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Research Article

Maximal extensions of a linear functional

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ABSTRACT. Extensions of a positive hermitian linear functional ω, defined on a dense *-subalgebra $\mathfrak{A}_o$ of a topological *-algebra $\mathfrak{A}[\tau]$ are analyzed. It turns out that their maximal extensions as linear functionals or hermitian linear functionals are everywhere defined. The situation however changes deeply if one looks for positive extensions. The case of fully positive and widely positive extensions considered in [2] is revisited from this point of view. Examples mostly taken from the theory of integration are discussed.

Keywords: Positive linear functionals, topological *-algebras, extension of linear functionals.

2020 Mathematics Subject Classification: 46H05, 46H35.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we continue the analysis, undertaken in [2], [3] of the possibility of extending a positive hermitian linear functional ω, defined on a dense *-subalgebra $\mathfrak{A}_0$ of a topological *-algebra (in general, without unit), with topology τ and continuous involution *, to some elements of $\mathfrak{A}$. Moreover, we resume the notion of positive regular slight extension that closely reminds the construction of the Lebesgue integral or Segal's construction of noncommutative integration [16].

If we take, for instance, as $\mathfrak{A}$ the *-algebra of Lebesgue measurable functions on a compact interval X of $\mathbb{R}$ with the topology of convergence in measure, and $\mathfrak{A}_0 := C(X)$ is the *-algebra of continuous functions on X, then the Lebesgue integral ω_L provides an extension of the Riemann integral on $\mathfrak{A}_0$, which we regard as a positive linear functional on $\mathfrak{A}_0$. This extension is not unique as the literature on Integration Theory shows (think of Denjoy, Perron or Henstock-Kurzweil integrals see e.g. [7, 8, 1]). Thus, in an abstract set-up it makes sense to consider extensions enjoying appropriate properties. As in [2] and [3], the starting point is the notion of *slight extension*, which is treated for general linear maps in Köthe's book [6]. As application of the developed ideas, we report interesting results concerning infinite sums (see [2]).

In this paper, after showing that maximal extensions of linear functionals are necessarily everywhere defined, we revisit widely positive, fully positive and absolute convergent extensions already discussed in [2] and prove several new features that emerge from the discussion. Applications to extensions of Riemann integral on continuous functions are also examined.

We will adopt the following definitions and terminology. If $\mathfrak{A}$ is an arbitrary *-algebra, we put

$$\mathfrak{A}_h = \{b \in \mathfrak{A} : b = b^*\}, \quad \mathcal{P}(\mathfrak{A}) = \left\{ \sum_{i=1}^{n} a_i^* a_i : a_i \in \mathfrak{A} \right\}.$$

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Elements of $\mathfrak{A}_h$ are called *self-adjoint*; elements of $\mathcal{P}(\mathfrak{A})$ are called *positive*. Clearly, $\mathcal{P}(\mathfrak{A}) \subseteq \mathfrak{A}_h$. A linear functional ω, defined on a subspace $D(\omega)$ of $\mathfrak{A}$, is called

hermitian: if $a \in D(\omega) \Leftrightarrow a^* \in D(\omega)$ and $\omega(a^*) = \overline{\omega(a)}$, for every $a \in D(\omega)$;
positive: if $\omega(b) \geq 0$, for every $b \in D(\omega) \cap \mathcal{P}(\mathfrak{A})$.

Throughout this paper, we denote by ω a positive hermitian linear functional defined on a dense *-subalgebra $\mathfrak{A}_0$ of a topological *-algebra $\mathfrak{A}[\tau]$, with continuous involution *.

2. Maximal extensions

The problem of finding extensions of ω to larger subspaces of $\mathfrak{A}$ has, in some situations, easy solutions, namely when ω is τ-continuous or equivalently, closable [3, 17, 19], as discussed in the Appendix. For this reason, we will only consider the case of nonclosable i.e., discontinuous ω and we denote by G_ω the graph of ω:

$$G_\omega = \{(a, \omega(a)) \in \mathfrak{A}_0 \times \mathbb{C};\ a \in \mathfrak{A}_0\}.$$

The linear functional ω is closable if $\overline{G_\omega}$, the closure of G_ω, does not contain couples $(0, \ell)$ with $\ell \neq 0$. It turns out that a linear functional is closable if and only if it is continuous (see the Appendix). Let $\mathcal{S}_\omega$ denote the collection of all subspaces H of $\mathfrak{A} \times \mathbb{C}$ such that

(g1) $G_\omega \subseteq H \subseteq \overline{G_\omega}$;
(g2) $(0, \ell) \in H$ if, and only if, $\ell = 0$.

If ω is nonclosable, i.e. $\overline{G_\omega}$ contains pairs $(0, \ell)$ with $\ell \neq 0$, then $\overline{G_\omega} \notin \mathcal{S}_\omega$. To every $H \in \mathcal{S}_\omega$, there corresponds an extension ω_H, to be called a *slight* extension of ω, defined on

$$D(\omega_H) = \{a \in \mathfrak{A} : (a, \ell) \in H\}$$

by

$$\omega_H(a) = \ell,$$

where, from (g2), ℓ is the unique complex number such that $(a, \ell) \in H$. Moreover, by applying Zorn's lemma to the family $\mathcal{S}_\omega$, one proves that ω *admits a maximal slight extension*.

Remark 2.1. *The construction relies on the fact that if $a \notin \mathfrak{A}_0$ and $(a, \ell) \in \overline{G_\omega}$, then $H := G_\omega \oplus \langle (a, \ell) \rangle \in \mathcal{S}_\omega$ and we can construct the extension ω_H. Now if $a' \notin D(\omega_H)$ and $(a', \ell') \in \overline{G_\omega}$, then $H' := H \oplus \langle (a', \ell') \rangle \in \mathcal{S}_\omega$ and we can construct a new extension $\omega_{H'}$. Continuing in this way, at the end (i.e. invoking Zorn's lemma), we will find a maximal extension of ω.*

Using the same notations of Köthe's book [6], we put

$$\mathcal{K}_\omega := \{a \in \mathfrak{A} : (a, \ell) \in \overline{G_\omega},\ \text{for some } \ell \in \mathbb{C}\}.$$

The following propositions hold [2, 3, 6].

