$

On the other hand, we say that $f(g)\xrightarrow{r-\mathscr{I}}s$ if and only if the condition

$$|f(g)-s|\leq r+\varepsilon$$

holds for every $\varepsilon>0$ and almost $g\in G$.

In this convergence r is named the roughness degree. For $r=0$, we get the $\mathscr{I}$-convergence.
If (2.1) holds, then s is an r-$\mathscr{I}$-limit point of the function $f \in w(G)$, which is usually no longer unique (for $r>0$). Hence, we have to think the so-called rough $\mathscr{I}$-limit set of the function $f \in w(G)$ defined by

$$\mathscr{I}-\mathrm{LIM}^r f := \{s : f(g) \stackrel{r-\mathscr{I}}{\longrightarrow} s\}.$$

For any FS $\{S_n\}$ for G, the function $f \in w(G)$ is said to be r-$\mathscr{I}$-convergent if

$$\mathscr{I}-\mathrm{LIM}^r f \neq \emptyset.$$

If $\mathscr{I}-\mathrm{LIM}^r f \neq \emptyset$ for a function $f \in w(G)$, then we have

$$\mathscr{I}-\mathrm{LIM}^r f = [\mathscr{I}-\limsup f - r,\ \mathscr{I}-\liminf f + r].$$

Remark 2.2. *If $\mathscr{I}$ is an admissible ideal, then for a function $f \in w(G)$, usual rough convergence implies rough $\mathscr{I}$-convergence for any FS $\{S_n\}$ of G.*

Definition 2.3. *A function $f \in w(G)$ is said to be rough $\mathscr{I}$-Cauchy sequence, for any FS $\{S_n\}$ for G if for every $\varepsilon > 0$, there exists an $h = h(\varepsilon) \in G$ such that*

$$\left\{g \in G : |f(g) - f(h)| \geq r + \varepsilon\right\} \in \mathscr{I}.$$

Theorem 2.4. *If $f \in w(G)$ is rough $\mathscr{I}$-convergent for any FS $\{S_n\}$ for G, then it is rough $\mathscr{I}$-Cauchy for same sequence.*

Proof. For any Folner sequence $\{S_n\}$ for G, let

$$r-\mathscr{I}-\lim f(g) = s.$$

Then, for every $\varepsilon > 0$, we have

$$A_\varepsilon = \left\{g \in G : |f(g) - s| \geq r + \varepsilon\right\} \in \mathscr{I}.$$

Since $\mathscr{I}$ is an admissible ideal there exists an $h \in G$ such that $h \notin A_\varepsilon$. Now, let

$$B_\varepsilon = \left\{g \in G : |f(g) - f(h)| \geq 2(r + \varepsilon)\right\}.$$

Taking into account the inequality

$$|f(g) - f(h)| \leq |f(g) - s| + |f(h) - s|,$$

we observe that if $g \in B_\varepsilon$, then

$$|f(g) - s| + |f(h) - s| \geq 2(r + \varepsilon).$$

On the other hand, since $h \notin A_\varepsilon$ we have

$$|f(h) - s| < r + \varepsilon$$

and so

$$|f(g) - s| > r + \varepsilon.$$

Hence, $g \in A_\varepsilon$ and so we have

$$B_\varepsilon \subset A_\varepsilon \in \mathscr{I}.$$

Thus, $B_\varepsilon \in \mathscr{I}$ that is, f is rough $\mathscr{I}$-Cauchy sequence. □

Definition 2.5. *A function $f \in w(G)$ is said to be rough $\mathscr{I}^*$-convergent to s, for any FS $\{S_n\}$ for G if there exists $M \subset G$, $M \in \mathscr{F}(\mathscr{I})$ (i.e., $G \setminus M \in \mathscr{I}$) and a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$*

$$|f(g) - s| < r + \varepsilon, \tag{2.2}$$

for all $n > k_0$ and all $g \in M \setminus S_n$. In this case, we write

$$r-\mathscr{I}^*-\lim f(g) = s.$$

In this convergence r is named the roughness degree. For $r=0$, we get the $\mathscr{I}^*$-convergence.
If (2.2) holds, then s is an r-$\mathscr{I}^*$-limit point of the function $f \in w(G)$, which is usually no longer unique (for $r>0$).
Hence, we have to think the so-called rough $\mathscr{I}^*$-limit set of the function $f \in w(G)$ defined by

$$\mathscr{I}^*-\mathrm{LIM}^r f := \{s : f(g) \stackrel{r-\mathscr{I}^*}{\longrightarrow} s\}.$$

