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A Study of ϕ-Pluriharmonicity in Quasi bi-slant Conformal $\xi^{\perp}$-Submersions From Kenmotsu Manifold

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Abstract

In the present research paper, we look into quasi bi-slant conformal $\xi^{\perp}$-submersions from Kenmotsu manifolds onto Riemannian manifolds as a generalisation of quasi hemi-slant conformal submersions. We investigate the geometry of distributions's leaves in order to explain integrability conditions for distributions. Furthermore, we study of certain decomposition theorems, additionaly provide non-trivial examples of quasi bi-slant conformal $\xi^{\perp}$-submersions from Kenmotsu manifolds. We also look at the ϕ-pluriharmonicity of quasi bi-slant conformal $\xi^{\perp}$-submersions.

Keywords: *Kenmotsu manifold, Riemannian submersions, bi-slant submersions, quasi bi-slant submersions.*
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1. Introduction

Immersions and submersions have numerous important applications in differential geometry. The characteristics of slant submersions have become a fascinating topic in differential geometry, as well as in complex and contact geometry. The study of Riemannian submersions between Riemannian manifold were initiated by O' Neill [26] and Gray [12], independently. Later on, the Riemannian submersions between almost Hermitian manifolds with name as almost Hermitian submersions were studied by Watson [40] in 1976. The Riemannian submersions consists many applications in mathematics and in physics, specially in Yang-Mills theory ([7], [41]), Kaluza-Klein theory ([18], [22]). The Riemannian submersions are very interesting tolls in geometry to study Riemannian manifolds having differentiable structures. A generalization of holomorphic submersions and anti-invariant submersions, Sahin [34] introduced the semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds in 2013. Subsequently, in 2016, Tatsan, Sahin, and Yanan started studying hemi-slant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds and came up with a few decomposition theorems for them [39]. R Prasad et al. examined quasi bi-slant submersions from almost contact metric manifolds onto Riemannian manifolds [28] and from Kenmotsu manifolds [29], which is a step ahead. Later on, many authors investigated different kinds of Riemannian submersions like anti-invariant submersions ([4], [33]), slant submersions [10], [35], semi-slant submersions ([23], [24], [16], [27]) and hemi-slant submersions ([38], [1]) from almost Hermitian manifolds as well as from almost contact metric manifolds.

The notion of Riemannian submersions from almost contact manifold was introduced by Chinea in [8]. He also studied the fibre space, base space and total space with differential geometric point of view. To consider the generalization of Riemannian submersions, Gundmundsson and Wood [14, 15] presented horizontally conformal submersion, which they defined as: Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of

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dimension m_1 and m_2, respectively. A smooth map $\vec{f}:(M_1,g_1)\rightarrow(M_2,g_2)$ is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_1(X_1,X_2)=g_2(\vec{f}_*X_1,\vec{f}_*X_2), \tag{1.1}$$

for all $X_1,X_2\in\Gamma(ker\vec{f}_*)^{\perp}$. Thus Riemannian submersion is a particular horizontally conformal submersion with $\lambda=1$. Later on, Fuglede [13] and Ishihara [20] separately studied horizontally conformal submersions. Akyol and Sahin studied conformal slant submersions [3], conformal anti-invariant submersions [36], conformal semi-slant submersions [2], conformal semi-invariant submersions [5]. Further, R. Prasad et. al. also studied conformal anti-invariant submersions [30]. To explore the geometry of hemi-slant conformal submersions from almost product Riemannian manifold. Recently, Shuaib and Fatima [37] studied conformal hemi-slant Riemannian submersions from almost product manifolds onto Riemannian manifolds.

In this paper, we study quasi bi-slant conformal $\xi^{\perp}$-submersions from Kenmotsu manifold onto a Riemannian manifold with taking ξ as a horizontal vector field. This paper is divided into six sections. Section 2 contains definitions of almost contact metric manifolds and, in particular, Kenmotsu manifolds. In section 3, we investigates some fundamental results for quasi bi-slant conformal submersion from Kenmotsu manifolds that are required for our main sections. The results of integrability and totally geodesicness of distributions are presented in Section 4. In section 5, we obtain condition under which a Kenmotsu manifold become twisted product manifold with quasi bi-slant conformal $\xi^{\perp}$-submersions. In the last section, we discuss ϕ-pluriharmonicity of quasi bi-slant conformal $\xi^{\perp}$-submersions.

Note: We will use some abbreviations throughout the paper as follows:
Horizontally conformal submersion- HCS, Riemannian Manifold- RM, Almost contact metric manifold-ACM manifold, Quasi bi-slant conformal $\xi^{\perp}$-submersion- $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, gradient- G.

2. Preliminaries

Let M be a $(2n+1)$-dimensional almost contact manifold with almost contact structures (ϕ,ξ,η), where a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2=-I+\eta\otimes\xi,\ \ \phi\xi=0,\ \ \eta\circ\phi=0,\ \ \eta(\xi)=1, \tag{2.1}$$

where I is the identity tensor. The almost contact structure is said to be normal if $N+d\eta\otimes\xi=0$, where N is the Nijenhuis tensor of ϕ. Suppose that a Riemannian metric tensor g is given in M and satisfies the condition

$$g(\phi\widehat{U},\phi\widehat{V})=g(\widehat{U},\widehat{V})-\eta(\widehat{U})\eta(\widehat{V}),\ \ \eta(\widehat{U})=g(\widehat{U},\xi). \tag{2.2}$$

Then (ϕ,ξ,η,g)-structure is called an almost contact metric structure. Define a tensor field Φ of type $(0,2)$ by $\Phi(\widehat{X},\widehat{Y})=g(\phi\widehat{X},\widehat{Y})$. If $d\eta=\Phi$, then an almost contact metric structure is said to be normal contact metric structure. Let Φ be the fundamental 2-form on M, i.e, $\Phi(\widehat{U},\widehat{V})=g(\widehat{U},\phi\widehat{V})$. If $\Phi=d\eta$, M is said to be a contact manifold. If ξ is a Killing vector field with respect to g, the contact metric structure is called a K-contact structure.

S.Tanno [32], classified connected almost contact metric manifolds whose automorphism groups posses the maximum dimension. For such a manifold, the sectional curvature of a plane section containing ξ is a constant c. One of the classes of this classification consists of warped product $R\times_f C^n$ with $c<0$. The tensorial equation of these manifolds are:

$$(\nabla_{\widehat{U}}\phi)\widehat{V}=g(\phi\widehat{U},\widehat{V})\xi-\eta(\widehat{V})\phi\widehat{U}. \tag{2.3}$$

Kenmotsu [21], explored some fundamental differential geometric properties of these spaces and therefore they are named as Kenmotsu manifolds. It can also be seen that on a Kenmotsu manifold M

$$\nabla_{\widehat{U}}\xi=-\phi^2\widehat{U}=\widehat{U}-\eta(\widehat{U})\xi. \tag{2.4}$$

The covariant derivative of ϕ is defined by

$$(\nabla_{\widehat{U}_1}\phi)\widehat{V}_1=\nabla_{\widehat{U}_1}\phi\widehat{V}_1-\phi\nabla_{\widehat{U}_1}\widehat{V}_1, \tag{2.5}$$

for any vector fields $\widehat{U}_1,\widehat{V}_1\in\Gamma(TM)$. Now, we provide a definition for conformal submersion and discuss some useful results that help us to achieve our main results.

Definition 2.1. [6] Let $\vec{f}$ be a horizontally conformal submersion (HCS) from an ACM manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (RM) (M_2, g_2). Then $\vec{f}$ is called a horizontally conformal submersion, if there is a positive function λ such that

$$g_1(\widehat{U}_1, \widehat{V}_1) = \frac{1}{\lambda^2} g_2(\vec{f}_* \widehat{U}_1, \vec{f}_* \widehat{V}_1), \tag{2.6}$$

for any $\widehat{U}_1, \widehat{V}_1 \in \Gamma(ker \vec{f}_*)^{\perp}$. It is obvious that every HCS is a particularly horizontally conformal submersion with $\lambda = 1$.

Let $\vec{f} : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a HCS. A vector field $\vec{X}$ on M_1 is called a basic vector field if $\vec{X} \in \Gamma(\ker \vec{f}_*)^{\perp}$ and $\vec{f}$-related with a vector field $\vec{X}$ on M_2 i.e $\vec{f}_*(\vec{X}(q)) = \vec{X}\vec{f}(q)$ for $q \in M_1$.
The formulas provide the two $(1, 2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ by O'Neill are

$$\mathcal{A}_{E_1} F_1 = \mathcal{H}\nabla_{\mathcal{H}E_1} \mathcal{V}F_1 + \mathcal{V}\nabla_{\mathcal{H}E_1} \mathcal{H}F_1, \tag{2.7}$$

$$\mathcal{T}_{E_1} F_1 = \mathcal{H}\nabla_{\mathcal{V}E_1} \mathcal{V}F_1 + \mathcal{V}\nabla_{\mathcal{V}E_1} \mathcal{H}F_1, \tag{2.8}$$

for any $E_1, F_1 \in \Gamma(TM_1)$ and ∇ is Levi-Civita connection of g_1. Note that a HCS $\vec{f} : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. From equations (2.4) and (2.8), we can deduce

$$\nabla_{\widehat{U}_1} \widehat{V}_1 = \mathcal{T}_{\widehat{U}_1} \widehat{V}_1 + \mathcal{V}\nabla_{\widehat{U}_1} \widehat{V}_1 \tag{2.9}$$

$$\nabla_{\widehat{U}_1} \widehat{X}_1 = \mathcal{T}_{\widehat{U}_1} \widehat{X}_1 + \mathcal{H}\nabla_{\widehat{U}_1} \widehat{X}_1 \tag{2.10}$$

$$\nabla_{\widehat{X}_1} \widehat{U}_1 = \mathcal{A}_{\widehat{X}_1} \widehat{U}_1 + \mathcal{V}_1\nabla_{\widehat{X}_1} \widehat{U}_1 \tag{2.11}$$

$$\nabla_{\widehat{X}_1} \widehat{Y}_1 = \mathcal{H}\nabla_{\widehat{X}_1} \widehat{Y}_1 + \mathcal{A}_{\widehat{X}_1} \widehat{Y}_1 \tag{2.12}$$

for any vector fields $\widehat{U}_1, \widehat{V}_1 \in \Gamma(ker \vec{f}_*)$ and $\widehat{X}_1, \widehat{Y}_1 \in \Gamma(ker \vec{f}_*)^{\perp}$ [11].
It is easily seen that $\mathcal{T}$ and $\mathcal{A}$ are skew-symmetric, that is

$$g(\mathcal{A}_{\widehat{X}} E_1, F_1) = -g(E_1, \mathcal{A}_{\widehat{X}} F_1), \;\; g(\mathcal{T}_{\widehat{V}} E_1, F_1) = -g(E_1, \mathcal{T}_{\widehat{V}} F_1), \tag{2.13}$$

for any vector fields $E_1, F_1 \in \Gamma(T_p M_1)$.

Definition 2.2. A horizontally conformally submersion $\vec{f} : M_1 \to M_2$ is called horizontally homothetic if the gradient (G) of its dilation λ is vertical, i.e.,

$$H(G\lambda) = 0, \tag{2.14}$$

at $p \in TM_1$, where H is the complement orthogonal distribution to $\nu = ker \vec{f}_*$ in $\Gamma(T_p M)$.

The second fundamental form of smooth map $\vec{f}$ is given by the formula

$$(\nabla \vec{f}_*)(\widehat{U}_1, \widehat{V}_1) = \nabla^{\vec{f}}_{\widehat{U}_1} \vec{f}_* \widehat{V}_1 - \vec{f}_* \nabla_{\widehat{U}_1} \widehat{V}_1, \tag{2.15}$$

and the map be totally geodesic if $(\nabla \vec{f}_*)(\widehat{U}_1, \widehat{V}_1) = 0$ for all $\widehat{U}_1, \widehat{V}_1 \in \Gamma(T_p M)$ where ∇ and $\nabla^{\vec{f}}$ are Levi-Civita and pullback connections.

