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Rough Convergence of Double Sequences in $n-$Normed Spaces

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Abstract

In this study, we introduced the concepts of rough convergence, rough Cauchy double sequence, and the set of rough limit points of a double sequence, as well as the rough convergence criteria associated with this set in n-normed spaces. Later, we proved that this set is both closed and convex. Finally, we presented the relationships between rough convergence and rough Cauchy double sequence in n-normed spaces.

Keywords: *Cauchy sequence, Double sequence, $n-$normed space, Rough convergence*

AMS Subject Classification (2020): *40A05; 40A35*

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1. Introduction

The concept of 2-normed spaces was initially introduced by Gähler [1, 2] in 1960. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [3] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Şahiner et al. [4] and Gürdal [5] studied $\mathcal{I}$-convergence in 2-normed spaces. Gürdal and Açık [6] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Also Çakallı and Ersan [7] studied new types of continuity in 2-normed spaces. Misiak [8] extended 2-normed spaces to $n-$normalized spaces. Since then, many researchers have studied this concept and obtained various results [9, 10]. Later, some studies on 2-normed spaces were transferred to $n-$normed spaces. For example, Reddy [11] investigated statistical convergence, the statistical Cauchy sequence and some properties of statistical convergence in $n-$normed spaces. Hazarika and Savaş [12] introduced the concept of λ-statistical convergence in $n-$normed spaces. They established some inclusion relations between the sets of statistically convergent and $\lambda-$statistically convergent sequences in [12]. Gürdal and Şahiner [13] studied ideal convergence in $n-$normed spaces and presented the main results.

In finite-dimensional normed spaces, Phu [14] was the first to present the concept of rough convergence. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in some normed linear space $(X, \|.\|)$ and r be a nonnegative real number, then $(x_i)_{i\in\mathbb{N}}$ is

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said to be $r-$convergent to x_*, denoted by $x_i \xrightarrow{r} x_*$, provided that

$$\forall\varepsilon > 0,\ \exists i_\varepsilon \in \mathbb{N}:\ i \geq i_\varepsilon \Rightarrow \|x_i - x_*\| < r + \varepsilon.$$

Also, the sequence (x_k) is said to be a rough Cauchy sequence satisfying

$$\forall\varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : k, m \geq K_\varepsilon \Rightarrow \|x_k - x_m\| < \rho + \varepsilon$$

for $\rho > 0$. ρ is roughness degree of (x_k). Shortly (x_k) is called a rough Cauchy sequence. ρ is also a Cauchy degree of (x_k). In [14], he showed that the set $\text{LIM}^r x$ is bounded, closed, and convex, and he introduced the notion of rough Cauchy sequence. He also investigated the relationships between rough convergence and other types of convergence, as well as the dependence of $\text{LIM}^r x$ on the roughness degree r. In another paper [15] related to this subject, he defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is $r-$continuous at every point $x \in X$ under the assumption $dimY < \infty$ and $r > 0$, where X and Y are normed spaces. In [16], he extended the results given in [14] to infinite-dimensional normed spaces. Aytar [17] studied rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [18] studied that the $r-$limit set of the sequence is equal to the intersection of these sets and that $r-$core of the sequence is equal to the union of these sets. In later times, Arslan and Dündar [19, 20] introduced the notions of rough convergence, rough Cauchy sequence, and the set of rough limit points of a sequence and obtained the rough convergence criteria associated with this set in 2-normed space first, then presented their work "On rough convergence in 2-normed spaces and some properties." They [21, 22] also examined rough statistical convergence and rough statistical cluster points in 2-normed spaces. Sunar and Arslan [23] introduced the concept of rough convergence in $n-$normed spaces by combining the concepts of rough convergence and $n-$normed spaces.

Pringsheim [24, 25] developed the idea of convergence for double sequences. He gave some examples of the convergence of double sequences with and without the usual convergence of rows and columns and defined the $P-$limit. $\mathbb{N}$ and $\mathbb{R}$ are used throughout the paper to denote the sets of all positive integers and all real numbers, respectively.