Proposition 2.1. *Let ω be nonclosable. If there exists $m \in \mathbb{C}$ such that $(a, m) \in \overline{G_\omega}$, then $(a, \ell) \in \overline{G_\omega}$ for every $\ell \in \mathbb{C}$, hence $\overline{G_\omega} = \mathcal{K}_\omega \times \mathbb{C}$.*

From this (see Remark 2.1), follows the next:

Proposition 2.2. *If ω is nonclosable and $\mathfrak{A}_0$ is a proper subspace of $\mathcal{K}_\omega$, then ω admits infinitely many maximal extensions.*

Furthermore,

Proposition 2.3. *For every maximal extension $\breve{\omega}$ of ω, $D(\breve{\omega}) = \mathcal{K}_\omega$.*

Corollary 2.1. *An extension $\breve{\omega}$ is maximal if and only if $D(\breve{\omega}) = \mathcal{K}_\omega$.*

Remark 2.2. *A stronger consequence actually comes from previous results. If $\widehat{\omega}$ is an extension of ω, and $a \in \mathcal{K}_\omega \setminus D(\widehat{\omega})$, then for any fixed $\ell \in \mathbb{C}$ there exists a maximal hermitian extension $\breve{\omega}$ of $\widehat{\omega}$ such that $\breve{\omega}(a) = \ell$; so we can choose arbitrarily the value that an extension takes at a.*

To construct hermitian extensions (see [2, 3]), we define $\mathcal{H}_\omega$ as the collection of all subspaces $H \in \mathcal{S}_\omega$ for which the following additional condition holds

(h3) $(a, \ell) \in H$ if and only if $(a^*, \overline{\ell}) \in H$,

and then we proceed like in the case of the construction of slight extensions. In [2], it is proved that all maximal hermitian extensions share the same domain. More precisely:

Proposition 2.4. *Every ω admits a maximal hermitian extension $\breve{\omega}$ which is, at once, a maximal extension so $D(\breve{\omega}) = \mathcal{K}_\omega$. Moreover, if ω is nonclosable and $\mathfrak{A}_0$ is a proper subspace of $\mathcal{K}_\omega$, then ω admits infinitely many maximal hermitian extensions.*

Remark 2.3. *Let ω_1 be a hermitian extension of ω and let $a \in \mathcal{K}_\omega \setminus D(\omega_1)$. If we want to extend ω_1, so that ω_2 is a hermitian extension of ω_1, in general we cannot choose arbitrarily the value $\omega_2(a)$. Indeed let $a = b + ic$ with $b = b^*$ $c = c^*$, and suppose $b \notin D(\omega_1)$. Then we can choose arbitrarily the real value $\ell_1 \in \mathbb{R}$ so that $\omega_2(b) = \ell_1$. Now if $c \notin \operatorname{span}\{D(\omega_1), b\}$, then we can choose arbitrarily the real value $\ell_2 = \omega_2(c)$, but if $c \in \operatorname{span}\{D(\omega_1), b\}$, then the value ℓ_2 is already fixed. The same argument can be made in the case $c \notin D(\omega_1)$.*

3. Some interesting situations

3.1. **Extensions of the Riemann integral.** Let $X = [0,1]$, $\mathfrak{A}$ be the *-algebra of Lebesgue measurable functions on X, τ be the topology of convergence in measure, $\mathfrak{A}_0 = C(X)$ be the *-algebra of all continuous functions on X and ω be the Riemann integral i.e.

$$\omega(f) := \int_0^1 f(x)\, dx.$$

It is well-known that the Riemann integral is nonclosable. To see this, let us consider the sequence

$$h_n(x) := \begin{cases} 2n(1-nx) & \text{if } 0 \le x \le 1/n, \\ 0 & \text{if } 1/n < x \le 1. \end{cases} \tag{3.1}$$

Then $h_n \to 0$ almost everywhere and hence in measure, but

$$\int_0^1 h_n(x)dx = 1, \; \forall n \ge 1.$$

Recall that we have defined

$$G_\omega = \{(a, \omega(a)) \in \mathfrak{A}_0 \times \mathbb{C};\ a \in \mathfrak{A}_0\}.$$

We will prove the following:

Theorem 3.1. *Let ω be the Riemann integral on a compact interval $I \subseteq \mathbb{R}$. Given $g \in \mathfrak{A}$, then $(g, \ell) \in \overline{G_\omega}$, for every $\ell \in \mathbb{C}$. Hence $\overline{G_\omega} = \mathfrak{A} \times \mathbb{C}$, thus $\mathcal{K}_\omega = \mathfrak{A}$.*

Proof. We can suppose, without loss of generality, that $I = [0,1]$ so we can use the previous sequence (3.1). As $\mathfrak{A}_0$ is dense in $\mathfrak{A}$, then there exists a sequence $(f_n) \subseteq \mathfrak{A}_0$, such that $f_n \to g$, and we put $\lambda_n := \omega(f_n)$. Fixed $\ell \in \mathbb{C}$, let $\alpha_n := \ell - \lambda_n$. Then $\alpha_n h_n \to 0$, $f_n + \alpha_n h_n \to g$ and $\omega(f_n + \alpha_n h_n) = \omega(f_n) + \alpha_n \omega(h_n) = \lambda_n + \ell - \lambda_n = \ell$, $\forall n$. Then $(g, \ell) \in \overline{G_\omega}$, hence $\overline{G_\omega} = \mathfrak{A} \times \mathbb{C}$. □