For any FS $\{S_n\}$ for G, the function $f \in w(G)$ is said to be r-$\mathscr{I}^*$-convergent if

$$\mathscr{I}^*-\mathrm{LIM}^r f \neq \emptyset.$$

Theorem 2.6. *If $f \in w(G)$ is rough $\mathscr{I}^*$-convergent to s, then f is rough $\mathscr{I}$-convergent to s for any FS $\{S_n\}$ for G.*

Proof. For any FS $\{S_n\}$ for G, let

$$r - \mathscr{I}^* - \lim f(g) = s.$$

Then, there exists $M \subset G$, $M \in \mathscr{F}(\mathscr{I})$ (i.e., $H = G \setminus M \in \mathscr{I}$) and a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$|f(g) - s| < r + \varepsilon,$$

for all $n > k_0$ and all $g \in M \setminus S_n$. Therefore obviously,

$$A(\varepsilon) = \left\{ g \in G : |f(g) - s| \geq r + \varepsilon \right\} \subset H \cup S_{k_0}.$$

Since $\mathscr{I}$ is admissible,

$$H \cup S_{k_0} \in \mathscr{I}$$

and so

$$A(\varepsilon) \in \mathscr{I}.$$

Hence,

$$r - \mathscr{I} - \lim f(g) = s.$$

□

Definition 2.7. *A function $f \in w(G)$ is said to be rough $\mathscr{I}^*$-Cauchy sequence, for any FS $\{S_n\}$ for G if there exists $M \subset G$, $M \in \mathscr{F}(\mathscr{I})$ (i.e., $G \setminus M \in \mathscr{I}$) and a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$*

$$|f(g) - f(h)| < r + \varepsilon,$$

for all $n > k_0$ and $g, h \in M \setminus S_n$.

Theorem 2.8. *If $f \in w(G)$ is rough $\mathscr{I}^*$-Cauchy for any FS $\{S_n\}$ for G, then it is rough $\mathscr{I}$-Cauchy for same sequence.*

Proof. Let $f \in w(G)$ be an rough $\mathscr{I}^*$-Cauchy for any FS $\{S_n\}$ for G. Then by definition, there exists $M \subset G$, $M \in \mathscr{F}(\mathscr{I})$ (i.e., $G \setminus M \in \mathscr{I}$) and a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$|f(g) - f(h)| < r + \varepsilon,$$

for all $n > k_0$ and $g, h \in M \setminus S_n$. Let $H = G \setminus M$. It is clearly $H \in \mathscr{I}$ and

$$A(\varepsilon) = \left\{ g \in G : |f(g) - f(h)| \geq r + \varepsilon \right\} \subset H \cup S_{k_0}.$$

Since $\mathscr{I}$ is admissible,

$$H \cup S_{k_0} \in \mathscr{I}$$

and so

$$A(\varepsilon) \in \mathscr{I}.$$

Consequently, f is rough $\mathscr{I}$-Cauchy for same sequence. □

Following theorems show relationships between $\mathscr{I}$-convergence and $\mathscr{I}^*$-convergence, between $\mathscr{I}$-Cauchy sequence and $\mathscr{I}^*$-Cauchy sequence. These theorems can be proved like in [19,25], these theorems are given without the proof.

Theorem 2.9. *Let $\mathscr{I} \subset 2^G$ be an admissible ideal with the property (AP). If $f(g) \in w(G)$ is rough $\mathscr{I}$-convergent to s, then f is rough $\mathscr{I}^*$-convergent to s for any FS $\{S_n\}$ for G.*

Theorem 2.10. *Let $\mathscr{I} \subset 2^G$ be an admissible ideal with the property (AP). If $f \in w(G)$ is rough $\mathscr{I}$-Cauchy for any FS $\{S_n\}$ for G, then it is rough $\mathscr{I}^*$-Cauchy for same sequence.*

3. Conclusion

In this paper, we introduced the concepts of rough $\mathscr{I}$-convergence, rough $\mathscr{I}^*$-convergence, rough $\mathscr{I}$-Cauchy sequence and rough $\mathscr{I}^*$-Cauchy sequence of a function defined on discrete countable amenable semigroups. Also, we investigated relations between them. Then after, The concepts given here can also be studied for double sequences.

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