A double sequence $(x_{tk})_{t,k\in\mathbb{N}}$ in some linear space $(X, \|.\|)$ is said to converge to a point $L \in X$ in Pringsheim's sense, denoted by $(x_{tk}) \to L$, if for any $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - L\| < \varepsilon \text{ for all } t, k \geq K_\varepsilon.$$

Further, a double sequence $(x_{tk})_{t,k\in\mathbb{N}}$ is said to be a Cauchy double sequence if for any $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - x_{mv}\| < \varepsilon \text{ for all } t, k, m, v \geq K_\varepsilon.$$

Contrary to the property of convergence in ordinary sequences, it is an important problem that convergent double sequences do not have to be bounded. Hardy [26] introduced the concept of regular convergence, which also needed the convergence of the rows and columns of a pair in addition to the Pringsheim convergence. Hence, this problem was eliminated. Later, many researchers used double sequences in their works in the area of summability theory. This work can be found in [27–32]. Malik and Maity [33] defined and examined rough convergence of double sequences, the set of $r-$limit points of double sequences and rough Cauchy double sequences. These concepts, given by Malik and Maity [33], are as follows:

Let (x_{tk}) be a double sequence in a normed space $(X, \|.\|)$ and r be a non-negative real number. (x_{tk}) is $r-$convergent to L in X, denoted by $x_{tk} \xrightarrow{\|.\|}_r L$ if

$$\forall\varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : t, k \geq K_\varepsilon \Rightarrow \|x_{tk} - L\| < r + \varepsilon.$$

A double sequence (x_{tk}) is called a rough Cauchy sequence with roughness degree ρ if for any $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - x_{mv}\| < \rho + \varepsilon, \text{ for all } t, k, m, v \geq K_\varepsilon.$$

Dündar and Çakan [34, 35] introduced the notions of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence and studied the notions of rough convergence and the set of rough limit points of a double sequence. Also, Kişi and Dündar [36] presented the notion of rough $\mathcal{I}_2-$lacunary statistical limit set of a double sequence.

By combining the concepts of rough convergence, double sequences and $n-$normed spaces, we introduce the concept of rough convergence of double sequences in $n-$normed spaces. We obtain two convergence criteria associated with the set of rough limit points of a double sequence in $n-$normed spaces. Later, we prove that this set is both closed and convex. Finally, we investigate the relationships between a double sequence's cluster points and its rough limit points. The results and proof techniques presented in this paper are analogous to those presented in Phu's [14] paper. The concept of convergent double sequences given in our paper is used in the sense of Pringsheim. So a convergent double sequence may not be bounded. Namely, the actual origin of most of these results and proof techniques are the papers. The following theorems and results are extensions of the theorems and results in [14]. Currently, we recall the idea of $n-$normed spaces, some fundamental definitions, and notations.(See [8, 10, 11, 30, 33, 37]).

Definition 1.1. [37] Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ on X^n satisfying the following properties for all $y, z, x_1, x_2, \cdots, x_{n-1}, x_n \in X$

(i) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, x_2 \cdots, x_n$ are linearly dependent,

(ii) $\|x_1, x_2, \cdots, x_n\|$ is invariant under any permutation of $x_1, x_2, \cdots, x_n$,

(iii) $\|x_1, x_2, \cdots, x_{n-1}, \alpha x_n\| = |\alpha| \|x_1, x_2, \cdots, x_{n-1}, x_n\|$ for all $\alpha \in \mathbb{R}$,

(iv) $\|x_1, x_2, \cdots, x_{n-1}, y + z\| \leq \|x_1, x_2, \cdots, x_{n-1}, y\| + \|x_1, x_2, \cdots, x_{n-1}, z\|$

is called an $n-$norm on X, and the pair $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is called an $n-$normed space.

An example of an $n-$normed space is $X = \mathbb{R}^n$ equipped with the followig Euclidean $n-$norm:

Example 1.1.

$$\|x_1, x_2 \cdots, x_{n-1}, x_n\|_E = | \, det(x_{ij}) \, | = abs \left(\begin{vmatrix} x_{11} \ldots & x_{1n} \\ x_{21} \ldots & x_{2n} \\ & \\ x_{n1} \ldots & x_{nn} \end{vmatrix} \right)$$

where $x_i = (x_{i1}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$.

In this study, we suppose X to be an $n-$normed space having dimension d; where $2 \leq d < \infty$.

Definition 1.2. [37] A sequence (x_k) in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to be convergent to L in X if

$$\lim_{k \to \infty} \|x_k - L, z_2, \cdots, z_n\| = 0$$

for every $z_2, \cdots, z_n \in X$. In such a case, we write $\lim\limits_{k \to \infty} x_k = L$ and call L the limit of (x_k).