From Proposition 2.2, Proposition 2.3, Proposition 2.4 and Theorem 3.1, it follows the next

Theorem 3.2. *Let ω be the Riemann integral on a compact interval $I \subseteq \mathbb{R}$, then ω admits infinitely many maximal hermitian extensions with domain the whole algebra and any maximal hermitian extension of the Riemann integral has the whole algebra $\mathfrak{A}$ as domain.*

Example 3.1. *Let ω be the Riemann integral on $[0,1]$, h_n given by (3.1) and let $c(x): \ [0,1] \to \mathbb{C}$ be the following function:*

$$c(x) := \begin{cases} 0 & \text{if } \ x = 0, \\ 1/x & \text{otherwise.} \end{cases}$$

Then $c \in \mathfrak{A} \setminus D(\omega)$.
Let $f_n(x): \ [0,1] \to \mathbb{C}, \ n \geq 1$, be the following sequence of functions:

$$f_n(x) := \begin{cases} n^2 x & \text{if } \ 0 \leq x \leq 1/n, \\ 1/x & \text{otherwise.} \end{cases}$$

Then $f_n \to c$ pointwise and hence in measure, $f_n \in \mathfrak{A}_0, \ \forall\, n \geq 1$ and

$$\omega(f_n) = \int_0^1 f_n(x)dx = 1/2 + \log(n).$$

Fixed $\ell \in \mathbb{C}$, let $\alpha_n := \ell - \omega(f_n) = \ell - (1/2 + \log(n))$. Then $\alpha_n h_n \to 0$, $f_n + \alpha_n h_n \to c$, and $\omega(f_n + \alpha_n h_n) = \omega(f_n) + \alpha_n \omega(h_n) = \ell$, $\forall n$. Since $f_n + \alpha_n h_n \in \mathfrak{A}_0$ $\forall n$, then $(c, \ell) \in \overline{G_\omega}$ and so, for any $\ell \in \mathbb{C}$, there exists a maximal hermitian extension $\widehat{\omega}$ of ω such that $\widehat{\omega}(c) = \ell$.

Remark 3.4. *The previous Example 3.1 first shows explicitly the construction used in Theorem 3.1, pointing out that any $a \in \mathfrak{A}$ is in $\mathcal{K}_\omega$; then, by Remark 2.2, it shows that, if $a \notin D(\omega)$, then $\forall \ell \in \mathbb{C}$ there exists a maximal hermitian extension $\breve{\omega}$ of ω such that $\breve{\omega}(a) = l$. We note that even if $c(x) \in \mathcal{P}(\mathfrak{A})$, we can choose $\ell < 0$. This shows that the previous construction could be inappropriate for most useful situations. As we will see later, we will be able to construct maximal positive extensions of the Riemann integral but, it is possible to prove that there are not positive extensions $\widehat{\omega}$ of the Riemann integral such that the function $c(x)$ is in the domain of $\widehat{\omega}$ (see Example 5.2).*

3.2. **The case of infinite sums.** Let $\mathfrak{S}$ denote the complex vector space of all infinite sequences of complex numbers. $\mathfrak{S}$ is a *-algebra if the product $\mathbf{a} \cdot \mathbf{b}$ of two sequences $\mathbf{a} = (a_k)$, $\mathbf{b} = (b_k)$, $k \geq 1$, is defined componentwise and the involution by $\mathbf{a}^* = (\overline{a_k})$. Let us endow $\mathfrak{S}$ with the topology defined by the set of seminorms

$$p_k(\mathbf{a}) = |a_k|, \quad \mathbf{a} = (a_k) \in \mathfrak{S}.$$

Let $\mathfrak{S}_0$ denote the *-subalgebra of $\mathfrak{S}$ consisting of all *finite* sequences in the sense that $\mathbf{a} = (a_k) \in \mathfrak{S}_0$ if, and only if, there exists $N \in \mathbb{N}$ such that $a_k = 0$ if $k > N$. We define

$$\omega(\mathbf{a}) = \sum_{k=1}^{\infty} a_k, \quad \mathbf{a} = (a_k) \in \mathfrak{S}_0.$$

The symbol of series is only *graphic* since all sums are finite. This functional, which is obviously positive hermitian, is nonclosable. To see this, let us consider the sequence of sequences $(\mathbf{a}_n) = ((a_{n,k})) \subseteq \mathfrak{S}_0$ with, for $n \geq 1$,

$$a_{n,k} := \delta_{n,k} \quad \text{(the Kronecker delta).}$$

For fixed k, clearly $\lim_{n\to\infty} a_{n,k} = 0$. Hence $\mathbf{a}_n \to \mathbf{0}$ as $n \to \infty$ and, applying ω, we get

$$\omega(\mathbf{a}_n) = \omega((a_{n,k})) = 1, \quad \forall n \geq 1.$$

We observe that any convergent series which converges to $l \in \mathbb{C}$, can be "rewritten" as a sequence of sequences $(\mathbf{a}_n) \subseteq \mathfrak{S}_0$, with $\mathbf{a}_n \to \mathbf{0}$ and $\omega(\mathbf{a}_n) \to l$, as $n \to \infty$. Indeed, given the series $c_1 + c_2 + c_3 \ldots$ converging to l, we define $(\mathbf{a}_n) = ((a_{n,k}))$, for $n \geq 1$, as follows:

$$a_{n,k} := \begin{cases} c_{n+1-k} & \text{if } k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

Clearly $((a_{n,k})) \subseteq \mathfrak{S}_0$ and $\omega((a_{n,k})) = c_1 + c_2 + \cdots + c_n$. Since the series is convergent, for fixed k, $a_{n,k} \to 0$ as $n \to \infty$ and, finally, $\omega((a_{n,k})) \to l$ as $n \to \infty$.