Lemma 2.1. *Let $\vec{f} : M_1 \to M_2$ be a horizontal conformal submersion. Then, we have*

(i) $(\nabla \vec{f}_*)(\widehat{X}_1, \widehat{Y}_1) = \widehat{X}_1(ln\lambda)\vec{f}_*(\widehat{Y}_1) + \widehat{Y}_1(ln\lambda)\vec{f}_*(\widehat{X}_1) - g_1(\widehat{X}_1, \widehat{Y}_1)\vec{f}_*(grad\; ln\lambda)$,

(ii) $(\nabla \vec{f}_*)(\widehat{U}_1, \widehat{V}_1) = -\vec{f}_*(\mathcal{T}_{\widehat{U}_1} \widehat{V}_1)$,

(iii) $(\nabla \vec{f}_*)(\widehat{X}_1, \widehat{U}_1) = -\vec{f}_*(\nabla_{\widehat{X}_1} \widehat{U}_1) = -\vec{f}_*(\mathcal{A}_{\widehat{X}_1} \widehat{U}_1)$

for any horizontal vector fields $\widehat{X}_1, \widehat{Y}_1$ and vertical vector fields $\widehat{U}_1, \widehat{V}_1$ [6].

3. Quasi bi-slant conformal $\xi^{\perp}$-submersions

Definition 3.1. Let $(M_1, \phi, \xi, \eta, g_1)$ be a ACM manifold and (M_2, g_2) a Riemannian manifold. A HCS $\vec{f} : M_1 \to M_2$ where $\xi \in \Gamma(ker f_*)^{\perp}$ is called quasi bi-slant conformal $\xi^{\perp}$-submersion ($\mathcal{QBSC}$ $\xi^{\perp}$-submersion) if there exists three mutually orthogonal distributions $\mathfrak{D}$, $\mathfrak{D}^{\theta_1}$ and $\mathfrak{D}^{\theta_2}$ such that

(i) $ker \vec{f}_* = \mathfrak{D} \oplus \mathfrak{D}^{\theta_1} \oplus \mathfrak{D}^{\theta_2}$,

(ii) $\mathfrak{D}$ is invariant. i.e., $\phi\mathfrak{D} = \mathfrak{D}$,

(iii) $\phi\mathfrak{D}^{\theta_1} \perp \mathfrak{D}^{\theta_2}$ and $\phi\mathfrak{D}^{\theta_2} \perp \mathfrak{D}^{\theta_1}$,

(iv) for any non-zero vector field $\widehat{V}_1 \in (\mathfrak{D}^{\theta_1})_p, p \in M_1$ the angle θ_1 between $(\mathfrak{D}^{\theta_1})_p$ and $\phi\widehat{V}_1$ is constant and independent of the choice of the point p and $\widehat{V}_1 \in (\mathfrak{D}^{\theta_1})_p$,

(v) for any non-zero vector field $\widehat{V}_1 \in (\mathfrak{D}^{\theta_2})_q, q \in M_1$ the angle θ_2 between $(\mathfrak{D}^{\theta_2})_q$ and $\phi\widehat{V}_1$ is constant and independent of the choice of the point q and $\widehat{V}_1 \in (\mathfrak{D}^{\theta_2})_q$,

where θ_1 and θ_2 are called the slant angles of submersion.

If we denote the dimensions of $\mathfrak{D}$, $\mathfrak{D}^{\theta_1}$ and $\mathfrak{D}^{\theta_2}$ by m_1, m_2 and m_3 respectively, then we have the following:

(i) If $m_1 \neq 0$, $m_2 = 0$ and $m_3 = 0$, then $\vec{f}$ is an invariant submersion.

(ii) If $m_1 \neq 0$, $m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 = 0$, then $\vec{f}$ is a proper semi-slant submersion.

(iii) If $m_1 = 0$, $m_2 = 0$ and $m_3 \neq 0, 0 < \frac{\pi}{2}$, then $\vec{f}$ is a slant submersion with slant angle θ_2.

(iv) If $m_1 = 0, m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0, \theta_2 = \frac{\pi}{2}$, then $\vec{f}$ proper hemi-slant submersion.

(v) If $m_1 = 0, m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0, 0 <, \theta_2 < \frac{\pi}{2}$, then $\vec{f}$ is proper bi-slant submersion with slant angles θ_1 and θ_2.

(vi) If $m_1 \neq 0, m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0, 0 < \theta_2 < \frac{\pi}{2}$, then $\vec{f}$ is proper quasi bi-slant submersion with slant angles θ_1 and θ_2.

Let $(x_i; y_i; z)$ be cartesian coordinates on R^{2n+1} for $i = 1, 2, 3, ..., n$. An almost contact metric structure (ϕ, ξ, η, g_1) is defined as follows:

$$\phi\left(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + ... + a_n\frac{\partial}{\partial x_n} + b_1\frac{\partial}{\partial y_1} + b_2\frac{\partial}{\partial y_2} + ... + b_n\frac{\partial}{\partial y_n} + c\frac{\partial}{\partial z}\right)$$
$$= \left(-b_1\frac{\partial}{\partial x_1} + a_1\frac{\partial}{\partial y_1} - b_2\frac{\partial}{\partial x_2} + a_2\frac{\partial}{\partial y_2} - ... + b_n\frac{\partial}{\partial x_n} + a_n\frac{\partial}{\partial y_n}\right),$$

where $\xi = \frac{\partial}{\partial z}$ and a_i, b_i, c are C^{∞}- real valued functions in R^{2n+1}. Let $\eta = dz$, g_1 is Euclidean metric and

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, ..., \frac{\partial}{\partial y_n}, \frac{\partial}{\partial z}\right)$$

is orthonormal base field of vectors on R^{2n+1}. We can easily show that (ϕ, ξ, η, g_1) is Kenmotsu structure on R^{2n+1}. Hence, it is Kenmotsu manifold.

Example 3.1. Define a map $\vec{f} : R^{13} \to R^7$ by

$$\vec{f}(x_1, ..., x_6, y_1, ..., y_6, z) \rightarrowtail \pi\left(\frac{x_2 - \sqrt{3}x_3}{2}, y_2, \frac{x_5 - x_4}{\sqrt{2}}, y_4, x_6, y_6, z\right)$$

which is a conformal quasi bi-slant submersion with dilation $\lambda = \pi$ such that

$$X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial y_1}, X_3 = \frac{1}{2}\left(\sqrt{3}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right), X_4 = \frac{\partial}{\partial y_3},$$

$$X_5 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_4}\right), X_6 = \frac{\partial}{\partial y_5}$$

and

$$ker\vec{f}_* = \mathfrak{D} \oplus \mathfrak{D}^{\theta_1} \oplus \mathfrak{D}^{\theta_2},$$

where the invariant distribution $\mathfrak{D}$ is spanned by the vectors $\{X_1, X_2\}$ and the slant distributions $\mathfrak{D}^{\theta_1}$ and $\mathfrak{D}^{\theta_2}$ are spanned by the vector fields $\{X_3, X_4\}$ and $\{X_5, X_6\}$ with slant angle $\theta_1 = \frac{\pi}{3}$ and $\theta_2 = \frac{\pi}{4}$, respectively. Furthermore,

$$(ker\vec{f}_*)^\perp = \left\langle Z_1 = \frac{1}{2}\left(\frac{\partial}{\partial x_2} - \sqrt{3}\frac{\partial}{\partial x_3}\right), Z_2 = \frac{\partial}{\partial y_2}, Z_3 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_4}\right), Z_4 = \frac{\partial}{\partial y_4}, \right\rangle$$

$$\left\langle Z_5 = \frac{\partial}{\partial x_6}, Z_6 = \frac{\partial}{\partial y_6}, Z_7 = \frac{\partial}{\partial z} = \xi \right\rangle$$

where the reeb vector field ξ is horizontal.

Example 3.2. Define a map $\vec{f} : R^{15} \to R^7$ by

$$\vec{f}(x_1, ..., x_6, x_7, y_1, ..., y_6, y_7, z) \rightarrowtail (x_1, x_3 \cos\alpha - y_4 \sin\alpha, y_3, x_4, x_5 \sin\beta - y_6 \cos\beta, y_7, z)$$

which is a conformal quasi bi-slant submersion with dilation $\lambda = 1$ such that

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y_1}, X_2 = \frac{\partial}{\partial x_2}, X_3 = \frac{\partial}{\partial y_2}, X_4 = \frac{\partial}{\partial x_3}\sin\alpha + \frac{\partial}{\partial y_4}\cos\alpha, \\ X_5 &= \frac{\partial}{\partial x_5}\cos\beta + \frac{\partial}{\partial y_6}\sin\beta, X_6 = \frac{\partial}{\partial y_5}, \\ X_7 &= \frac{\partial}{\partial x_6}, X_8 = \frac{\partial}{\partial x_7}. \end{aligned}$$

More precisely,

$$\mathfrak{D} =< X_1, X_2, X_6, X_8 >, \quad \mathfrak{D}^{\theta_1} =< X_5, X_7 >, \quad \mathfrak{D}^{\theta_2} =< X_3, X_4 >,$$

such that $ker\vec{f}_* = \mathfrak{D} \oplus \mathfrak{D}^{\theta_1} \oplus \mathfrak{D}^{\theta_2}$, where the distribution $\mathfrak{D}^{\theta_1}$ and $\mathfrak{D}^{\theta_2}$ are slant with slant angles β and α, respectively. Moreover,

$$(ker\vec{f}_*)^\perp = \left\langle Z_1 = \frac{\partial}{\partial x_1}, Z_2 = \frac{\partial}{\partial x_3}\cos\alpha - \frac{\partial}{\partial y_4}\sin\alpha, Z_3 = \frac{\partial}{\partial y_3}, Z_4 = \frac{\partial}{\partial x_4}, Z_5 = \frac{\partial}{\partial x_5}\sin\beta - \frac{\partial}{\partial y_6}\cos\beta, \right\rangle$$

$$\left\langle Z_6 = \frac{\partial}{\partial y_7} Z_7 = \frac{\partial}{\partial z} = \xi \right\rangle$$

where the reeb vector field ξ is horizontal.

Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from an ACM manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2). Then, for any $U \in (ker\vec{f}_*)$, we have

$$\widehat{U} = \mathfrak{A}\widehat{U} + \mathfrak{B}\widehat{U} + \mathfrak{C}\widehat{U} \tag{3.1}$$

where $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are the projections morphism onto $\mathfrak{D}, \mathfrak{D}^{\theta_1}$, and $\mathfrak{D}^{\theta_2}$. Now, for any $\widehat{U} \in (ker\vec{f}_*)$, we have

$$\phi\widehat{U} = \delta\widehat{U} + \zeta\widehat{U} \tag{3.2}$$

where $\delta\widehat{U} \in \Gamma(ker\vec{f}_*)$ and $\zeta\widehat{U} \in \Gamma(ker\vec{f}_*)^\perp$. From equations (3.1) and (3.2), we have

$$\begin{aligned} \phi\widehat{U} =& \phi(\mathfrak{A}\widehat{U}) + \phi(\mathfrak{B}\widehat{U}) + \phi(\mathfrak{C}\widehat{U}) \\ =& \delta(\mathfrak{A}\widehat{U}) + \zeta(\mathfrak{A}\widehat{U}) + \delta(\mathfrak{B}\widehat{U}) + \zeta(\mathfrak{B}\widehat{U}) + \delta(\mathfrak{C}\widehat{U}) + \zeta(\mathfrak{C}\widehat{U}). \end{aligned}$$

Since $\phi\mathfrak{D} = \mathfrak{D}$ and $\zeta(\mathfrak{A}\widehat{U}) = 0$, we have

$$\phi\widehat{U} = \delta(\mathfrak{A}\widehat{U}) + \delta(\mathfrak{B}\widehat{U}) + \zeta(\mathfrak{B}\widehat{U}) + \delta(\mathfrak{C}\widehat{U}) + \zeta(\mathfrak{C}\widehat{U}).$$

Hence we have the decomposition as :

$$\phi(ker\vec{f_*}) = \delta\mathfrak{D} \oplus \delta\mathfrak{D}^{\theta_1} \oplus \delta\mathfrak{D}^{\theta_2} \oplus \zeta\mathfrak{D}^{\theta_1} \oplus \zeta\mathfrak{D}^{\theta_2}. \tag{3.3}$$

From equations (3.3), we have the following decomposition

$$(ker\vec{f_*})^{\perp} = \zeta\mathfrak{D}^{\theta_1} \oplus \zeta\mathfrak{D}^{\theta_2} \oplus \mu, \tag{3.4}$$

where μ is the orthogonal complement to $\zeta\mathfrak{D}^{\theta_1} \oplus \zeta\mathfrak{D}^{\theta_2}$ in $(ker\vec{f_*})^{\perp}$ such that $\mu = (\phi\mu) \oplus < \xi >$ and μ is invariant with respect to ϕ. Now, for any $\widehat{X} \in \Gamma(ker\vec{f_*})^{\perp}$, we have

$$\phi\widehat{X} = P\widehat{X} + Q\widehat{X} \tag{3.5}$$

where $P\widehat{X} \in \Gamma(ker\vec{f_*})$ and $Q\widehat{X} \in \Gamma(ker\vec{f_*})^{\perp}$.