Example 1.2. [23] Let $x = (x_k) = (\frac{k}{k+1}, \frac{1}{k}, \ldots, \frac{1}{k})$, $L = (1, 0, \ldots, 0)$ and $z = (z_1, z_2, \ldots, z_n)$. It is clear that (x_k) is convergent to $L = (1, 0, \ldots, 0)$ in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$.

Definition 1.3. [37] A sequence (x_k) in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to be a Cauchy sequence in X if for every $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_k - x_m, z_2, z_3, \ldots, z_n\| < \varepsilon$$

for all $k, m \geq K_\varepsilon$ and every $z_2, z_3, \ldots, z_n \in X$.

Definition 1.4. [23] Let (x_k) be a sequence in $n-$normed linear space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ and r be a non-negative real number. (x_k) is said to be rough convergent (r-convergent) to L if

$$\forall \varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : k \geq K_\varepsilon \Rightarrow \|x_k - L, z_2, \cdots, z_n\| < r + \varepsilon$$

for every $z_2, \cdots, z_n \in X$.

Definition 1.5. [23] Let (x_k) be a sequence in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$. (x_k) is said to be a rough Cauchy sequence satisfying

$$\forall \varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : k, m \geq K_\varepsilon \Rightarrow \|x_k - x_m, z_2, \cdots, z_n\| < \rho + \varepsilon$$

for $\rho > 0$ and every $z_2, \cdots, z_n \in X$. ρ is roughness degree of (x_k).

Definition 1.6. (cf. [33]) A double sequence (x_{tk}) in $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to be bounded if there exists a non-negative real number M such that $\|x_{tk}, z_2, \cdots, z_n\| < M$ for all $t, k \in \mathbb{N}$.

Definition 1.7. [30] A double sequence (x_{tk}) in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to be convergent to $L \in X$ if for each $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - L, z_2, \cdots, z_n\| < \varepsilon$$

for all $t, k \geq K_\varepsilon$ and every $z_2, \cdots, z_n \in X$.

Definition 1.8. [30] A double sequence (x_{tk}) in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to be a Cauchy sequence if for each $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$

$$\|x_{tk} - x_{mv}, z_2, \cdots, z_n\| < \varepsilon$$

for all $t, k, m, v \geq K_\varepsilon$ and every $z_2, \cdots, z_n \in X$.

2. Main results

We introduced the concepts of rough convergence, rough Cauchy double sequence and the set of rough limit points set of a double sequence in this work and we obtained the rough convergence criteria associated with this set in n-normed space. We later demonstrated that this set is both closed and convex. Finally, we investigated the relationships between rough convergence and rough Cauchy double sequence in n-normed spaces.

Definition 2.1. Let (x_{tk}) be a double sequence in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ and r be a non-negative real number. (x_{tk}) is said to be rough convergent ($r-$convergent) to L denoted by $x_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L$ if

$$\forall \varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : t, k \geq K_\varepsilon \Rightarrow \|x_{tk} - L, z_2 \cdots, z_n\| < r + \varepsilon \tag{2.1}$$

for every $z_2, \cdots, z_n \in X$.

If (2.1) holds, L is an $r-$limit point of (x_{tk}), which is usually no more unique (for $r > 0$). So, we have to consider the so-called $r-$limit set (or shortly $r-$limit) of (x_{tk}) defined by

$$\mathrm{LIM}_n^r x_{tk} := \{L \in X : x_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L\}. \tag{2.2}$$

A double sequence (x_{tk}) is said to be $r-$convergent if $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$. In this case, r is called the convergence degree of the double sequence (x_{tk}). For $r = 0$ we have the classical convergence in $n-$normed space again. But our proper interest is the case $r > 0$. There are several reasons for this interest. For instance, since an originally convergent double sequence (y_{tk}) (with $y_{tk} \to L$) in $n-$normed space often cannot be determined (i.e., measured or calculated) exactly, one has to do with an approximated double sequence (x_{tk}) satisfying

$$\|x_{tk} - y_{tk}, z_2, \cdots, z_n\| \leq r$$

for all n and every $z_2, z_3, \ldots, z_n \in X$, where $r > 0$ is an upper bound of approximation error. Then, (x_{tk}) is no more convergent in the classical sense, but for every $z_2, \cdots z_n \in X$,

$$\|x_{tk} - L, z_2, \cdots, z_n\| \leq \|x_{tk} - y_{tk}, z_2, \cdots, z_n\| + \|y_{tk} - L, z_2, \cdots, z_n\| \leq r + \|y_{tk} - L, z_2, \cdots, z_n\|$$

implies that (x_{tk}) is $r-$convergent in the sense of (2.1).