The next proposition shows that in this case $\mathcal{K}_\omega$ is not a proper subset of the algebra.

Proposition 3.5. *Let $\mathfrak{S}$ and ω be as above. Then $\mathfrak{S}_0$ is a dense subalgebra of $\mathfrak{S}$ and $\mathcal{K}_\omega = \mathfrak{S}$.*

Proof. See Proposition 4.2 of [2]. □

Now, it seems interesting to us to show another example in which $\mathcal{K}_\omega$ coincides with the entire algebra $\mathfrak{A}$. Starting with a subalgebra of $\mathfrak{S}$ and changing the topology with a finer one, we will find a new topological *-algebra $\mathfrak{S}_1$. Then, taking the closure of $\mathfrak{S}_0$ in $\mathfrak{S}_1$, we will obtain the required algebra $\mathfrak{A} \subseteq \mathfrak{S}_1$.

We point out that in the following we will adopt notations that are not the usual ones. Let us consider the subalgebra $\mathfrak{S}_1 \subseteq \mathfrak{S}$ of all bounded sequences $x = (x_k)$, endowed with the norm

$$\|x\|_\infty = \sup_k |x_k|.$$

Then $\mathfrak{S}_1$ is a topological (precisely, a Banach) *-algebra with $\mathfrak{S}_0 \subseteq \mathfrak{S}_1$. In [2], it is shown first that the closure of $\mathfrak{S}_0$ in $\mathfrak{S}_1$ is the algebra $\mathfrak{A} := \{(c_k) \in \mathfrak{S}_1 \ : \ |c_k| \to 0 \text{ as } k \to \infty\}$; then it is shown that ω is a nonclosable positive hermitian linear functional defined on $\mathfrak{S}_0$ (a dense *-subalgebra of the topological *-algebra $\mathfrak{A}$); finally it is shown (see Proposition 4.10 of [2]) that, even in this case, $\mathcal{K}_\omega = \mathfrak{A}$.

4. The domain of maximal extensions

As seen in Theorem 3.2 for the case of the Riemann integral and in Proposition 3.5 for the infinite sums, all maximal extensions of a nonclosable linear functional have the same domain. The following theorem generalizes this statement to the abstract case.

Theorem 4.3. *Let ω be nonclosable. Then $\overline{G_\omega} = \mathfrak{A} \times \mathbb{C}$. Hence $\mathcal{K}_\omega = \mathfrak{A}$.*

Proof. As ω is nonclosable, then there exists a net $(a_\alpha)_{\alpha \in \Gamma} \subseteq \mathfrak{A}_0$, such that $a_\alpha \xrightarrow{\tau} 0$ and $\omega(a_\alpha) \to l \in \mathbb{C}$, with $l \neq 0$. Now let $b \in \mathfrak{A}$. Since $\mathfrak{A}_0$ is a dense subalgebra of $\mathfrak{A}$, there exists a net $(b_\alpha)_{\alpha \in \Gamma} \subseteq \mathfrak{A}_0$, such that $b_\alpha \xrightarrow{\tau} b$: indeed, since $\mathfrak{A}$ is a topological vector space, we can choose as unique set of indices Γ, the class of all neighbourhoods of 0, directed by inclusion. Now, since $l \neq 0$, there exist subnets $(a_\gamma) \subseteq (a_\alpha)$, $(b_\gamma) \subseteq (b_\alpha)$, $\gamma \in \Gamma_1 \subseteq \Gamma$, such that:

- $a_\gamma \xrightarrow{\tau} 0$;
- $\omega(a_\gamma) \neq 0$, $\forall\, \gamma$;
- $b_\gamma \xrightarrow{\tau} b$.

Let $\lambda \in \mathbb{C}$ and, for each γ, let $\lambda'_\gamma := \lambda - \omega(b_\gamma) \in \mathbb{C}$. We assert that there exists a monotone function $h : \Gamma_1 \to \Gamma_1$, such that

$$\lambda'_\gamma \cdot a_{h(\gamma)} \xrightarrow{\tau} 0.$$

Indeed if $\lambda'_\gamma = 0$ we put $h(\gamma) = \gamma$; otherwise, for every neighbourhood U of the origin, $U' := 1/\lambda'_\gamma\, U$ is still a neighbourhood of the origin, so there exists $\gamma' \geq \gamma$ such that $a_{\gamma'} \in U'$, and therefore $\lambda'_\gamma \cdot a_{\gamma'} \in U$. Now, since $a_\gamma \xrightarrow{\tau} 0$, put $h(\gamma) := \gamma'$, we have $\lambda'_\gamma \cdot a_{h(\gamma)} \xrightarrow{\tau} 0$. Since, obviously, $\omega(a_{h(\gamma)}) \to l$ then:

$$\frac{\lambda'_\gamma}{\omega(a_{h(\gamma)})} \cdot a_{h(\gamma)} \to 0.$$

Hence:

$$\frac{\lambda'_\gamma}{\omega(a_{h(\gamma)})} \cdot a_{h(\gamma)} + b_\gamma \to b \tag{4.2}$$

and

$$\omega\left(\frac{\lambda'_\gamma}{\omega(a_{h(\gamma)})} \cdot a_{h(\gamma)} + b_\gamma\right) = \frac{\lambda - \omega(b_\gamma)}{\omega(a_{h(\gamma)})} \cdot \omega(a_{h(\gamma)}) + \omega(b_\gamma) = \lambda, \ \forall\gamma. \tag{4.3}$$

Therefore (3.1) and (4.2) imply that $\forall b \in \mathfrak{A}$, and $\forall\lambda \in \mathbb{C}$, $(b, \lambda) \in \overline{G_\omega}$, from which the statement follows. □

By Proposition 2.4, the analogue of Theorem 3.2 is the following:

Theorem 4.4. *Let ω be nonclosable. Then ω admits infinitely many maximal hermitian extensions with domain the whole algebra $\mathfrak{A}$ and any maximal hermitian extension of ω has the whole algebra $\mathfrak{A}$ as domain.*

5. Widely positive and fully positive extensions

We have proved that all maximal extensions of a nonclosable linear functional ω are defined on the whole algebra $\mathfrak{A}$. This leads to a significant simplification on the the notion of *widely positive* and *fully positive* extension introduced in [2]. By Theorem 4.3, the definitions can be lightened and, in this way, several new developments emerge.