Lemma 3.1. *Let $(M_1, \phi, \xi, \eta, g_1)$ be an ACM manifold and (M_2, g_2) be a RM. If $\vec{f} : M_1 \to M_2$ is a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, then we have*

$$-\widehat{U} = \delta^2\widehat{U} + P\zeta\widehat{U},\ \zeta\delta\widehat{U} + Q\zeta\widehat{U} = 0,$$
$$-\widehat{X} + \eta(\widehat{X})\xi = \zeta P\widehat{X} + Q^2\widehat{X},\ \delta P\widehat{X} + PQ\widehat{X} = 0,$$

for $\widehat{U} \in \Gamma(ker\vec{f_*})$ and $\widehat{X} \in \Gamma(ker\vec{f_*})^{\perp}$.

Proof. By using equations (2.1), (3.2) and (3.5), we get the desired results. □

Since $\vec{f} : M_1 \to M_2$ is a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, then let's give some useful results that will use all along the paper.

Lemma 3.2. *Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion from an ACM manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2), then we have*

(i) $\delta^2\widehat{U} = -\cos^2\theta_1\, U$,

(ii) $g_1(\delta\widehat{U}, \delta\widehat{V}) = \cos^2\theta_1\, g_1(\widehat{U}, \widehat{V})$,

(iii) $g_1(\zeta\widehat{U}, \zeta\widehat{V}) = \sin^2\theta_1\, g_1(\widehat{U}, \widehat{V})$,

for any vector fields $\widehat{U}, \widehat{V} \in \Gamma(\mathfrak{D}^{\theta_1})$.

Lemma 3.3. *Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion from an ACM manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2), then we have*

(i) $\delta^2\widehat{Z} = -\cos^2\theta_2\, \widehat{Z}$,

(ii) $g_1(\delta\widehat{Z}, \delta\widehat{W}) = \cos^2\theta_2\, g_1(\widehat{Z}, \widehat{W})$,

(iii) $g_1(\zeta\widehat{Z}, \zeta\widehat{W}) = \sin^2\theta_2\, g_1(\widehat{Z}, \widehat{W})$,

for any vector fields $\widehat{Z}, \widehat{W} \in \Gamma(\mathfrak{D}^{\theta_2})$.

Proof. The proof of above Lemmas is similar to the proof of the Theorem (2.2) of [9]. □

Let (M_2, g_2) is a Riemannian manifold and that $(M_1, \phi, \xi, \eta, g_1)$ is a Kenmotsu manifold. We now look at how the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion $\vec{f} : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ are affected by the Kenmotsu structure on M_1.

Lemma 3.4. *Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion from an Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2), then we have*

$$\mathcal{A}_{\widehat{X}}P\widehat{Y} + \mathcal{H}\nabla_{\widehat{X}}Q\widehat{Y} = Q\mathcal{H}\nabla_{\widehat{X}}\widehat{Y} + \zeta\mathcal{A}_{\widehat{X}}\widehat{Y} + g_1(Q\widehat{X}, \widehat{Y})\xi - \eta(\widehat{Y})Q\widehat{X} \tag{3.6}$$

$$\mathcal{V}\nabla_{\widehat{X}}P\widehat{Y} + \mathcal{A}_{\widehat{X}}Q\widehat{Y} = P\mathcal{H}\nabla_{\widehat{X}}\widehat{Y} + \delta\mathcal{A}_{\widehat{X}}\widehat{Y} - \eta(\widehat{Y})P\widehat{X} \tag{3.7}$$

$$\mathcal{V}\nabla_{\widehat{X}}\delta\widehat{V} + \mathcal{A}_{\widehat{X}}\zeta\widehat{V} = P\mathcal{A}_{\widehat{X}}\widehat{V} + \delta\mathcal{V}\nabla_{\widehat{X}}\widehat{V} \tag{3.8}$$

$$\mathcal{A}_{\widehat{X}}\delta\widehat{V} + \mathcal{H}\nabla_{\widehat{X}}\zeta\widehat{V} = Q\mathcal{A}_{\widehat{X}}\widehat{V} + \zeta\mathcal{V}\nabla_{\widehat{X}}\widehat{V} + g_1(P\widehat{X}, \widehat{V})\xi \tag{3.9}$$

$$\mathcal{V}\nabla_{\widehat{V}}P\widehat{X} + \mathcal{T}_{\widehat{V}}Q\widehat{X} = \delta\mathcal{T}_{\widehat{V}}\widehat{X} + P\mathcal{H}\nabla_{\widehat{V}}\widehat{X} - \eta(\widehat{X})\delta\widehat{V} \tag{3.10}$$

$$\mathcal{T}_{\widehat{V}}P\widehat{X} + \mathcal{H}\nabla_{\widehat{V}}Q\widehat{X} = \zeta\mathcal{T}_{\widehat{V}}\widehat{X} + Q\mathcal{H}\nabla_{\widehat{V}}\widehat{X} + g_1(\zeta\widehat{V}, \widehat{X})\xi - \eta(\widehat{X})\zeta\widehat{V} \tag{3.11}$$

$$\mathcal{V}\nabla_{\widehat{U}}\delta\widehat{V} + \mathcal{T}_{\widehat{U}}\zeta\widehat{V} = P\mathcal{T}_{\widehat{U}}\widehat{V} + \delta\mathcal{V}\nabla_{\widehat{U}}\widehat{V} \tag{3.12}$$

$$\mathcal{T}_{\widehat{U}}\delta\widehat{V} + \mathcal{H}\nabla_{\widehat{U}}\zeta\widehat{V} + \eta(\widehat{V})\zeta\widehat{U} = Q\mathcal{T}_{\widehat{U}}\widehat{V} + \zeta\mathcal{V}\nabla_{\widehat{U}}\widehat{V} + g_1(\delta\widehat{U}, \widehat{V})\xi, \tag{3.13}$$

for any vector fields $\widehat{U}, \widehat{V} \in \Gamma(\ker \vec{f}_*)$ *and* $\widehat{X}, \widehat{Y} \in \Gamma(\ker \vec{f}_*)^\perp$.

Proof. By direct calculation by using equations (3.5), (2.12) and (2.3), (2.5), we can easily obtains relations given by (3.6) and (3.7). In a similar way, from equations (3.2), (3.5), (2.9)-(2.12) and (2.3), (2.5), we obtains all parts of the result. □

Now, we discuss some basic results which are useful to explore the geometry of $\mathcal{QBSC}$ $\xi^\perp$-submersion $\vec{f}: M_1 \to M_2$. For this, define the following :

$$(\nabla_{\widehat{U}}\delta)\widehat{V} = \mathcal{V}\nabla_{\widehat{U}}\delta\widehat{V} - \delta\mathcal{V}\nabla_{\widehat{U}}\widehat{V} \tag{3.14}$$

$$(\nabla_{\widehat{U}}\zeta)\widehat{V} = \mathcal{H}\nabla_{\widehat{U}}\zeta\widehat{V} - \zeta\mathcal{V}\nabla_{\widehat{U}}\widehat{V} \tag{3.15}$$

$$(\nabla_{\widehat{X}}P)\widehat{Y} = \mathcal{V}\nabla_{\widehat{X}}P\widehat{Y} - P\mathcal{H}\nabla_{\widehat{X}}\widehat{Y} \tag{3.16}$$

$$(\nabla_{\widehat{X}}Q)\widehat{Y} = \mathcal{H}\nabla_{\widehat{X}}Q\widehat{Y} - Q\mathcal{H}\nabla_{\widehat{X}}\widehat{Y}, \tag{3.17}$$

for any vector fields $\widehat{U}, \widehat{V} \in \Gamma(\ker \vec{f}_*)$ and $\widehat{X}, \widehat{Y} \in \Gamma(\ker \vec{f}_*)^\perp$.

Lemma 3.5. *Let* $(M_1, \phi, \xi, \eta, g_1)$ *be Kenmotsu manifold and* (M_2, g_2) *be a RM. If* $\vec{f}: M_1 \to M_2$ *is a* $\mathcal{QBSC}$ $\xi^\perp$-*submersion, then we have*

$$(\nabla_{\widehat{U}}\delta)\widehat{V} = P\mathcal{T}_{\widehat{U}}\widehat{V} - \mathcal{T}_{\widehat{U}}\zeta\widehat{V}$$

$$(\nabla_{\widehat{U}}\zeta)\widehat{V} = Q\mathcal{T}_{\widehat{U}}\widehat{V} - \mathcal{T}_{\widehat{U}}\delta\widehat{V} + g_1(\delta\hat{U}, \hat{V})\xi$$

$$(\nabla_{\widehat{X}}P)\widehat{Y} = \delta\mathcal{A}_{\widehat{X}}\widehat{Y} - \mathcal{A}_{\widehat{X}}Q\widehat{Y} - \eta(\hat{Y}P\hat{X})$$

$$(\nabla_{\widehat{X}}Q)\widehat{Y} = \zeta\mathcal{A}_{\widehat{X}}\widehat{Y} - \mathcal{A}_{\widehat{X}}P\widehat{Y} + g_1(Q\hat{X}, \hat{Y})\xi - \eta(\hat{Y})Q\hat{X}$$

for all vector fields $\widehat{U}, \widehat{V} \in \Gamma(ker\vec{f}_*)$ *and* $\widehat{X}, \widehat{Y} \in \Gamma(ker\vec{f}_*)^\perp$.

Proof. By using equations (2.5), (2.9)- (2.12) and equations (3.14)-(3.17), we get the proof of the lemma. □

If the tenors δ and ζ are parallel with respect to the connection ∇ of M_1 then, we have

$$P\mathcal{T}_{\widehat{U}}\widehat{V} = \mathcal{T}_{\widehat{U}}\zeta\widehat{V},\ Q\mathcal{T}_{\widehat{U}}\widehat{V} = \mathcal{T}_{\widehat{U}}\delta\widehat{V} - g_1(\delta\hat{U}, \hat{V})\xi$$

for any vector fields $\widehat{U}, \widehat{V} \in \Gamma(TM_1)$.

4. Integrability and totally geodesicness of distributions

Since, $\vec{f}: M_1 \to M_2$ is a $\mathcal{QBSC}$ $\xi^\perp$-submersion, where $(M_1, \phi, \xi, \eta, g_1)$ representing a Kenmotsu manifold and (M_2, g_2) denoting a Riemannian manifold. The existence of three mutually orthogonal distributions, including an invariant distribution $\mathfrak{D}$, a pair of slant distributions $\mathfrak{D}^{\theta_1}$ and $\mathfrak{D}^{\theta_2}$, is guaranteed by the definition of $\mathcal{QBSC}$ $\xi^\perp$-submersion. We begin the subject of distributions integrability by determining the integrability of the slant distributions as follows:

Theorem 4.1. *Let $\vec{f}$ be a QBSC $\xi^{\perp}$-submersion from Kenmotsu manifold (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2). Then slant distribution $\mathfrak{D}^{\theta_1}$ is integrable if and only if*

$$\begin{aligned}&\frac{1}{\lambda^2}\{g_2(\nabla^{\vec{f}}_{\widehat{U}_1}\vec{f}_*\zeta\widehat{V}_1+\nabla^{\vec{f}}_{\widehat{V}_1}\vec{f}_*\zeta\widehat{U}_1,\vec{f}_*\zeta\mathfrak{C}\widehat{Z})\}\\&=\frac{1}{\lambda^2}\{g_2((\nabla\vec{f}_*)(\widehat{U}_1,\zeta\widehat{V}_1)+(\nabla\vec{f}_*)(\widehat{V}_1,\zeta\widehat{U}_1),\vec{f}_*\zeta\mathfrak{C}\widehat{Z})\}\\&\quad-g_1(\mathcal{T}_{\widehat{V}_1}\zeta\delta\widehat{U}_1-\mathcal{T}_{\widehat{U}_1}\zeta\delta\widehat{V}_1,\widehat{Z})-g_1(\mathcal{T}_{\widehat{U}_1}\zeta\widehat{V}_1-\mathcal{T}_{\widehat{V}_1}\zeta\widehat{U}_1,\phi\mathfrak{A}\widehat{Z}+\delta\mathfrak{C}\widehat{Z}),\end{aligned}\tag{4.1}$$

for any $\widehat{U}_1,\widehat{V}_1\in\Gamma(\mathfrak{D}^{\theta_1})$ and $\widehat{Z}\in\Gamma(\mathfrak{D}\oplus\mathfrak{D}^{\theta_2})$.