Example 2.1. The double sequence $(x_{tk}) = ((-1)^{tk}, (-1)^{tk}, \ldots, (-1)^{tk})$ is not convergent in n-normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$, but it is rough convergent to $L = (0, 0, \ldots, 0)$ for every $z_2, \cdots, z_n \in X$. It is clear that

$$\mathrm{LIM}_n^r x_{tk} = \begin{cases} \emptyset, \text{ if } \quad r < 1 \\ [(-r, -r, \ldots, -r), (r, r, \ldots, r)], \text{ otherwise.} \end{cases}$$

Sometimes we are interested in the set of $r-$limit points lying in a given subset $D \subset X$, which is called $r-$limit in D and denoted by

$$\mathrm{LIM}_n^{D,r} x_{tk} := \{L \in D : x_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L\}. \tag{2.3}$$

It is clear that

$$\mathrm{LIM}_n^{X,r} x_{tk} = \mathrm{LIM}_n^{r} x_{tk} \ \ and \ \ \mathrm{LIM}_n^{D,r} x_{tk} = D \cap \mathrm{LIM}_n^{r} x_{tk}.$$

First, let us transform some properties of classical convergence to rough convergence in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$. It is well known if a sequence converges then its limit is unique. This property is maintained for rough convergence with roughness degree $r > 0$, but only has the following analogy.

Theorem 2.1. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. We have $diam(\mathrm{LIM}_n^r x_{tk}) \leq 2r$. In general, $diam(\mathrm{LIM}_n^r x_{tk})$ has no smaller bound.*

Proof. We have to show that

$$diam(\mathrm{LIM}_n^r x_{tk}) = sup \left\{\|x_1 - x_2, z_2, \cdots, z_n\| : x_1, x_2 \in \mathrm{LIM}_n^r x_{tk} \leq 2r\right\}, \tag{2.4}$$

where $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is an $n-$normed space and for every $z_2, \cdots, z_n \in X$. Assume the contrary that

$$diam(\mathrm{LIM}_n^r x_{tk}) > 2r.$$

Then, there exist $x_1, x_2 \in \mathrm{LIM}_n^r x_{tk}$ satisfying

$$d := \|x_1 - x_2, z_2, \ldots, z_n\| > 2r$$

for every $z_2, z_3, \cdots, z_n \in X$. For an arbitrary $\varepsilon \in (0, \frac{d-2r}{2})$, it follows from (2.1) and (2.2) that there is a $K_\varepsilon \in \mathbb{N}$ such that for $t, k \geq K_\varepsilon$,

$$\|x_{tk} - x_1, z_2, \ldots, z_n\| < r + \varepsilon \ \ and \ \ \|x_{tk} - x_2, z_2, \ldots, z_n\| < r + \varepsilon$$

for every $z_2, z_3, \ldots, z_n \in X$. This implies

$$\begin{aligned} \|x_1 - x_2, z_2, \ldots, z_n\| &\leq \|x_{tk} - x_1, z_2, \ldots, z_n\| + \|x_{tk} - x_2, z_2, \ldots, z_n\| \\ &< 2(r + \varepsilon) \\ &< 2r + 2(\frac{d - 2r}{2}) \\ &= d \end{aligned}$$

for every $z_2, z_3, \ldots, z_n \in X$, which conflicts with $d = \|x_1 - x_2, z_2, \ldots, z_n\|$. Hence, (2.4) must be true. Consider a convergent double sequence (x_{tk}) with $\lim\limits_{t,k\to\infty} x_{tk} = L$. Then, for

$$\overline{B}_r(L) := \{x_1 \in X : \|x_1 - L, z_2, z_3, \ldots, z_n\| \leq r\}$$

it follows from

$$\begin{aligned} \|x_{tk} - x_1, z_2, z_3, \ldots, z_n\| &\leq \|x_{tk} - L, z_2, z_3, \ldots, z_n\| + \|L - x_1, z_2, z_3, \ldots, z_n\| \\ &\leq \|x_{tk} - L, z_2, z_3, \ldots, z_n\| + r \end{aligned}$$

for every $z_2, z_3, \ldots, z_n \in X$ and for $x_1 \in \overline{B}_r(L)$, from (2.1) and (2.2) that

$$\mathrm{LIM}_n^r x_{tk} = \overline{B}_r(L).$$

Since $diam(\overline{B}_r(L)) = 2r$, this shows that in general the upper bound $2r$ of the diameter of an $r-$limit set cannot be decreased anymore. □