Definition 5.1. *Given ω, we define $\mathcal{P}_\omega$ as the collection of all subspaces $K \in \mathcal{H}_\omega$ satisfying the following additional condition*

(p4) *$(a, \ell) \in K$ and $a \in \mathcal{P}(\mathfrak{A})$, implies $\ell \geq 0$.*

Since ω is positive, then $\mathcal{P}_\omega \neq \emptyset$ and $G_\omega \subseteq K \subseteq \mathfrak{A}$ for every $K \in \mathcal{P}_\omega$. To every $K \in \mathcal{P}_\omega$, there corresponds a hermitian extension ω_K of ω, defined on

$$D(\omega_K) = \{a \in \mathfrak{A} : (a, \ell) \in K\}$$

by

$$\omega_K(a) = \ell, \quad a \in D(\omega_K),$$

where, from $(g2)$ of Section 2, ℓ is the unique complex number such that $(a, \ell) \in K$. By (p4), ω_K is a positive hermitian extension of ω. We observe that $\mathfrak{A}_0 \subseteq D(\omega_K) \subseteq \mathfrak{A}$ as vector spaces. Since $\mathcal{P}_\omega$ satisfies the assumptions of Zorn's lemma, we have the following:

Theorem 5.5. *Every positive hermitian linear functional ω admits a maximal positive hermitian extension.*

Definition 5.2. *Let $\widehat{\omega}$ be an extension of ω defined on the domain $D(\widehat{\omega})$ with $\mathfrak{A}_0 \subseteq D(\widehat{\omega}) \subseteq \mathfrak{A}$. We say that $\widehat{\omega}$ is* fully positive *if $\widehat{\omega}$ is positive and $D(\widehat{\omega}) \supseteq \mathcal{P}(\mathfrak{A})$.*

For $a, b \in \mathfrak{A}_h$, we define

$$a \leq b \Leftrightarrow b - a \in \mathcal{P}(\mathfrak{A}).$$

Remark 5.5. *Let $\widehat{\omega}$ be a hermitian extension of ω, $a \in D(\widehat{\omega})$ and $c \in \mathfrak{A}_h$. If $b := \pm(a-c) \in \mathfrak{A}_h$, then $\widehat{\omega}(a) \in \mathbb{R}$. Indeed if $b \in \mathfrak{A}_h$, then $a = c \pm b \in \mathfrak{A}_h$ and so, by the hermiticity of $\widehat{\omega}$, $\widehat{\omega}(a) \in \mathbb{R}$. Moreover if $\widehat{\omega}$ is a positive hermitian extension of ω and $a, c \in D(\widehat{\omega}) \cap \mathfrak{A}_h$ with $a \geq c$, put $b := a - c$, then $b \in \mathcal{P}(\mathfrak{A}) \cap D(\widehat{\omega})$, so $\widehat{\omega}(a) = \widehat{\omega}(c) + \widehat{\omega}(b) \geq \widehat{\omega}(c)$. Hence $\widehat{\omega}$ is monotone on $D(\widehat{\omega}) \cap \mathfrak{A}_h$.*

If $\widehat{\omega}$ is a positive hermitian extension of ω and $c \in \mathfrak{A}_h$, then by Remark 5.5, we can introduce (see [2]) the following notations that will use both to characterize the elements for which it is possible to find a positive hermitian extension and, given such an element, the values this extension may assume.

$$\mu_{c,\widehat{\omega}} := \inf \{\widehat{\omega}(a) : \; a \in D(\widehat{\omega}),\; a \geq c\},$$

where we put $\mu_{c,\widehat{\omega}} := +\infty$ if the set in the right hand side of the definition is the empty set;

$$\lambda_{c,\widehat{\omega}} := \sup \{\widehat{\omega}(a) : \; a \in D(\widehat{\omega}),\; a \leq c\}.$$

Definition 5.3. *Let $\widehat{\omega}$ be a positive hermitian extension of ω and let*

$$\mathcal{K}^{\ddagger}_{\widehat{\omega}} := \{c \in \mathcal{P}(\mathfrak{A}) : \lambda_{c,\widehat{\omega}} \text{ is finite}\}. \tag{5.4}$$

We say that $\widehat{\omega}$ is widely positive *if $\widehat{\omega}$ is positive and $D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A}) = \mathcal{K}^{\ddagger}_{\omega}$.*

The following statements hold (see [2]).

Lemma 5.1. *Let $\widehat{\omega}$ be a positive hermitian extension of ω and let $c \in \mathcal{P}(\mathfrak{A})$. Then $0 \leq \lambda_{c,\widehat{\omega}} \leq \mu_{c,\widehat{\omega}}$.*

Theorem 5.6. *Let ω be nonclosable, $\widehat{\omega}$ a positive hermitian extension of ω and $c \in \mathcal{K}^{\ddagger}_{\widehat{\omega}}$ with $c \notin D(\widehat{\omega})$. Then, $\forall\, \gamma \in \mathbb{R}$ such that $\lambda_{c,\widehat{\omega}} \leq \gamma \leq \mu_{c,\widehat{\omega}}$, there exists a positive hermitian extension ω_1 of $\widehat{\omega}$, such that $c \in D(\omega_1)$ and $\omega_1(c) = \gamma$.*

Theorem 5.7. *Let $c \in \mathcal{P}(\mathfrak{A}) \setminus \mathcal{K}^{\ddagger}_{\omega}$. Then there is no positive hermitian extension $\widehat{\omega}$ of ω such that $c \in D(\widehat{\omega})$.*

In the following examples, we use the notation introduced in Section 3.1.