Proof. For all $\widehat{U}_1,\widehat{V}_1\in\Gamma(\mathfrak{D}^{\theta_1})$ and $\widehat{Z}\in\Gamma(\mathfrak{D}\oplus\mathfrak{D}^{\theta_2})$ by using equations (2.1), (2.2), (2.5), (2.14) and (3.2), we get

$$g_1([\widehat{U}_1,\widehat{V}_1],\widehat{Z})=g_1(\nabla_{\widehat{U}_1}\delta\widehat{V}_1,\phi\widehat{Z})+g_1(\nabla_{\widehat{U}_1}\zeta\widehat{V}_1,\phi\widehat{Z})-g_1(\nabla_{\widehat{V}_1}\delta\widehat{U}_1,\phi\widehat{Z})-g_1(\nabla_{\widehat{V}_1}\zeta\widehat{U}_1,\phi\widehat{Z}).$$

By using equations (2.1), (2.2), (2.5), and (3.2), we have

$$\begin{aligned}g_1([\widehat{U}_1,\widehat{V}_1],\widehat{Z})=&-g_1(\nabla_{\widehat{U}_1}\delta^2\widehat{V}_1,\widehat{Z})-g_1(\nabla_{\widehat{U}_1}\zeta\delta\widehat{V}_1,\widehat{Z})+g_1(\nabla_{\widehat{V}_1}\delta^2\widehat{U}_1,\widehat{Z})\\&+g_1(\nabla_{\widehat{V}_1}\zeta\delta\widehat{U}_1,\widehat{Z})+g_1(\nabla_{\widehat{U}_1}\zeta\widehat{V}_1,\phi\mathfrak{A}\widehat{Z}+\delta\mathfrak{C}\widehat{Z}+\zeta\mathfrak{C}\widehat{Z})\\&-g_1(\nabla_{\widehat{V}_1}\zeta\widehat{U}_1,\phi\mathfrak{A}\widehat{Z}+\delta\mathfrak{C}\widehat{Z}+\zeta\mathfrak{C}\widehat{Z}).\end{aligned}$$

Taking account the fact of Lemma 3.2 with equation (2.10), we get

$$\begin{aligned}g_1([\widehat{U}_1,\widehat{V}_1],\widehat{Z})=&cos^2\theta_1g_1([\widehat{U}_1,\widehat{V}_1],\widehat{Z})+g_1(\mathcal{T}_{\widehat{V}_1}\zeta\delta\widehat{U}_1-\mathcal{T}_{\widehat{U}_1}\zeta\delta\widehat{V}_1,\widehat{Z})\\&+g_1(\mathcal{T}_{\widehat{U}_1}\zeta\widehat{V}_1-\mathcal{T}_{\widehat{V}_1}\zeta\widehat{U}_1,\phi\mathfrak{A}\widehat{Z}+\delta\mathfrak{C}\widehat{Z})\\&+g_1(\mathcal{H}\nabla_{\widehat{U}_1}\zeta\widehat{V}_1-\mathcal{H}\nabla_{\widehat{V}_1}\zeta\widehat{U}_1,\zeta\mathfrak{C}\widehat{Z}).\end{aligned}$$

By using equation (2.6), formula (2.15) with Lemma 2.1, we finally get

$$\begin{aligned}sin^2\theta_1g_1([\widehat{U}_1,\widehat{V}_1],\widehat{Z})=&\frac{1}{\lambda^2}\{g_2(\nabla^{\vec{f}}_{\widehat{U}_1}\vec{f}_*\zeta\widehat{V}_1-\nabla^{\vec{f}}_{\widehat{V}_1}\vec{f}_*\zeta\widehat{U}_1,\vec{f}_*\zeta\mathfrak{C}\widehat{Z})\}\\&+\frac{1}{\lambda^2}\{-g_2((\nabla\vec{f}_*)(\widehat{U}_1,\zeta\widehat{V}_1),\vec{f}_*\zeta\mathfrak{C}\widehat{Z})+g_2((\nabla\vec{f}_*)(\widehat{V}_1,\zeta\widehat{U}_1),\vec{f}_*\zeta\mathfrak{C}\widehat{Z})\}\\&+g_1(\mathcal{T}_{\widehat{U}_1}\zeta\widehat{V}_1-\mathcal{T}_{\widehat{V}_1}\zeta\widehat{U}_1,\delta\mathfrak{A}\widehat{Z}+\delta\mathfrak{C}\widehat{Z})+g_1(\mathcal{T}_{\widehat{V}_1}\zeta\delta\widehat{U}_1-\mathcal{T}_{\widehat{U}_1}\zeta\delta\widehat{V}_1,\widehat{Z}).\end{aligned}$$

□

In a similar way, we can examine the condition of integrability for slant distribution $\mathfrak{D}^{\theta_2}$ as follows:

Theorem 4.2. *Let $\vec{f}:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ be a QBSC $\xi^{\perp}$-submersion, where (M_1,ϕ,ξ,η,g_1) a Kenmotsu manifold and (M_2,g_2) a RM. Then slant distribution $\mathfrak{D}^{\theta_2}$ is integrable if and only if*

$$\begin{aligned}&\frac{1}{\lambda^2}\{g_2((\nabla\vec{f}_*)(\widehat{U}_2,\zeta\widehat{V}_2)-(\nabla\vec{f}_*)(\widehat{V}_2,\zeta\widehat{U}_2),\vec{f}_*\zeta\mathfrak{B}\widehat{Z})\}\\&=g_1(\mathcal{T}_{\widehat{V}_2}\zeta\delta\widehat{U}_2-\mathcal{T}_{\widehat{U}_2}\zeta\delta\widehat{V}_2,\widehat{Z})+g_1(\mathcal{T}_{\widehat{U}_2}\zeta\widehat{V}_2-\mathcal{T}_{\widehat{V}_2}\zeta\widehat{U}_2,\phi\mathfrak{A}\widehat{Z}+\delta\mathfrak{B}\widehat{Z})\\&\quad+\frac{1}{\lambda^2}\{g_2(\nabla^{\vec{f}}_{\widehat{U}_2}\vec{f}_*\zeta\widehat{V}_2-\nabla^{\vec{f}}_{\widehat{V}_2}\vec{f}_*\zeta\widehat{U}_2,\vec{f}_*\zeta\mathfrak{B}\widehat{Z})\}.\end{aligned}$$

for any $\widehat{U}_2,\widehat{V}_2\in\Gamma(\mathfrak{D}^{\theta_2})$ and $\widehat{Z}\in\Gamma(\mathfrak{D}\oplus\mathfrak{D}^{\theta_1})$.

Proof. By using equations (2.1), (2.2), (2.5) and (3.2), we have

$$\begin{aligned}g_1([\widehat{U}_2,\widehat{V}_2],\widehat{Z})=&g_1(\nabla_{\widehat{V}_2}\delta^2\widehat{U}_2,\widehat{Z})+g_1(\nabla_{\widehat{V}_2}\zeta\delta\widehat{U}_2,\widehat{Z})-g_1(\nabla_{\widehat{U}_2}\delta^2\widehat{V}_2,\widehat{Z})\\&-g_1(\nabla_{\widehat{U}_2}\zeta\delta\widehat{V}_2,\widehat{Z})+g_1(\nabla_{\widehat{U}_2}\zeta\widehat{V}_2-\nabla_{\widehat{V}_2}\zeta\widehat{U}_2,\phi\widehat{Z}),\end{aligned}$$

for any $\widehat{U}_2,\widehat{V}_2\in\Gamma(\mathfrak{D}^{\theta_2})$ and $\widehat{Z}\in\Gamma(\mathfrak{D}\oplus\mathfrak{D}^{\theta_1})$. From equation (2.10) and Lemma 3.3, we get

$$\begin{aligned}sin^2\theta_2g_1([\widehat{U}_2,\widehat{V}_2],\widehat{Z})=&g_1(\mathcal{T}_{\widehat{V}_2}\zeta\delta\widehat{U}_2-\mathcal{T}_{\widehat{U}_2}\zeta\delta\widehat{V}_2,\widehat{Z})+g_1(\mathcal{T}_{\widehat{U}_2}\zeta\widehat{V}_2-\mathcal{T}_{\widehat{V}_2}\zeta\widehat{U}_2,\phi\mathfrak{A}\widehat{Z}+\delta\mathfrak{B}\widehat{Z})\\&+g_1(\mathcal{H}\nabla_{\widehat{U}_2}\zeta\widehat{V}_2-\mathcal{H}\nabla_{\widehat{V}_2}\zeta\widehat{U}_2,\zeta\mathfrak{B}\widehat{Z}).\end{aligned}$$

Since $\vec{f}$ is $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, using conformality condition with equations (2.6) and (2.15), we finally get

$$\begin{aligned} sin^2\theta_2 g_1([\widehat{U}_2,\widehat{V}_2],\widehat{Z}) &= g_1(\mathcal{T}_{\widehat{V}_2}\zeta\delta\widehat{U}_2 - \mathcal{T}_{\widehat{U}_2}\zeta\delta\widehat{V}_2,\widehat{Z}) + g_1(\mathcal{T}_{\widehat{U}_2}\zeta\widehat{V}_2 - \mathcal{T}_{\widehat{V}_2}\zeta\widehat{U}_2,\phi\mathfrak{A}\widehat{Z}+\delta\mathfrak{B}\widehat{Z}) \\ &+ \frac{1}{\lambda^2}\{g_2(\nabla^{\vec{f}}_{\widehat{U}_2}\vec{f}_*\zeta\widehat{V}_2 - \nabla^{\vec{f}}_{\widehat{V}_2}\vec{f}_*\zeta\widehat{U}_2,\vec{f}_*\zeta\mathfrak{B}\widehat{Z})\} \\ &- \frac{1}{\lambda^2}\{g_2((\nabla\vec{f}_*)(\widehat{U}_2,\zeta\widehat{V}_2)+(\nabla\vec{f}_*)(\widehat{V}_2,\zeta\widehat{U}_2),\vec{f}_*\zeta\mathfrak{B}\widehat{Z})\}. \end{aligned}$$

This completes the proof of the theorem. □

Since, the invariant distribution is mutually orthogonal to the slant distributions in accordance with the concept of $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, this led us to investigate the necessary and sufficient condition for the invariant distribution to be integrable.

Theorem 4.3. *Let $\vec{f}:(M_1,\phi,\xi,\eta,g_1)\rightarrow(M_2,g_2)$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, where (M_1,ϕ,ξ,η,g_1) a Kenmotsu manifold and (M_2,g_2) a RM. Then the invariant distribution $\mathfrak{D}$ is integrable if and only if*

$$\begin{aligned} &g_1(\mathcal{T}_{\widehat{U}}\delta\mathfrak{A}\widehat{V} - \mathcal{T}_{\widehat{V}}\delta\mathfrak{A}\widehat{U},\zeta\mathfrak{B}\widehat{Z}+\zeta\widehat{\mathfrak{C}W}) \\ &+ g_1(\mathcal{V}\nabla_{\widehat{U}}\delta\mathfrak{A}\widehat{V} - \mathcal{V}\nabla_{\widehat{V}}\delta\mathfrak{A}\widehat{U},\delta\mathfrak{B}\widehat{Z}+\delta\mathfrak{C}\widehat{Z}) = 0, \end{aligned} \tag{4.2}$$

for any $\widehat{U},\widehat{V}\in\Gamma(\mathfrak{D})$ and $\widehat{Z}\in\Gamma(\mathfrak{D}^{\theta_1}\oplus\mathfrak{D}^{\theta_2})$.

Proof. For all $\widehat{U},\widehat{V}\in\Gamma(\mathfrak{D})$ and $\widehat{Z}\in\Gamma(\mathfrak{D}^{\theta_1}\oplus\mathfrak{D}^{\theta_2})$ with using equations (2.1), (2.2), and decomposition (3.1), we have

$$g_1([\widehat{U},\widehat{V}],\widehat{Z}) = g_1(\nabla_{\widehat{U}}\delta\mathfrak{A}\widehat{V},\phi\mathfrak{B}\widehat{Z}+\phi\mathfrak{C}\widehat{Z}) - g_1(\nabla_{\widehat{V}}\delta\mathfrak{A}\widehat{U},\phi\mathfrak{B}\widehat{Z}+\phi\mathfrak{C}\widehat{Z}).$$

By using equations (2.9), (3.2), we finally have

$$\begin{aligned} g_1([\widehat{U},\widehat{V}],\widehat{Z}) &= g_1(\mathcal{T}_{\widehat{U}}\delta\mathfrak{A}\widehat{V} - \mathcal{T}_{\widehat{V}}\delta\mathfrak{A}\widehat{U},\zeta\mathfrak{B}\widehat{Z}+\zeta\mathfrak{C}\widehat{Z}) \\ &+ g_1(\mathcal{V}\nabla_{\widehat{U}}\delta\mathfrak{A}\widehat{V} - \mathcal{V}\nabla_{\widehat{V}}\delta\mathfrak{A}\widehat{U},\delta\mathfrak{B}\widehat{Z}+\delta\mathfrak{C}\widehat{Z}). \end{aligned}$$

This completes the proof of theorem. □

After discussing the prerequisites for distribution's integrability, it is time to examine the necessary and sufficient conditions that must also exists in order for distributions to be totally geodesic. We begin by looking at the condition of totally geodesicness for invariant distribution.