Obviously the uniqueness of limit (of classical convergence) can be regarded as a special case of latter property, because if $r = 0$ then $diam(\mathrm{LIM}_n^r x_{tk}) = 2r = 0$, that is, $\mathrm{LIM}_n^r x_{tk}$ is either empty or a singleton.

The following property shows an analogy between boundedness and rough convergence of a double sequence in $n-$normed space.

Theorem 2.2. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. If the double sequence (x_{tk}) is bounded then there exists an $r \geq 0$, such that $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$.*

$$\mathrm{LIM}_n^{(x_{t_v k_s}),r} x_{t_v k_s} \neq \emptyset.$$

Proof. For every $z_2, z_3, \ldots, z_n \in X$ if

$$s := sup\{\|x_{tk}, z_2, z_3, \ldots, z_n\| : t, k \in \mathbb{N}\} < \infty.$$

Then, $\mathrm{LIM}_n^s x_{tk}$ contains the origin of X. So, $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$. □

The converse of the previous theorem might not hold true since a convergent double sequence is not always bounded. Let's now introduce the notion of loosely boundedness for $n-$normed spaces, which is analogous to [33].

Definition 2.2. A double sequence (x_{tk}) in X is said to be loosely bounded if there exist an $M \in \mathbb{R}^+$ and a $K \in \mathbb{N}$ such that $\|x_{tk}, z_2, z_3, \ldots, z_n\| < M$ for all $t, k \geq K$.

Every bounded double sequence is obviously loosely bounded, but the converse is not true.

Theorem 2.3. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. The double sequence (x_{tk}) is loosely bounded if and only if there exists an $r \geq 0$, such that $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$.*

Proof. Let (x_{tk}) be a loosely bounded double sequence. Then there exist an $M \in \mathbb{R}^+$ and a $K \in \mathbb{N}$ such that $\|x_{tk}, z_2, z_3, \ldots, z_n\| < M$ for all $t, k \geq K$. Then, $\mathrm{LIM}_n^M x_{tk}$ contains the origin of X. So, $\mathrm{LIM}_n^M x_{tk} \neq \emptyset$.

Conversely, let $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$ for some $r \geq 0$. Let $L \in \mathrm{LIM}_n^r x_{tk}$. We take $\varepsilon = 1$. Then there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - L, z_2, z_3, \ldots, z_n\| < r + 1 \text{ for all } t, k \geq K_\varepsilon.$$

So, (x_{tk}) is loosely bounded. □

Now let $(t_i)_{i \in \mathbb{N}}$ and $(k_j)_{j \in \mathbb{N}}$ be two strictly increasing sequences of natural numbers. If $(x_{tk})_{t,k \in \mathbb{N}}$ is a double sequence in $(X, \|\bullet, \bullet, \ldots, \bullet\|)$, then we can define $(x_{t_i k_j})_{i,j \in \mathbb{N}}$ as a subsequence of $(x_{tk})_{t,k \in \mathbb{N}}$. (See, [33]).

Proposition 2.1. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. If $(x_{t_i k_j})$ is a subsequence of (x_{tk}) then,*

$$\mathrm{LIM}_n^r x_{tk} \subseteq \mathrm{LIM}_n^r x_{t_i k_j}$$

in n-normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$.