Example 5.2. *Let us consider again the function $c(x) : \; [0,1] \to \mathbb{C}$*

$$c(x) := \begin{cases} 0 & \text{if } \; x = 0 \\ 1/x & \text{otherwise} \end{cases},$$

and let, like in Example 3.1, $f_n(x) : \; [0,1] \to \mathbb{C}, \; n \geq 1$ be the sequence

$$f_n(x) := \begin{cases} n^2\, x & \text{if } \; 0 \leq x \leq 1/n, \\ 1/x & \text{otherwise.} \end{cases}$$

If ω is the Riemann integral on $[0,1]$, then $\omega(f_n) = 1/2 + \log(n)$ and $0 \leq f_n(x) \leq c(x)$, $\forall x \in [0,1]$, $\forall n \geq 1$. Since $\omega(f_n) \to +\infty$, as $n \to \infty$ then, by definition, $\lambda_{c,\omega} = +\infty$, so there is no positive hermitian extension ω' of Riemann integral such that $c \in D(\omega')$.

Remark 5.6. *The previous Example 3.1 shows that if we impose to an extension $\widehat{\omega}$ the constraint to be positive, differently from the case of Theorem 3.2, the domain of the extension is, in general, a proper subset of the algebra $\mathfrak{A}$: $D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A}) \subsetneq \mathcal{P}(\mathfrak{A})$. In particular, the following result holds true.*

Theorem 5.8. *There are no fully positive extensions of the Riemann integral.*

Remark 5.7. *We note that in the case of the Riemann integral with $c \in \mathfrak{A}_h$, $\lambda_{c,\omega}$ and $\mu_{c,\omega}$ correspond to the lower and upper Riemann integral, respectively.*

Let us now consider the following function $c_1(x) : [0,1] \to \mathbb{C}$:

$$c_1(x) := \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1/x & \text{otherwise.} \end{cases}$$

From the density of $\mathbb{Q}$ follows that $\lambda_{c_1,\omega} = 0$, so $c_1 \in \mathcal{K}^{\ddagger}_{\omega}$; but, since of course c_1 is not Lebesgue integrable, then we have proved the following:

Theorem 5.9. *The Lebesgue integral (as an extension of the Riemann integral on $\mathfrak{A}_0$) is not widely positive.*

At this point one might ask whether there exists any extension of the Lebesgue integral which is widely positive. From [2] (see Remark 3.15) it follows that, if ω_1, ω_2 are positive hermitian extensions of the Riemann integral ω, with $D(\omega_1) \subseteq D(\omega_2)$, then $\mathcal{K}^{\ddagger}_{\omega_2} \subseteq \mathcal{K}^{\ddagger}_{\omega_1} \subseteq \mathcal{K}^{\ddagger}_{\omega}$. Hence given ω_1 the Lebesgue integral on $[0,1]$, since the previous function $c_1 \in \mathcal{K}^{\ddagger}_{\omega}$, if we prove that $c_1 \notin \mathcal{K}^{\ddagger}_{\omega_1}$, then we will have proved the following:

Theorem 5.10. *There are no widely positive extensions of the Lebesgue integral, considered as an extension of the the Riemann integral on $\mathfrak{A}_0$.*

Proof. Let ω_1 be the Lebesgue integral and let $g_n(x) : [0,1] \to \mathbb{C},\ n \geq 1$ be the sequence

$$g_n(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq 1/n \ \vee \ x \in \mathbb{Q}, \\ 1/x & \text{otherwise.} \end{cases}$$

Then $\omega_1(g_n) = \log(n)$ and $0 \leq g_n(x) \leq c_1(x)$, $\forall x \in [0,1]$, $\forall n \geq 1$. Since $\omega_1(g_n) \to +\infty$ as $n \to \infty$, by definition, $\lambda_{c_1,\omega_1} = +\infty$, and so $c_1 \notin \mathcal{K}^{\ddagger}_{\omega_1}$. From this, the statement follows. ☐

Corollary 5.2. *There are no widely positive extensions of the Henstock-Kurzweil integral.*

On the other hand since the Lebesgue integral belongs to the family $\mathcal{P}_\omega$ of Definition 5.1, by Zorn's lemma, there exists $\breve{\omega}$, a maximal positive hermitian extension of the Riemann integral that is actually a positive hermitian extension of the Lebesgue integral. Hence the existence of $\breve{\omega}$ shows that even if a positive hermitian linear functional is a maximal extension, it is not necessarily widely positive. In other words, we have proved the following:

Proposition 5.6. *Given a positive hermitian linear functional ω, there are maximal positive hermitian extensions of ω that are not widely positive.*

Now, we want to analyse the case where we start from the the Lebesgue integral.
Notation: From now on, ω is the Lebesgue integral on a compact interval I of $\mathbb{R}$ and $\mathfrak{A}_0 := L^\infty(I) \subseteq \mathfrak{A}$ is the algebra of all measurable functions which are essentially bounded on I.