Theorem 4.4. *Let $\vec{f}:(M_1,\phi,\xi,\eta,g_1)\rightarrow(M_2,g_2)$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, where (M_1,ϕ,ξ,η,g_1) a Kenmotsu manifold and (M_2,g_2) a RM. Then invariant distribution $\mathfrak{D}$ defines totally geodesic foliation on M_1 if and only if*

(i) $\lambda^{-2}g_2\{((\nabla\vec{f}_*)(\widehat{U},\phi\widehat{V}),\vec{f}_*\zeta\widehat{Z})\} = g_1(\mathcal{V}\nabla_{\widehat{U}}\phi\widehat{V},\delta\widehat{Z})$

(ii) $\lambda^{-2}\{g_2((\nabla\vec{f}_*)(\widehat{U},\phi\widehat{V}),\vec{f}_*Q\widehat{X})\} = g_1(\mathcal{V}\nabla_{\widehat{U}}\phi\widehat{V},P\widehat{X}) - g_1(\widehat{U},\widehat{V})\eta(\widehat{X})$,

for any $\widehat{U},\widehat{V}\in\Gamma(\mathfrak{D})$ and $\widehat{Z}\in\Gamma(\mathfrak{D}^{\theta_1}\oplus\mathfrak{D}^{\theta_2})$, $\widehat{X}\in(ker f_*)^{\perp}$.

Proof. For any $\widehat{U},\widehat{V}\in\Gamma(\mathfrak{D})$ and $\widehat{Z}\in\Gamma(\mathfrak{D}^{\theta_1}\oplus\mathfrak{D}^{\theta_2})$ by using equations (2.2), (2.5) and (3.2), we may write

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{Z}) = g_1(\mathcal{V}\nabla_{\widehat{U}}\phi\widehat{V},\delta\widehat{Z}) + g_1(\mathcal{T}_{\widehat{U}}\phi\widehat{V},\zeta\widehat{Z}).$$

By using the horizontal conformality of $\vec{f}$ with equation (2.6) and (2.15), we get

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{Z}) = g_1(\mathcal{V}\nabla_{\widehat{U}}\phi\widehat{V},\delta\widehat{Z}) - \lambda^{-2}g_2((\nabla\vec{f}_*)(\widehat{U},\phi\widehat{V}),\vec{f}_*\zeta\widehat{Z}) - .$$

On the other hand, using equations (2.2), (2.5) with horizontal conformality of $\vec{f}$, we finally have

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = g_1(\mathcal{V}\nabla_{\widehat{U}}\phi\widehat{V},P\widehat{X}) - \lambda^{-2}g_2((\nabla\vec{f}_*)(\widehat{U},\phi\widehat{V}),\vec{f}_*Q\widehat{X}) - g_1(\widehat{U},\widehat{V})\eta(\widehat{X}).$$

This completes the proof of the theorem. □

In similar way, we can discuss the geometry of leaves of slant distribution $\mathfrak{D}^{\theta_1}$ as follows:

Theorem 4.5. *Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion from Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2). Then slant distribution $\mathfrak{D}^{\theta_1}$ defines totally geodesic foliation on M_1 if and only if*

$$\begin{aligned}&\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{Z},\zeta\mathfrak{B}\widehat{W}),\vec{f}_*\zeta\mathfrak{C}\widehat{U})\\&=\frac{1}{\lambda^2}g_2(\nabla^{\vec{f}}_{\widehat{Z}}\vec{f}_*\zeta\mathfrak{B}\widehat{W},\vec{f}_*\zeta\mathfrak{C}\widehat{U})+\cos^2\theta_1 g_1(\nabla_{\widehat{Z}}\mathfrak{B}\widehat{W},\widehat{U})\\&\quad-g_1(\mathcal{T}_{\widehat{Z}}\zeta\delta\mathfrak{B}\widehat{W},\widehat{U})+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},\delta\widehat{A}\widehat{U})+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},\delta\widehat{C}\widehat{U})\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}&\lambda^{-2}\{g_2(\nabla^{\vec{f}}_{\widehat{Z}}\vec{f}_*\zeta\delta\mathfrak{B}\widehat{W},\vec{f}_*\widehat{X})-g_2(\nabla^{\vec{f}}_{\widehat{Z}}\vec{f}_*\zeta\mathfrak{B}\widehat{W},\vec{f}_*Q\widehat{X})\}\\&=\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{Z},\zeta\delta\mathfrak{B}\widehat{W}),\vec{f}_*\widehat{X}))-\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{Z},\zeta\mathfrak{B}\widehat{W}),\vec{f}_*Q\widehat{X}))\\&\quad+cos^2\theta_1 g_1(\nabla_{\widehat{Z}}\mathfrak{B}\widehat{W},\widehat{X})+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},P\widehat{X})-\sin^2\theta_1(\widehat{Z},\widehat{W})\eta(\widehat{X}),\end{aligned}\tag{4.4}$$

for any $\widehat{Z},\widehat{W}\in\Gamma(\mathfrak{D}^{\theta_1}),\widehat{U}\in\Gamma(\mathfrak{D}\oplus\mathfrak{D}^{\theta_2})$ and $\widehat{X}\in\Gamma(ker\vec{f}_*)^{\perp}$.

Proof. By using equations (2.1), (2.2), (2.5) and (3.2), we get

$$g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{U})=g_1(\nabla_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},\phi(\mathfrak{A}\widehat{U}+\mathfrak{C}\widehat{U}))-g_1(\phi\nabla_{\widehat{Z}}\delta\mathfrak{B}\widehat{W},\widehat{U}),$$

for $\widehat{Z},\widehat{W}\in\Gamma(\mathfrak{D}^{\theta_1})$ and $\widehat{U}\in\Gamma(\mathfrak{D}\oplus\mathfrak{D}^{\theta_2})$. Again using equations (2.3), (2.5), (3.2), (2.10) with Lemma 3.2, we may write

$$\begin{aligned}g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{U})=&\,cos^2\theta_1 g_1(\nabla_{\widehat{Z}}\mathfrak{B}\widehat{W},\widehat{U})-g_1(\mathcal{T}_{\widehat{Z}}\zeta\delta\mathfrak{B}\widehat{W},\widehat{U})+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},\delta\mathfrak{A}\widehat{U})\\&+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},\delta\mathfrak{C}\widehat{U})+g_1(\mathcal{H}\nabla_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},\zeta\mathfrak{C}\widehat{U}).\end{aligned}$$

Since, $\vec{f}$ is HCS, using Lemma 2.1 with equations (2.6) and (2.15), we have

$$\begin{aligned}g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{U})=&\,cos^2\theta_1 g_1(\nabla_{\widehat{Z}}\mathfrak{B}\widehat{W},\widehat{U})-g_1(\mathcal{T}_{\widehat{Z}}\zeta\delta\mathfrak{B}\widehat{W},\widehat{U})+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},\delta\mathfrak{A}\widehat{U})\\&+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},\delta\mathfrak{C}\widehat{U})+\frac{1}{\lambda^2}g_2(\nabla^{\vec{f}}_{\widehat{Z}}\vec{f}_*\zeta\mathfrak{B}\widehat{W},\vec{f}_*\zeta\mathfrak{C}\widehat{U})\\&-\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{Z},\zeta\mathfrak{B}\widehat{W}),\vec{f}_*\zeta\mathfrak{C}\widehat{U}).\end{aligned}\tag{4.5}$$

On the other hand, for $\widehat{Z},\widehat{W}\in\Gamma(\mathfrak{D}^{\theta_1})$ and $\widehat{X}\in\Gamma(ker\vec{f}_*)^{\perp}$, by using equations (2.2), (2.4), (2.5) and (3.2), we get

$$g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{X})=g_1(\nabla_{\widehat{Z}}\delta\mathfrak{B}\widehat{W},\phi\widehat{X})+g_1(\nabla_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},\phi\widehat{X})-g_1(\widehat{Z},\widehat{W})\eta(\widehat{X}).$$

From Lemma 3.2 with equations (2.10) and (3.5), the above equation takes the form

$$\begin{aligned}g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{X})=&\,cos^2\theta_1 g_1(\nabla_{\widehat{Z}}\mathfrak{B}\widehat{W},\widehat{X})-g_1(\mathcal{H}\nabla_{\widehat{Z}}\zeta\delta\mathfrak{B}\widehat{W},\widehat{X})+\cos^2\theta_1 g_1(\widehat{Z},\widehat{W})\eta(\widehat{X})\\&+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},P\widehat{X})+g_1(\mathcal{H}\nabla_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},Q\widehat{X})-g_1(\widehat{Z},\widehat{W})\eta(\widehat{X}).\end{aligned}$$

Since $\vec{f}$ is horizontally conformal and from equations (2.6) and (2.15), we have

$$\begin{aligned}g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{X})=&\,cos^2\theta_1 g_1(\nabla_{\widehat{Z}}\mathfrak{B}\widehat{W},\widehat{X})+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{B}\widehat{W},P\widehat{X})\\&+\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{Z},\zeta\delta\mathfrak{B}\widehat{W}),\vec{f}_*\widehat{X})-\frac{1}{\lambda^2}g_2(\nabla^{\vec{f}}_{\widehat{Z}}\vec{f}_*\zeta\delta\mathfrak{B}\widehat{W},\vec{f}_*\widehat{X})\\&-\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{Z},\zeta\mathfrak{B}\widehat{W}),\vec{f}_*Q\widehat{X})+\frac{1}{\lambda^2}g_2(\nabla^{\vec{f}}_{\widehat{Z}}\vec{f}_*\zeta\delta\mathfrak{B}\widehat{W},\vec{f}_*Q\widehat{X})\\&-(1-\cos^2\theta_1)g_1(\widehat{Z},\widehat{W})\eta(\widehat{X}).\end{aligned}$$

This completes the proof of theorem. □

In the following theorem, we study the necessary and sufficient conditions for slant distribution $\mathfrak{D}^{\theta_2}$ to be totally geodesic.

Theorem 4.6. *Let $\vec{f}:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, where (M_1,ϕ,ξ,η,g_1) a Kenmotsu manifold and (M_2,g_2) a RM. Then slant distribution $\mathfrak{D}^{\theta_2}$ defines totally geodesic foliation on M_1 if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{Z},\zeta\mathfrak{C}\widehat{W}),\vec{f}_*\zeta\mathfrak{B}\widehat{V})-cos^2\theta_2 g_1(\nabla_{\widehat{Z}}\mathfrak{C}\widehat{W},\widehat{V})\\
&=-g_1(\mathcal{T}_{\widehat{Z}}\zeta\delta\mathfrak{C}\widehat{W},\widehat{V})+\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\zeta\mathfrak{C}\widehat{W},\widehat{Z}),\vec{f}_*\zeta\mathfrak{B}\widehat{V})\\
&\quad+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{C}\widehat{W},\delta\mathfrak{A}\widehat{V})+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{C}\widehat{W},\delta\mathfrak{B}\widehat{V})
\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}
&\lambda^{-2}\{g_2(\nabla^{\vec{f}}_{\widehat{Z}}\vec{f}_*\zeta\delta\mathfrak{C}\widehat{W},\vec{f}_*\widehat{Y})-g_2(\nabla^{\vec{f}}_{\widehat{Z}}\vec{f}_*\zeta\mathfrak{C}\widehat{W},\vec{f}_*Q\widehat{Y})\}\\
&=\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{Z},\zeta\delta\mathfrak{C}\widehat{W}),\vec{f}_*\widehat{Y})-\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{Z},\zeta\mathfrak{C}\widehat{W}),\vec{f}_*Q\widehat{Y})\\
&\quad+cos^2\theta_2 g_1(\nabla_{\widehat{Z}}\mathfrak{C}\widehat{W},\widehat{Y})+g_1(\mathcal{T}_{\widehat{Z}}\zeta\mathfrak{C}\widehat{W},P\widehat{Y})-\sin^2\theta_2 g_1(\widehat{Z},\widehat{W})\eta(\widehat{Y}),
\end{aligned}\tag{4.7}$$

for any $\widehat{Z},\widehat{W}\in\Gamma(\mathfrak{D}^{\theta_2})$, $\widehat{V}\in\Gamma(\mathfrak{D}\oplus\mathfrak{D}^{\theta_1})$ and $\widehat{Y}\in\Gamma(ker\vec{f}_*)^{\perp}$.