Proof. Let $L \in \mathrm{LIM}_n^r x_{tk}$. Then for any $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - L, z_2, z_3, \ldots, z_n\| < r + \varepsilon$$

for all $t, k \geq K_\varepsilon$ and every $z_2, z_3, \ldots, z_n \in X$. Since (t_i) and (k_j) are strictly increasing sequences, so there exists a $k_0 \in \mathbb{N}$ such that $t_{k_0} > K_\varepsilon$ and $k_{k_0} > K_\varepsilon$. Therefore, we get

$$\|x_{t_i k_j} - L, z_2, z_3, \ldots, z_n\| < r + \varepsilon$$

for all $t_i, k_j \geq K_\varepsilon$ and every $z_2, z_3, \ldots, z_n \in X$. So, $L \in \mathrm{LIM}_n^r x_{t_i k_j}$. □

Theorem 2.4. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. For all $r \geq 0$, the $r-$limit set $\mathrm{LIM}_n^r x_{tk}$ of an arbitrary double sequence (x_{tk}) is closed.*

Proof. Let (y_{sv}) be an arbitrary double sequence in $\mathrm{LIM}_n^r x_{tk}$ which converges to some point L. For each $\varepsilon > 0$ and every $z_2, z_3, \ldots, z_n \in X$, by definition there exist $m_{\varepsilon/2}, k_{\varepsilon/2} \in \mathbb{N}$ such that

$$\|y_{m_{\varepsilon/2}} - L, z_2, z_3, \ldots, z_n\| < \frac{\varepsilon}{2} \ \ and \ \ \|x_{tk} - y_{m_{\varepsilon/2}}, z_2, z_3, \ldots, z_n\| < r + \frac{\varepsilon}{2}$$

whenever $k \geq k_{\varepsilon/2}$. Consequently for every $z_2, z_3, \ldots, z_n \in X$,

$$\begin{aligned} \|x_{tk} - L, z_2, \ldots, z_n\| &\leq \|x_{tk} - y_{m_{\varepsilon/2}}, z_2, \ldots, z_n\| + \|y_{m_{\varepsilon/2}} - L, z_2, \ldots, z_n\| \\ &< r + \varepsilon \end{aligned}$$

for $k \geq k_{\varepsilon/2}$. That means $L \in \mathrm{LIM}_n^r x_{tk}$, too. Hence, $\mathrm{LIM}_n^r x_{tk}$ is closed. □

Theorem 2.5. *Let* $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ *be an* $n-$*normed space and consider a double sequence* $(x_{tk}) \in X$. *If*

$$y_0 \in \mathrm{LIM}_n^{r_0} x_{tk} \ \ and \ \ y_1 \in \mathrm{LIM}_n^{r_1} x_{tk},$$

then,

$$y_\alpha := (1-\alpha)y_0 + \alpha y_1 \in \mathrm{LIM}_n^{(1-\alpha)r_0+\alpha r_1} x_{tk}, \quad for \ \ \alpha \in [0,1].$$

Proof. By definition, for every $\varepsilon > 0$, $r_0, r_1 > 0$ and every $z_2, z_3, \ldots, z_n \in X$ there exists a $K_\varepsilon \in \mathbb{N}$ such that $t, k > K_\varepsilon$ implies

$$\|x_{tk} - y_o, z_2, \ldots, z_n\| < r_0 + \varepsilon \ \ and \ \ \|x_{tk} - y_1, z_2, \ldots, z_n\| < r_1 + \varepsilon,$$

which yields also, for every $z_2, z_3, \ldots, z_n \in X$,

$$\begin{aligned} \|x_{tk} - y_\alpha, z_2, z_3, \ldots, z_n\| &\leq (1-\alpha)\|x_{tk} - y_o, z_2, z_3, \ldots, z_n\| + \alpha\|x_{tk} - y_1, z_2, z_3, \ldots, z_n\| \\ &< (1-\alpha)(r_0+\varepsilon) + \alpha(r_1+\varepsilon) \\ &= (1-\alpha)r_0 + \alpha r_1 + \varepsilon. \end{aligned}$$

Hence, we have

$$y_\alpha \in \mathrm{LIM}_n^{(1-\alpha)r_0+\alpha r_1} x_{tk}.$$

□

Theorem 2.6. *Let* $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ *be an* $n-$*normed space and consider a double sequence* $(x_{tk}) \in X$. $\mathrm{LIM}_n^r x_{tk}$ *is convex.*

Proof. In particular, for $r = r_0 = r_1$, Theorem 2.5 yields immediately that $\mathrm{LIM}_n^r x_{tk}$ is convex. □

Theorem 2.7. *If* $x_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L_1$ *and* $y_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L_2$. *Then,*

(i) $(x_{tk} + y_{tk}) \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r (L_1 + L_2)$ *and*

(ii) $\alpha(x_{tk}) \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r \alpha L_1$, $(\alpha \in \mathbb{R})$.