Theorem 5.11. *Let $I \subseteq \mathbb{R}$ be a compact interval. Then the Lebesgue integral ω on I is widely positive. Hence any positive hermitian extension of the Lebesgue integral is widely positive.*

Proof. By Lemma 3.19 of [2], we will just prove that $\mathcal{K}^{\ddagger}_{\omega} \subseteq D(\omega) \cap \mathcal{P}(\mathfrak{A})$. Let $c \in \mathcal{K}^{\ddagger}_{\omega}$, then $\lambda_0 := \lambda_{c,\omega} < +\infty$. Since $\mathcal{K}^{\ddagger}_{\omega} \subseteq \mathcal{P}(\mathfrak{A})$ and c is measurable, then c is the limit of a sequence (b_n) of simple functions such that $b_n \geq 0$, with (b_n) increasing and $b_n \leq c, \quad \forall n \geq 1$. Since simple functions on I are Lebesgue integrable, then $\forall n \geq 1,\ b_n \in D(\omega)$, with $\omega(b_n) \leq \omega(b_{n+1})$. Then the limit $\bar{\lambda} := \lim_n \omega(b_n)$ exists and, by definition of $\lambda_{c,\omega}$, $\bar{\lambda} \leq \lambda_0 < +\infty$. Hence

$$\liminf_n \omega(b_n) = \lim_n \omega(b_n) < +\infty;$$

so, by Fatou's lemma,

$$\omega(c) = \omega(\lim_n b_n) = \omega(\liminf_n b_n) \leq \liminf_n \omega(b_n) = \lim_n \omega(b_n) = \bar{\lambda} < +\infty.$$

Hence c is Lebesgue integrable with $\omega(c) \leq \bar{\lambda}$. □

From Example 5.2, we have the next

Corollary 5.3. *There are no fully positive extensions of the Lebesgue integral. In particular, the Lebesgue integral is not fully positive.*

From Corollary 5.3 and from Theorem 5.11, it follows the next interesting:

Remark 5.8. *Let $\widehat{\omega}$ be an extension of the Lebesgue integral and let $a \in D(\widehat{\omega}) \setminus \mathfrak{A}_0$, then $a \notin \mathcal{P}(\mathfrak{A})$.*

Finally, we recall (see [3]) that the Henstock-Kurzweil integral is a positive extension of the Lebesgue integral that is not maximal, so (see [2]) we have:

Theorem 5.12. *There exists a maximal positive hermitian extension of the Henstock-Kurzweil integral.*

Returning to the general case, let $\mathfrak{A}$ be a *-algebra. We say that $\mathfrak{A}$ has the property (D) if, for every $a \in \mathfrak{A}_h$, there exists a unique pair (a_+, a_-) of elements of $\mathfrak{A}$, with $a_+, a_- \in \mathcal{P}(\mathfrak{A})$ such that

(D1) $a = a_+ - a_-$;
(D2) $a_+a_- = a_-a_+ = 0$;
(D3) $(\lambda a)_+ = \lambda a_+, \quad \forall a \in \mathfrak{A}_h, \lambda \in \mathbb{R}^+$;

then we put

$$|a| := a_+ + a_-.$$

If $\mathfrak{A}$ has the property (D), one has:

$$|a| \in \mathcal{P}(\mathfrak{A}), \quad \forall a \in \mathfrak{A}_h.$$

We remind that a positive hermitian linear functional $\bar{\omega}$ defined on a subspace of $\mathfrak{A}$ is called *absolutely convergent* if for all $a \in D(\bar{\omega}) \cap \mathfrak{A}_h$, $a_+, a_- \in D(\bar{\omega})$, and so $|a| \in D(\bar{\omega})$.

Several examples that guarantee the existence of absolutely convergent extensions, are given in [20]. Now, we state the following theorem and corollary (see [2]).

Theorem 5.13. *Let $\breve{\omega}$ be an absolutely convergent extension of ω. If $\breve{\omega}$ is widely positive, then $\breve{\omega}$ is a maximal absolutely convergent extension of ω.*

Corollary 5.4. *Let $\breve{\omega}$ be an absolutely convergent extension of ω. If $\breve{\omega}$ is fully positive, then $\breve{\omega}$ is a maximal absolutely convergent extension of ω.*

Concluding remark

The problem of extending the Riemann integral defined on continuous functions is probably as old as the Riemann integral itself. In [2] , [3] and in the present paper, this question has been cast into an abstract framework, looking for extensions of a positive hermitian linear functional ω, defined on a dense *-subalgebra $\mathfrak{A}_0$ of a topological *-algebra (in general, without unit), with topology τ and continuous involution *, to a larger family of elements of $\mathfrak{A}$. Several particular cases have been discussed in those papers; among them *positive regular slight extension* arouse interest since it closely reminds the construction of the Lebesgue integral or Segal's construction of noncommutative integration [16]. We have first proved that there are maximal extension of the Riemann integral defined on the whole *-algebra of Lebesgue measurable functions on a compact interval, and then this result has been shown to hold also in the abstract case for certain functional. Of course there is a price to pay for this: for instance several familiar properties of the integral are missing for this maximal everywhere defined extension (e.g., positivity). In the end, our reader may legitimately wonder how does this extension of the integral work. This aspect is matter of further investigations.

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APPENDIX A. A DIFFERENT APPROACH

The proof of Theorem 4.3 might be modified using a different approach [13]. The starting point is observing that the closability and continuity are equivalent. Let V be a complex topological vector space with topology τ (for short, $V[\tau]$). Let ω be a nonzero linear functional defined on V. We collect some elementary (and well-known) facts.

Lemma A.2. *The following statement hold:*

(a) *The range $\omega(V)$ coincides with $\mathbb{C}$.*
(b) *The kernel $Ker\ \omega$ of ω is a proper maximal subspace of V.*
(c) *$Ker\ \omega$ is either closed or dense in $V[\tau]$.*
(d) *ω is continuous if, and only if, $Ker\ \omega$ is closed in $V[\tau]$.*
(e) *If θ is another nonzero linear functionals on V, $Ker\ \omega = Ker\ \theta$ if, and only if, θ is a multiple of ω.*

We consider the graph of ω, i.e.,

$$G_\omega = \{(x, \omega(x)) \in V \times \mathbb{C}\}.$$

The functional ω is said to be closable if one of the two equivalent statements which follow is satisfied.