Proof. The proof of above theorem is similar to the proof of Theorem 4.5. □

Since, $\vec{f}$ is $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, its vertical and horizontal distribution are $(ker\vec{f}_*)$ and $(ker\vec{f}_*)^{\perp}$, respectively. Now, we examine the necessary and sufficient conditions under which distributions defines totally geodesic foliation on M_1. With regards to the totally geodesicness of vertical distribution, we have

Theorem 4.7. *Let $\vec{f}:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, where (M_1,ϕ,ξ,η,g_1) a Kenmotsu manifold and (M_2,g_2) a RM. Then $ker\vec{f}_*$ defines totally geodesic foliation on M_1 if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2}\{g_2(\nabla^{\vec{f}}_{\widehat{U}}\vec{f}_*\zeta\delta\mathfrak{B}\widehat{V}+\nabla^{\vec{f}}_{\widehat{U}}\vec{f}_*\zeta\delta\mathfrak{C}\widehat{V},\vec{f}_*\widehat{X})\}\\
&=g_1(\mathcal{T}_{\widehat{U}}\mathfrak{A}\widehat{V}+cos^2\theta_1\mathcal{T}_{\widehat{U}}\mathfrak{B}\widehat{V}+cos^2\theta_2\mathcal{T}_{\widehat{U}}\mathfrak{C}\widehat{V},\widehat{X})+g_1(\mathcal{T}_{\widehat{U}}\zeta\widehat{V},P\widehat{X})\\
&\quad+\frac{1}{\lambda^2}\{g_2((\nabla\vec{f}_*)(\widehat{U},\zeta\delta\mathfrak{B}\widehat{V})+(\nabla\vec{f}_*)(\widehat{U},\zeta\delta\mathfrak{C}\widehat{V}),\vec{f}_*\widehat{X})\}\\
&\quad+\frac{1}{\lambda^2}\{g_2(\nabla^{\vec{f}}_{\widehat{U}}\vec{f}_*\zeta\widehat{V}-(\nabla\vec{f}_*)(\widehat{U},\zeta\widehat{V}),\vec{f}_*Q\widehat{X})\}+g_1(\zeta\widehat{U},\zeta\widehat{V})\eta(\widehat{X}).
\end{aligned}\tag{4.8}$$

for any $\widehat{U},\widehat{V}\in\Gamma(ker\vec{f}_*)$ and $\widehat{X}\in\Gamma(ker\vec{f}_*)^{\perp}$.

Proof. For any $\widehat{U},\widehat{V}\in\Gamma(ker\vec{f}_*)$ and $\widehat{X}\in\Gamma(ker\vec{f}_*)^{\perp}$ by using equations (2.2), (2.4), (2.5) with decomposition (3.1), we get

$$\begin{aligned}
g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X})&=g_1(\nabla_{\widehat{U}}\phi\mathfrak{A}\widehat{V},\phi\widehat{X})+g_1(\nabla_{\widehat{U}}\phi\mathfrak{B}\widehat{V},\phi\widehat{X})\\
&\quad+g_1(\nabla_{\widehat{U}}\phi\mathfrak{C}\widehat{V},\phi\widehat{X})-g_1(\widehat{U},\widehat{V})\eta(\widehat{X}).
\end{aligned}$$

By using (2.1) and the equation (3.2) with Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned}
g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X})&=g_1(\nabla_{\widehat{U}}\mathfrak{A}\widehat{V},\widehat{X})+g_1(\delta\widehat{U},\delta\mathfrak{A}\widehat{V})\eta(\widehat{X})+cos^2\theta_1 g_1(\nabla_{\widehat{U}}\mathfrak{B}\widehat{V},\widehat{X})\\
&\quad+g_1(\delta\widehat{U},\delta\mathfrak{B}\widehat{V})\eta(\widehat{X})+cos^2\theta_2 g_1(\nabla_{\widehat{U}}\mathfrak{C}\widehat{V},\widehat{X})+g_1(\delta\widehat{U},\delta\mathfrak{C}\widehat{V})\eta(\widehat{X})\\
&\quad+g_1(\nabla_{\widehat{U}}\zeta\mathfrak{B}\widehat{V},\phi\widehat{X})-g_1(\nabla_{\widehat{U}}\zeta\delta\mathfrak{B}\widehat{V},\widehat{X})-g_1(\nabla_{\widehat{U}}\zeta\delta\mathfrak{C}\widehat{V},\widehat{X})\\
&\quad+g_1(\nabla_{\widehat{U}}\zeta\mathfrak{C}\widehat{V},\phi\widehat{X})-g_1(\widehat{U},\widehat{V})\eta(\widehat{X}).
\end{aligned}$$

From equations (2.9), (2.10) and (3.5), we may yields

$$\begin{aligned}
g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X})&=g_1(\mathcal{T}_{\widehat{U}}\mathfrak{A}\widehat{V}+cos^2\theta_1\mathcal{T}_{\widehat{U}}\mathfrak{B}\widehat{V}+cos^2\theta_2\mathcal{T}_{\widehat{U}}\mathfrak{C}\widehat{V},\widehat{X})-g_1(\widehat{U},\widehat{V})\eta(\widehat{X})\\
&\quad-g_1(\mathcal{H}\nabla_{\widehat{U}}\zeta\delta\mathfrak{B}\widehat{V}+\mathcal{H}\nabla_{\widehat{U}}\zeta\delta\mathfrak{C}\widehat{V},\widehat{X})+g_1(\mathcal{T}_{\widehat{U}}\zeta\mathfrak{B}\widehat{V}+\mathcal{T}_{\widehat{U}}\zeta\mathfrak{C}\widehat{V},P\widehat{X})\\
&\quad+g_1(\mathcal{H}\nabla_{\widehat{U}}\zeta\mathfrak{B}\widehat{V}+\mathcal{H}\nabla_{\widehat{U}}\zeta\mathfrak{C}\widehat{V},Q\widehat{X})+g_1(\delta\widehat{U},\delta\widehat{V})\eta(\widehat{X}).
\end{aligned}$$

From decomposition (3.1), the above equation takes the form

$$\begin{aligned} g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) &= g_1(\mathcal{T}_{\widehat{U}}\mathfrak{A}\widehat{V} + cos^2\theta_1\mathcal{T}_{\widehat{U}}\mathfrak{B}\widehat{V} + cos^2\theta_2\mathcal{T}_{\widehat{U}}\mathfrak{C}\widehat{V},\widehat{X}) + g_1(\mathcal{T}_{\widehat{U}}\zeta\widehat{V},P\widehat{X}) \\ &- g_1(\mathcal{H}\nabla_{\widehat{U}}\zeta\delta\mathfrak{B}\widehat{V} + \mathcal{H}\nabla_{\widehat{U}}\zeta\delta\mathfrak{C}\widehat{V},\widehat{X}) + g_1(\mathcal{H}\nabla_{\widehat{U}}\zeta\widehat{V},Q\widehat{X}) - g_1(\zeta\widehat{U},\zeta\widehat{V})\eta(\widehat{X}). \end{aligned}$$

Using the horizontal conformality of $\vec{f}$ with equations (2.6) and (2.15), we have

$$\begin{aligned} g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) &= g_1(\mathcal{T}_{\widehat{U}}\mathfrak{A}\widehat{V} + cos^2\theta_1\mathcal{T}_{\widehat{U}}\mathfrak{B}\widehat{V} + cos^2\theta_2\mathcal{T}_{\widehat{U}}\mathfrak{C}\widehat{V},\widehat{X}) + g_1(\mathcal{T}_{\widehat{U}}\zeta\widehat{V},P\widehat{X}) \\ &+ \frac{1}{\lambda^2}\{g_2((\nabla\vec{f}_*)(\widehat{U},\zeta\delta\mathfrak{B}\widehat{V}) + (\nabla\vec{f}_*)(\widehat{U},\zeta\delta\mathfrak{C}\widehat{V}),\vec{f}_*\widehat{X})\} \\ &- \frac{1}{\lambda^2}\{g_2(\nabla^{\vec{f}}_{\widehat{U}}\vec{f}_*\zeta\delta\mathfrak{B}\widehat{V} + \nabla^{\vec{f}}_{\widehat{U}}\vec{f}_*\zeta\delta\mathfrak{C}\widehat{V},\vec{f}_*\widehat{X})\} \\ &+ \frac{1}{\lambda^2}\{g_2(\nabla^{\vec{f}}_{\widehat{U}}\vec{f}_*\zeta\widehat{V} - (\nabla\vec{f}_*)(\widehat{U},\zeta\widehat{V}),\vec{f}_*Q\widehat{X})\} - g_1(\zeta\widehat{U},\zeta\widehat{V})\eta(\widehat{X}). \end{aligned}$$

This completes the proof of the theorem. □

We can now talk about the geometry of leaves of horizontal distribution. The following theorem presents the necessary and sufficient condition under which horizontal distribution defines totally geodesic foliation on M_1.

Theorem 4.8. *Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion from Kenmotsu manifold (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2),. Then $(ker\vec{f}_*)^{\perp}$ defines totally geodesic foliation on M_1 if and only if*

$$\begin{aligned} &\frac{1}{\lambda^2}g_2(\nabla^{\vec{f}}_{\widehat{X}}\vec{f}_*Q\widehat{Y},\vec{f}_*\zeta\widehat{Z}) + \frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{X},\zeta P\widehat{Y}),\vec{f}_*\widehat{Z}) \\ &g_1(\mathcal{V}\nabla_{\widehat{X}}\delta P\widehat{Y},\widehat{Z}) - g_1(\mathcal{A}_{\widehat{X}}Q\widehat{Y},\delta\widehat{Z}) + g_1(\widehat{X},grad\ln\lambda)g_1(Q\widehat{Y},\zeta\widehat{Z}) \\ &\quad + g_1(Q\widehat{Y},grad\ln\lambda)g_1(\widehat{X},\zeta\widehat{Z}) - g_1(\zeta\widehat{Z},grad\ln\lambda)g_1(\widehat{X},Q\widehat{Y}), \end{aligned} \tag{4.9}$$

for any $\widehat{X},\widehat{Y}\in\Gamma(ker\vec{f}_*)^{\perp}$ and $\widehat{Z}\in\Gamma(ker\vec{f}_*)$.