Proof. *(i)* By definition for every $z_2, z_3, \ldots, z_n \in X$,

$$\forall \varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : t, k \geq K_\varepsilon \Rightarrow \|x_{tk} - L_1, z_2, z_3, \ldots, z_n\| < r_1 + \frac{\varepsilon}{2}$$

and

$$\forall \varepsilon > 0, \exists J_\varepsilon \in \mathbb{N} : t, k \geq J_\varepsilon \Rightarrow \|y_{tk} - L_2, z_2, z_3, \ldots, z_n\| < r_2 + \frac{\varepsilon}{2}.$$

Let $j = max\{K_\varepsilon, J_\varepsilon\}$ and $r_1 + r_2 = r$. For every $t, k > j$ and every $z_2, z_3, \ldots, z_n \in X$ we have

$$\begin{aligned} \|(x_{tk} + y_{tk}) - (L_1 + L_2), z_2, z_3, \ldots, z_n\| &= \|x_{tk} - L_1, z_2, z_3, \ldots, z_n\| + \|y_{tk} - L_2, z_2, z_3, \ldots, z_n\| \\ &< r_1 + \frac{\varepsilon}{2} + r_2 + \frac{\varepsilon}{2} \\ &= r + \varepsilon \end{aligned}$$

and so

$$(x_{tk} + y_{tk}) \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r (L_1 + L_2).$$

(ii) It is obvious for $\alpha = 0$. Let $\alpha \neq 0$. Since

$$x_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L_1$$

for every $\varepsilon > 0$ and every $z_2, z_3, \ldots, z_n \in X$, $\exists K_\varepsilon \in \mathbb{N}$ such that for every $t, k \geq K_\varepsilon$, we have

$$\|x_{tk} - L_1, z_2, z_3, \ldots, z_n\| < \frac{r + \varepsilon}{|\alpha|}.$$

According to this, for $\forall t, k \geq K_\varepsilon$ and every $z_2, z_3, \ldots, z_n \in X$, we can write

$$\begin{aligned} \|\alpha x_{tk} - \alpha L_1, z_2, z_3, \ldots, z_n\| &= |\alpha| \|x_{tk} - L_1, z_2, z_3, \ldots, z_n\| \\ &< |\alpha| \frac{r+\varepsilon}{|\alpha|} \\ &= r + \varepsilon. \end{aligned}$$

So,

$$(\alpha x_{tk}) \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r \alpha L_1.$$

□

Following, we give some relations between convergence and rough convergence of double sequences in $n-$normed space.

Theorem 2.8. *Let $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ be an $n-$normed space and condiser a double sequence (x_{tk}) in X. If c is a cluster point of (x_{tk}), then $\|L - c, z_2, \cdots, z_n\| \leq r$ for every $L \in \mathrm{LIM}_n^r x_{tk}$.*

Proof. Let $L \in \mathrm{LIM}_n^r x_{tk}$. Assume the contrary that $d := \|L - c, z_2, \cdots, z_n\| > r$. Let $\varepsilon = \dfrac{d-r}{2}$. Since $L \in \mathrm{LIM}_n^r x_{tk}$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - L, z_2, \cdots, z_n\| < r + \varepsilon$$

for all $t, k \geq K_\varepsilon$ and every $z_2, \cdots, z_n \in X$. Then we write

$$\|L - c, z_2, \cdots, z_n\| \leq \|x_{tk} - L, z_2, \cdots, z_n\| + \|x_{tk} - c, z_2, \cdots, z_n\|$$

for all $t, k \geq K_\varepsilon$ and every $z_2, \cdots, z_n \in X$. If we rewrite the inequality, we get

$$\begin{aligned} \|x_{tk} - c, z_2, \cdots, z_n\| &\geq \|L - c, z_2, \cdots, z_n\| - \|x_{tk} - L, z_2, \cdots, z_n\| \\ &> d - (r + \frac{d-r}{2}) \\ &= \varepsilon \end{aligned}$$

for all $t, k \geq K_\varepsilon$ and every $z_2, \cdots, z_n \in X$ which contradicts that c is a cluster point. So $\|L - c, z_2, \cdots, z_n\| \leq r$ for every $L \in \mathrm{LIM}_n^r x_{tk}$. □

Theorem 2.9. *Let $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ be an $n-$normed space and condiser a double sequence (x_{tk}) in X. Then (x_{tk}) converges to $L \in X$ if and only if $\mathrm{LIM}_n^r x_{tk} = \overline{B}_r(L)$.*

Proof. The first part of the proof is obtained directly from the second part of Theorem 2.1, that is, if (x_{tk}) converges to $L \in X$, then $\mathrm{LIM}_n^r x_{tk} = \overline{B}_r(L)$. Let us now show the second part of the theorem.