- If $x_\alpha \to 0$ w.r. to τ and $\omega(a_\alpha) \to \ell$, then $\ell = 0$.
- $\overline{G_\omega}$, the closure of G_ω, does not contain couples $(0, \ell)$ with $\ell \neq 0$.

It turns out that in the case of linear functionals closability and continuity are equivalent. As a consequence, a discontinuous linear functional is *never* closable.

Remark A.9. *So far, we have considered the case of functionals that are everywhere defined on V. Suppose that Y is a dense subspace of $V[\tau]$, and let ω be a linear functional on Y. As stated before, if ω is closable in Y it is continuous on Y and then it extends by continuity to the whole space V and, of course, the extension is continuous.*

On $V \oplus \mathbb{C}$ define $\Omega(a, \ell) = \omega(a) - \ell$. It is easily seen that Ω is a linear functional on $V \oplus \mathbb{C}$.

Lemma A.3. *Ω is continuous on $V \oplus \mathbb{C}$ if, and only if, ω is continuous on V.*

Proof. Let Ω be continuous and let $(x_\alpha)_{\alpha\in\Gamma} \subseteq V$ be a net converging to $x \in V$. Then $\omega(x_\alpha) = \Omega(x_\alpha, 0) \to \Omega(x, 0) = \omega(x)$; i.e., ω is continuous. Conversely, assume that ω is continuous and that $(x_\alpha, \lambda_\alpha) \to (x, \lambda)$. Then $x_\alpha \to x$ and $\lambda_\alpha \to \lambda$. Hence,

$$\Omega(x_\alpha, \lambda\alpha) = \omega(x_\alpha) - \lambda\alpha \to \omega(x) - \lambda = \Omega(x, \lambda).$$

□

Proposition A.7. *If ω, defined on V, is discontinuous, then G_ω is dense in $V \oplus \mathbb{C}$.*

Proof. By Lemma A.3, Ω is linear and discontinuous then its kernel is dense in $V \oplus \mathbb{C}$. It is easily seen that $Ker\,\Omega = G_\omega$. Thus $\overline{G_\omega} = V \oplus \mathbb{C}$. □

Proposition A.8. *Let V be a vector space and W a proper subspace of V. Then, for every $x \in V \setminus W$, there exists a linear functional ω on V such that $\omega(y) = 0$ for every $y \in W$ and $\omega(x) = 1$.*

Proof. Let $x \in V \setminus W$ then the span $\mathbb{C} \cdot x$ is a subspace of V with $W \cap \mathbb{C} \cdot x = \{0\}$. Then on $W \oplus \mathbb{C} \cdot x$ we can define $\omega(y + \lambda x) = \lambda$; then, $\omega(y) = 0$, for every $y \in W$ and $\omega(x) = 1$. If $\{e_j\}$ is a Hamel basis of W, we can find linearly independent vectors $\{h_k\}$ of $V \setminus W \oplus \mathbb{C} \cdot x$ such that $\{e_j\} \cup \{h_k\} \cup \{x\}$ is a Hamel basis for V. It is now sufficient to define $\omega(h_k) = 0$, for every k. □

Let V_0 be a dense subspace of $V[\tau]$ and ω a linear functional defined on V_0.

A linear functional $\widehat{\omega}$ defined on a vector subspace $D(\widehat{\omega})$ of V is called an *extension* of ω if

$$V_0 \subseteq D(\widehat{\omega})$$
$$\widehat{\omega}(x) = \omega(x), \ \forall x \in V_0.$$

In this case, we write $\omega \subseteq \widehat{\omega}$. It is clear that $\omega \subseteq \widehat{\omega}$ if, and only if, $G_\omega \subseteq G_{\widehat{\omega}}$. If ω is continuous on V_0, it has a unique continuous extension to V. In what follows, we will consider the case when ω is discontinuous (equivalently, nonclosable) in V_0.

As in Section 2, an extension $\widehat{\omega}$ of ω is a *slight extension* if $G_{\widehat{\omega}} \subseteq \overline{G_\omega}$. We denote by $\mathcal{S}_\omega$ the collection of all subspaces H of $V \times \mathbb{C}$ such that

(g1) $G_\omega \subseteq H \subseteq \overline{G_\omega}$;
(g2) $(0, \ell) \in H$ if, and only if, $\ell = 0$.

$\mathcal{S}_\omega$ is nonempty, since it contains G_ω and each $H \in \mathcal{S}_\omega$ defines an extension ω_H as follows

$$D(\omega_H) = \{x \in V : (x, \ell) \in H\}$$
$$\omega_H(x) = \ell.$$

It is clear on the other hand that every slight extension $\widehat{\omega}$ of ω defines a subspace $H' \in \mathcal{S}_\omega$. Namely, $H' = G(\widehat{\omega})$. By Proposition A.7, it follows that $\overline{G_\omega} = V_0 \oplus \mathbb{C}$. The density of V_0 in V implies that $V_0 \oplus \mathbb{C}$ is dense in $V \oplus \mathbb{C}$. Then, we conclude that $\overline{G_\omega} = V \oplus \mathbb{C}$. Therefore, the set

$$\mathcal{K}_\omega = \{x \in V : (x, \ell) \in \overline{G_\omega}, \text{ for some } \ell \in \mathbb{C}\}$$

coincides evidently with V. From these considerations, it follows also that *every extension* $\widehat{\omega}$ *of* ω *is a slight* one.

As discussed in Section 2, the existence of maximal extensions of ω can be proved by using Zorn's lemma. Let $\breve{\omega}$ denote a maximal extension of ω. Then as proved in [2] $D(\breve{\omega}) = \mathcal{K}_\omega$. Thus, in conclusion,

Proposition A.9. *An extension $\breve{\omega}$ of ω is maximal if, and only if, $D(\breve{\omega}) = V$.*

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