Proof. Note that, we have from **??**,

$$g_1((\nabla_{\widehat{X}}\phi)\widehat{Y},\phi\widehat{Z}) = g_1(\phi\widehat{X},\widehat{Y})\eta(\phi\widehat{Z}) - \eta(\widehat{Y})g_1(\phi\widehat{X},\phi\widehat{Z}) = 0,$$

where we used $\eta\circ\phi = 0$ and $g_1(\phi\widehat{X},\phi\widehat{Z}) = g_1(\widehat{X},\widehat{Z}) - \eta(\widehat{X})\eta(\widehat{Z}) = 0$. Similarly, we have that

$$g_1((\nabla_{\widehat{X}}\phi)P\widehat{Y},\phi\widehat{Z}) = g_1(\phi\widehat{X},P\widehat{Y})\eta(\phi\widehat{Z}) - \eta(P\widehat{Y})g_1(\phi\widehat{X},\phi\widehat{Z}) = 0,$$

where we have used that $\eta(\widehat{Z})=\eta(P\widehat{Y})=0$. We used these in next computation. By applying (**??**), (**??**) and (**??**), we compute as follows:

$$\begin{aligned}
g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{Z}) =& g_1(\phi\nabla_{\widehat{X}}\widehat{Y},\phi\widehat{Z})-\eta(\nabla_{\widehat{X}}\widehat{Y})\eta(\widehat{Z})\\
&= g_1(\nabla_{\widehat{X}}\phi\widehat{Y}-(\nabla_{\widehat{X}}\phi)\widehat{Y},\phi\widehat{Z})\\
&= -g_1(\phi^2\nabla_{\widehat{X}}P\widehat{Y},\phi\widehat{Z})+g_1(\nabla_{\widehat{X}}Q\widehat{Y},\phi\widehat{Z})\\
&= -g_1(\phi\nabla_{\widehat{X}}P\widehat{Y},\widehat{Z})+g_1(\nabla_{\widehat{X}}Q\widehat{Y},\delta\widehat{Z}+\zeta\widehat{Z})\\
&= -g_1(\nabla_{\widehat{X}}\phi P\widehat{Y}-(\nabla_{\widehat{X}}\phi)P\widehat{Y},\widehat{Z})+g_1(\nabla_{\widehat{X}}Q\widehat{Y},\delta\widehat{Z}+\zeta\widehat{Z})\\
&= -g_1(\nabla_{\widehat{X}}\delta P\widehat{Y},\widehat{Z})-g_1(\nabla_{\widehat{X}}\zeta P\widehat{Y},\widehat{Z})\\
&+g_1(\nabla_{\widehat{X}}Q\widehat{Y},\delta\widehat{Z})+g_1(\nabla_{\widehat{X}}Q\widehat{Y},\zeta\widehat{Z})\\
&= -g_1(\mathcal{V}\nabla_X\delta P\widehat{Y},\widehat{Z})-g_1(\mathcal{A}_{\widehat{X}}\zeta P\widehat{Y},\widehat{Z})\\
&+g_1(\mathcal{H}\nabla_X QY,\zeta\widehat{Z})+g_1(\mathcal{A}_{\widehat{X}}Q\widehat{Y},\delta\widehat{Z})\\
&= -g_1(\mathcal{V}\nabla_X\delta P\widehat{Y},\widehat{Z})-\frac{1}{\lambda^2}g_2(\vec{f}_*\mathcal{A}_{\widehat{X}}\zeta P\widehat{Y},\vec{f}_*\widehat{Z})\\
&+\frac{1}{\lambda^2}g_2(\nabla^{\vec{f}_*}_{\widehat{X}}\vec{f}_*Q\widehat{Y}-(\nabla\vec{f}_*)(\widehat{X},Q\widehat{Y}),\vec{f}_*\zeta\widehat{Z})+g_1(\mathcal{A}_{\widehat{X}}Q\widehat{Y},\delta\widehat{Z})\\
&= -g_1(\mathcal{V}\nabla_{\widehat{X}}\delta P\widehat{Y},\widehat{Z})+\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{X},\zeta P\widehat{Y}),\vec{f}_*\widehat{Z})\\
&+\frac{1}{\lambda^2}g_2(\nabla^{\vec{f}_*}_{\widehat{X}}\vec{f}_*Q\widehat{Y},\vec{f}_*\zeta\widehat{Z})+g_1(\mathcal{A}_{\widehat{X}}Q\widehat{Y},\delta\widehat{Z})\\
&-g_1(\widehat{X},grad\ln\lambda)g_1(Q\widehat{Y},\zeta\widehat{Z})-g(Q\widehat{Y},grad\ln\lambda)g_1(\widehat{X},\zeta\widehat{Z})\\
&+g(\zeta\widehat{Z},grad\ln\lambda)g_1(\widehat{X},Q\widehat{Y}).
\end{aligned}\tag{4.10}$$

This completes the proof of theorem. ☐

We now have some necessary and sufficient conditions for $\mathcal{QBSC}$ $\xi^{\perp}$-submersion $\vec{f}:M_1\to M_2$ to be totally geodesic map. In this regard, we are presenting the following result.

Theorem 4.9. *Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion from Kenmotsu manifold (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2). Then $\vec{f}:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ is totally geodesic map if and only if*

(i) $\vec{f}_*\{cos^2\theta_1\nabla_{\widehat{U}}\mathfrak{B}\widehat{V}+cos^2\theta_2\nabla_{\widehat{U}}\mathfrak{C}\widehat{V}-\nabla_{\widehat{U}}\zeta\delta\mathfrak{B}\widehat{V}-\nabla_{\widehat{U}}\zeta\delta\mathfrak{C}\widehat{V}\}$
$=\vec{f}_*\{Q(\mathcal{H}\nabla_{\widehat{U}}\zeta\mathfrak{B}\widehat{V}+\mathcal{H}\nabla_{\widehat{U}}\zeta\mathfrak{C}\widehat{V}+\mathcal{T}_{\widehat{U}}\delta\mathfrak{A}\widehat{V})\}$
$+\vec{f}_*\{\zeta(\mathcal{T}_{\widehat{U}}\zeta\mathfrak{B}\widehat{V}+\mathcal{T}_{\widehat{U}}\zeta\mathfrak{C}\widehat{V}+\mathcal{V}\nabla_{\widehat{U}}\delta\mathfrak{A}\widehat{V})\}$
$+g_1(\widehat{U},\widehat{V})-g_1(\delta\widehat{U},\delta B\widehat{V})-g_1(\delta\widehat{U},\delta C\widehat{V})\vec{f}_*\xi$

(ii) $\vec{f}_*\{cos^2\theta_1\nabla_{\widehat{X}}\mathfrak{B}\widehat{U}+cos^2\theta_2\nabla_{\widehat{X}}\mathfrak{C}\widehat{U}-\nabla_{\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U}-\nabla_{\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U}\}$
$=\vec{f}_*\{Q(\mathcal{A}_{\widehat{X}}\delta\mathfrak{A}\widehat{U}+\mathcal{H}\nabla_{\widehat{X}}\zeta\mathfrak{B}\widehat{U}+\mathcal{H}\nabla_{\widehat{X}}\zeta\mathfrak{C}\widehat{U})\}$
$-g_1(P\widehat{X},\delta\mathfrak{B}\widehat{U})\vec{f}_*-g_1(P\widehat{X},\delta\mathfrak{C}\widehat{U})\vec{f}_*$

for any $\widehat{U},\widehat{V}\in\Gamma(ker\vec{f}_*)$ and $\widehat{X},\widehat{Y}\in\Gamma(ker\vec{f}_*)^{\perp}$.

Proof. Now, using equations (2.15), (2.5), (2.3)and (2.1). we can write

$$(\nabla\vec{f}_*)(\widehat{U},\widehat{V})=-\vec{f}_*\{\eta(\nabla_{\widehat{U}}\widehat{V})\xi-\phi\nabla_{\widehat{U}}\phi\widehat{V}\},$$

for any $\widehat{U},\widehat{V}\in\Gamma(ker\vec{f}_*)$. From decomposition (3.1) and equation (3.2), we have

$$\begin{aligned}
(\nabla\vec{f}_*)(\widehat{U},\widehat{V})=\vec{f}_*\{&\phi\nabla_{\widehat{U}}\delta\mathfrak{A}\widehat{V}+\phi\nabla_{\widehat{U}}\delta\mathfrak{B}\widehat{V}+\phi\nabla_{\widehat{U}}\zeta\mathfrak{B}\widehat{V}\\
&+\phi\nabla_{\widehat{U}}\delta\mathfrak{C}\widehat{V}+\phi\nabla_{\widehat{U}}\zeta\mathfrak{C}\widehat{V}+g_1(\widehat{U},\widehat{V})\xi\}.
\end{aligned}$$

By using equations (2.3), (2.5), (2.9) and (2.10), the above equation takes the form

$$\begin{aligned}
(\nabla\vec{f}_*)(\widehat{U},\widehat{V})=&\vec{f}_*\{\phi\mathcal{T}_{\widehat{U}}\delta\mathfrak{A}\widehat{V}+\phi\mathcal{V}\nabla_{\widehat{U}}\delta\mathfrak{A}\widehat{V}\}+\vec{f}_*(\nabla_{\widehat{U}}\phi\delta\mathfrak{B}\widehat{V})-g_1(\delta\widehat{U},\delta\mathfrak{B}\widehat{V})\vec{f}_*\xi\\
&+\vec{f}_*(\phi\mathcal{T}_{\widehat{U}}\zeta\mathfrak{B}\widehat{V}+\phi\mathcal{H}\nabla_{\widehat{U}}\zeta\mathfrak{B}\widehat{V})+\vec{f}_*(\nabla_{\widehat{U}}\phi\delta\mathfrak{B}\widehat{V})-g_1(\delta\widehat{U},\delta\mathfrak{C}\widehat{V})\vec{f}_*\xi\\
&+\vec{f}_*\{\phi\mathcal{T}_{\widehat{U}}\zeta\mathfrak{C}\widehat{V}+\phi\mathcal{H}\nabla_{\widehat{U}}\zeta\mathfrak{C}\widehat{V}+g_1(\widehat{U},\widehat{V})\xi\}.
\end{aligned}$$

Since $\vec{f}$ is HCS, by using Lemma 3.2 and Lemma 3.3 with equation (3.2), we finally get

$$\begin{aligned}(\nabla \vec{f}_*)(\widehat{U},\widehat{V}) = \vec{f}*\{&Q(\mathcal{H}\nabla_{\widehat{U}}\zeta\mathfrak{B}\widehat{V} + \mathcal{H}\nabla_{\widehat{U}}\zeta\mathfrak{C}\widehat{V} + \mathcal{T}_{\widehat{U}}\delta\mathfrak{A}\widehat{V})\\ &+ \zeta(\mathcal{V}\nabla_{\widehat{U}}\delta\mathfrak{A}\widehat{V} + \mathcal{T}_{\widehat{U}}\zeta\mathfrak{B}\widehat{V} + \mathcal{T}_{\widehat{U}}\zeta\mathfrak{C}\widehat{V})\}\\ &- \vec{f}_*\{cos^2\theta_1\nabla_{\widehat{U}}\mathfrak{B}\widehat{V} + cos^2\theta_2\nabla_{\widehat{U}}\mathfrak{C}\widehat{V} - \nabla_{\widehat{U}}\zeta\delta\mathfrak{B}\widehat{V} - \nabla_{\widehat{U}}\zeta\delta\mathfrak{C}\widehat{V}\}\\ &+ \{g_1(\widehat{U},\widehat{V})\} - g_1(\delta\widehat{U},\delta\mathfrak{B}\widehat{V}) - g_1(\delta\widehat{U},\delta\mathfrak{C}\widehat{V})\vec{f}_*\xi.\end{aligned}$$

From this, the (i) part of theorem proved. On the other hand, for any $\widehat{U}\in\Gamma(ker\vec{f}_*)$ and $\widehat{X}\in\Gamma(ker\vec{f}_*)^{\perp}$ by using equations (2.15), (2.5), (2.3) (2.4)and (2.1), we can write

$$(\nabla \vec{f}_*)(\widehat{X},\widehat{U}) = \vec{f}_*(\phi\nabla_{\widehat{X}}\phi\widehat{U}).$$

By using decomposition (3.1) with equation (3.2), we have

$$(\nabla \vec{f}_*)(\widehat{X},\widehat{U}) = \vec{f}_*\{\phi(\nabla_{\widehat{X}}\delta\mathfrak{A}\widehat{U} + \nabla_{\widehat{X}}\delta\mathfrak{B}\widehat{U} + \nabla_{\widehat{X}}\zeta\mathfrak{B}\widehat{U} + \nabla_{\widehat{X}}\delta\mathfrak{C}\widehat{U} + \nabla_{\widehat{X}}\zeta\mathfrak{C}\widehat{U})\}.$$

By taking account the fact from equations (2.11) and (2.12), we get

$$\begin{aligned}(\nabla \vec{f}_*)(\widehat{X},\widehat{U}) = \vec{f}_*\{&\phi(\mathcal{A}_{\widehat{X}}\delta\mathfrak{A}\widehat{U} + \mathcal{V}\nabla_{\widehat{X}}\delta\mathfrak{A}\widehat{U}) + \nabla_{\widehat{X}}\phi\delta\mathfrak{B}\widehat{U} - g_1(P\widehat{X},\delta\mathfrak{B}\widehat{U})\xi\\ &+ \phi(\mathcal{H}\nabla_{\widehat{X}}\zeta\mathfrak{B}\widehat{U} + \mathcal{A}_{\widehat{X}}\zeta\mathfrak{B}\widehat{U}) + \nabla_{\widehat{X}}\phi\delta\mathfrak{C}\widehat{U} - g_1(P\widehat{X},\delta\mathfrak{C}\widehat{U})\xi\\ &+ \phi(\mathcal{H}\nabla_{\widehat{X}}\zeta\mathfrak{C}\widehat{U} + \mathcal{A}_{\widehat{X}}\zeta\mathfrak{C}\widehat{U})\}.\end{aligned}$$