Conversely, let $\mathrm{LIM}_n^r x_{tk} = \overline{B}_r(L)$. Now let's show that (x_{tk}) converges to L, that is, for every $\alpha > 0$, there exists a $K_\alpha \in \mathbb{N}$ such that $\|x_{tk} - L, z_2, \cdots, z_n\| \leq \alpha$ for all $t, k \geq K_\alpha$ and every $z_2, \cdots, z_n \in X$. Now we can take a fixed $\alpha > 0$, such that $r + \varepsilon < \alpha$ for $r > 0$ and $\varepsilon > 0$. For $L \in \mathrm{LIM}_n^r x_{tk}$, there exists a $K_\alpha \in \mathbb{N}$ such that

$$\|x_{tk} - L, z_2, \cdots, z_n\| < r + \varepsilon < \alpha$$

for all $t, k \geq K_\alpha$ and every $z_2, \cdots, z_n \in X$. Therefore (x_{tk}) converges to $L \in X$. □

Definition 2.3. Let (x_{tk}) be a double sequence in $n-$normed space $(X, \|\bullet,\bullet,\ldots,\bullet\|)$. (x_{tk}) is said to be a rough Cauchy double sequence with roughness degree ρ, if

$$\forall\varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : m, v, t, k \geq K_\varepsilon \Rightarrow \|x_{mv} - x_{tk}, z_2, z_3, \ldots, z_n\| < \rho + \varepsilon$$

is hold for $\rho > 0$, $L \in X$ and every $z_2, z_3, \ldots, z_n \in X$. ρ is also called a Cauchy degree of (x_{tk}).

Proposition 2.2. *(i) Monotonicity: Assume $\rho' > \rho$. If ρ is a Cauchy degree of a given double sequence (x_{tk}) in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$, so ρ' is a Cauchy degree of (x_{tk}).*

(ii) Boundedness: A double sequence (x_{tk}) is loosely bounded if and only if there exists a $\rho \geq 0$ such that (x_{tk}) is a $\rho-$Cauchy double sequence in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$.

Theorem 2.10. *If (x_{tk}) is rough convergent in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$, i.e., $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$ if and only if (x_{tk}) is a ρ-Cauchy double sequence for every $\rho \geq 2r$. This bound for the Cauchy degree cannot be generally decreased.*

Proof. A Cauchy double sequence is loosely bounded. By Theorem 2.3, (x_{tk}) is rough convergent, that is, $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$. So, it is sufficient to prove the first part of the theorem. Let L be any point in $\mathrm{LIM}_n^r x_{tk}$. Then, for all $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that $m, v, t, k \geq K_\varepsilon$ implies

$$\|x_{mv} - L, z_2, z_3, \ldots, z_n\| \leq r + \frac{\varepsilon}{2} \;\; \textit{and} \;\; \|x_{tk} - L, z_2, z_3, \ldots, z_n\| \leq r + \frac{\varepsilon}{2}$$

for every $z_2, z_3, \ldots, z_n \in X$. Therefore, for $m, v, t, k \geq K_\varepsilon$, we have

$$\begin{aligned} \|x_{mv} - x_{tk}, z_2, z_3, \ldots, z_n\| &= \|x_{mv} - L + L - x_{tk}, z_2, z_3, \ldots, z_n\| \\ &\leq \|x_{mv} - L, z_2, z_3, \ldots, z_n\| + \|L - x_{tk}, z_2, z_3, \ldots, z_n\| \\ &\leq r + \frac{\varepsilon}{2} + r + \frac{\varepsilon}{2} \\ &= 2r + \varepsilon \end{aligned}$$

for every $z_2, z_3, \ldots, z_n \in X$. Hence, (x_{tk}) is a ρ-Cauchy double sequence for $\rho \geq 2r$. By Proposition 2.2, every $\rho \geq 2r$ is also a Cauchy degree of (x_{tk}). It is clear that this bound $2r$ can not be generally decreased, similar to Proposition 5.1 in [16]. □

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