Finally, since $\vec{f}_*$ is a HCS, and from Lemma 3.2, Lemma 3.3, we can write

$$\begin{aligned}(\nabla \vec{f}_*)(\widehat{X},\widehat{U}) = &\vec{f}_*\{Q(\mathcal{A}_{\widehat{X}}\delta\mathfrak{A}\widehat{U} + \mathcal{H}\nabla_{\widehat{X}}\zeta\mathfrak{B}\widehat{U} + \mathcal{H}\nabla_{\widehat{X}}\zeta\mathfrak{C}\widehat{U})\}\\ &+ \vec{f}_*\{\zeta(\mathcal{V}\nabla_{\widehat{X}}\delta\mathfrak{A}\widehat{U} + \mathcal{A}_{\widehat{X}}\zeta\mathfrak{B}\widehat{U} + \mathcal{A}_{\widehat{X}}\zeta\mathfrak{C}\widehat{U})\}\\ &- \vec{f}_*(cos^2\theta_1\nabla_{\widehat{X}}\mathfrak{B}\widehat{U} + cos^2\theta_2\nabla_{\widehat{X}}\mathfrak{C}\widehat{U} - \nabla_{\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U} - \nabla_{\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U})\\ &- g_1(P\widehat{X},\delta\mathfrak{B}\widehat{U})\vec{f}_*\xi - g_1(P\widehat{X},\delta\mathfrak{C}\widehat{U})\vec{f}_*\xi.\end{aligned}$$

From which we obtain (ii) part of theorem. This completes the proof of theorem. □

5. Decomposition Theorems

In this section, we recall the following result from [31] and discuss some decomposition theorems by using prior theorems. Let g be a Riemannian metric tensor on the manifold $M = M_1 \times M_2$, then

(i) $M = M_1 \times_{\lambda} M_2$ is a locally product if and only if M_1 and M_2 are totally geodesic foliations,
(ii) a warped product $M_1 \times_{\lambda} M_2$ if and only if M_1 is a totally geodesic foliation and M_2 is a spherics foliation, i.e., it is umbilic and its mean curvature vector field is parallel,
(iii) $M = M_1 \times_{\lambda} M_2$ is a twisted product if and only if M_1 is a totally geodesic foliation and M_2 is a totally umbilic foliation.

The presence of three orthogonal complementary distributions $\mathfrak{D}$, $\mathfrak{D}^{\theta_1}$, and $\mathfrak{D}^{\theta_2}$, which are integrable and totally geodesic under the conditions that we have stated previously, is ensured by the fact that $\vec{f}:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ is $\mathcal{QBSC}$ $\xi^{\perp}$-submersion. It makes sense to now look for the conditions in which the total space M_1 converts into locally twisted product manifolds. In order to explore the geometry of $\mathcal{QBSC}$ $\xi^{\perp}$-submersion $\vec{f}$, we are providing the following result.

Theorem 5.1. *Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion from Kenmotsu manifold (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2). Then M_1 is locally twisted product of the form $M_{1(\ker\vec{f}_*)}\times M_{1(\ker\vec{f}_*)^{\perp}}$ if and only if*

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{U},\zeta\widehat{V}),\vec{f}_*Q\widehat{X}) &= g_1(\mathcal{T}_{\widehat{U}}\delta\widehat{V},Q\widehat{X}) + g_1(\mathcal{V}\nabla_{\widehat{U}}\delta\widehat{V} + \mathcal{T}_{\widehat{U}}\zeta\widehat{V},P\widehat{X})\\ &+ \frac{1}{\lambda^2}g_2(\nabla^{\vec{f}}_{U}\vec{f}_*\zeta\widehat{V},\vec{f}_*Q\widehat{X}) - g_1(\widehat{U},\widehat{V})\eta(\widehat{X})\end{aligned} \tag{5.1}$$

and

$$g_1(\widehat{X},\widehat{Y})H = Q(\nabla_{\widehat{X}}\phi\widehat{Y}) + \vec{f}_*|^{-1}_{(\ker \vec{f}_*)^\perp}(\nabla^{\vec{f}}_{\widehat{X}}\vec{f}_*Q\widehat{Y}) + \mathcal{A}_{\widehat{X}}P\widehat{Y} + \zeta(grad\ln\lambda)g_1(\widehat{X},Q\widehat{Y}) - Q(G\ln\lambda)g_1(Q\widehat{Y},\zeta\widehat{U}) - Q(G\ln\lambda)g_1(\widehat{X},\zeta\widehat{U}). \tag{5.2}$$

where H is a mean curvature vector and for any $\widehat{U},\widehat{V}\in\Gamma(ker\vec{f}_*)$ and $\widehat{X}\in\Gamma(ker\vec{f}_*)^\perp$.

Proof. For any $\widehat{U},\widehat{V}\in\Gamma(ker\vec{f}_*)$ and $\widehat{X}\in\Gamma(ker\vec{f}_*)^\perp$ and using equations (2.2), (2.5), (2.11) and (2.12), we have

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = g_1(\mathcal{T}_{\widehat{U}}\delta\widehat{V},Q\widehat{X}) + g_1(\mathcal{V}\nabla_{\widehat{U}}\delta\widehat{V} + \mathcal{T}_{\widehat{U}}\zeta\widehat{V},P\widehat{X}) - g_1(\widehat{U},\widehat{V})\eta(\widehat{X}) + g_1(\mathcal{H}\nabla_{\widehat{U}}\zeta\widehat{V},Q\widehat{X})$$

From using formula (2.6), (2.15) and since $\vec{f}$ is HCS, the above equation finally takes the form

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = g_1(\mathcal{T}_{\widehat{U}}\delta\widehat{V},Q\widehat{X}) + g_1(\mathcal{V}\nabla_{\widehat{U}}\delta\widehat{V} + \mathcal{T}_{\widehat{U}}\zeta\widehat{V},P\widehat{X}) - g_1(\widehat{U},\widehat{V})\eta(\widehat{X}) - \frac{1}{\lambda^2}g_2((\nabla\vec{f}_*)(\widehat{U},\zeta\widehat{V}),f_*Q\widehat{X}) + \frac{1}{\lambda^2}g_2(\nabla^{\vec{f}_*}_{\widehat{U}}\zeta\widehat{V},\vec{f}_*Q\widehat{X})$$

It follows that the equation (5.1) satisfies if and only if $M_{1_{(ker\vec{f}_*)}}$ is totally geodesic. On the other hand, for $\widehat{U}\in\Gamma(ker\vec{f}_*)$ and $\widehat{X},\widehat{Y}\in\Gamma(ker\vec{f}_*)^\perp$ by using equations (2.2), (2.5) and (3.5), we get

$$\begin{aligned} g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{U}) &= g_1(\nabla_{\widehat{X}}P\widehat{Y},\phi\widehat{U}) + g_1(\nabla_{\widehat{X}}Q\widehat{Y},\phi\widehat{U}) \\ &= g_1(\mathcal{V}\nabla_{\widehat{X}}P\widehat{Y} + \mathcal{A}_{\widehat{X}}Q\widehat{Y},\delta\widehat{U}) \\ &+ g_1(\mathcal{A}_{\widehat{X}}P\widehat{Y}\zeta\widehat{U})g_1(\mathcal{H}\nabla_{\widehat{X}}Q\widehat{Y},\zeta\widehat{U}). \end{aligned}$$

By using the equations (2.10) with the horizontal conformality of $\vec{f}$, we may write

$$g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{U}) = g_1(\mathcal{V}\nabla_{\widehat{X}}P\widehat{Y} + \mathcal{A}_{\widehat{X}}Q\widehat{Y},\delta\widehat{U}) + \frac{1}{\lambda^2}g_2(\nabla^{\vec{f}}_{\widehat{X}}\vec{f}_*Q\widehat{Y},\vec{f}_*\zeta\widehat{U}) \\ g_1(\mathcal{A}_{\widehat{X}}P\widehat{Y},\zeta\widehat{U}) - g_1(X,G\ln\lambda)g_1(Q\widehat{Y},\zeta\widehat{U}) - g_1(Q\widehat{Y},G\ln\lambda)g_1(\widehat{X},\zeta\widehat{U}) + g_1(\zeta\widehat{U},grad\ln\lambda)g_1(\widehat{X},\zeta Q\widehat{Y}).$$

From the above equation we conclude that $M_{1_{(ker\vec{f}_*)^\perp}}$ is totally umbilical if and only if equation (5.2) satisfied □

6. ϕ-Pluriharmonicity of Quasi bi-slant Conformal $\xi^\perp$-Submersion

Y. Ohnita established J-pluriharminicity from a almost hermitian manifold in [25]. In this section, we extend the concept of ϕ-pluriharmonicity to almost contact metric manifolds.

Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from Kenmotsu manifold (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2) with slant angles θ_1 and θ_2. Then $\mathcal{QBSC}$ submersion is ϕ-pluriharmonic, $\mathfrak{D}$-ϕ-pluriharmonic, $\mathfrak{D}^{\theta_i}$-ϕ-pluriharmonic, $(\mathfrak{D}-\mathfrak{D}^{\theta_i})$-$\phi$ pluriharmonic (where $i=1,2$), $ker\vec{f}_*$-ϕ-pluriharmonic, $(ker\vec{f}_*)^\perp$-ϕ-pluriharmonic and $((ker\vec{f}_*)^\perp - ker\vec{f}_*)$-$\phi$-pluriharmonic if

$$(\nabla\vec{f}_*)(\widehat{U},\widehat{V}) + (\nabla\vec{f}_*)(\phi\widehat{U},\phi\widehat{V}) = 0, \tag{6.1}$$

for any $\widehat{U},\widehat{V}\in\Gamma(\mathfrak{D})$, for any $\widehat{U},\widehat{V}\in\Gamma(\mathfrak{D}^{\theta_i})$, for any $\widehat{U}\in\Gamma(\mathfrak{D}),\widehat{V}\in\Gamma(\mathfrak{D}^{\theta_i})$ (where $i=1,2$), for any $\widehat{U},\widehat{V}\in\Gamma(ker\vec{f}_*)$, for any $\widehat{U},\widehat{V}\in\Gamma(ker\vec{f}_*)^\perp$ and for any $\widehat{U}\in\Gamma(ker\vec{f}_*)^\perp,\widehat{V}\in\Gamma(ker\vec{f}_*)$ respectively.

Theorem 6.1. *Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from Kenmotsu manifold (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2) with slant angles θ_1 and θ_2. Suppose that $\vec{f}$ is $\mathfrak{D}^{\theta_1}$-ϕ-pluriharmonic. Then $\mathfrak{D}^{\theta_1}$ defines totally geodesic foliation M_1 if and only if*

$$\begin{aligned} &\vec{f}_*(\zeta\mathcal{T}_{\delta\widehat{U}}\zeta\delta\widehat{V} + Q\mathcal{H}\nabla_{\delta\widehat{U}}\zeta\delta\widehat{V}) - \vec{f}_*(\mathcal{A}_{\zeta\widehat{U}}\delta\widehat{V} + \mathcal{H}\nabla_{\delta\widehat{U}}\zeta\widehat{V}) \\ &= \cos^2\theta_1\vec{f}_*(Q\mathcal{T}_{\delta\widehat{U}}\widehat{V} + \zeta\mathcal{V}\nabla_{\delta\widehat{U}}\widehat{V}) - \nabla^{\vec{f}}_{\phi\widehat{U}}\vec{f}_*\phi\widehat{V} - \cos^2\theta_1 g_1(\widehat{U},\widehat{V})\vec{f}_*\xi + \nabla^{\vec{f}}_{\zeta\widehat{U}}\vec{f}_*\zeta\widehat{V} \\ &\quad - \zeta\widehat{U}(\ln\lambda)\vec{f}_*\zeta\widehat{V} - \zeta\widehat{V}(\ln\lambda)\vec{f}_*\zeta\widehat{U} + \sin^2\theta_1 g_1(\widehat{U},\widehat{V})\vec{f}_*(grad\ln\lambda) \end{aligned}$$

for any $\widehat{U},\widehat{V}\in\Gamma(\mathfrak{D}^{\theta_1})$.

Proof. For any $\widehat{U}, \widehat{V} \in \Gamma(\mathfrak{D}^{\theta_1})$ and since, $\vec{f}$ is $\mathfrak{D}^{\theta_1}$-ϕ-pluriharmonic, then by using equation (2.9) and (2.15), we have

$$\begin{aligned} 0 =& (\nabla \vec{f}_*)(\widehat{U}, \widehat{V}) + (\nabla \vec{f}_*)(\phi\widehat{U}, \phi\widehat{V}) \\ \vec{f}_*(\nabla_{\widehat{U}}\widehat{V}) \Leftrightarrow& -\vec{f}_*(\nabla_{\phi\widehat{U}}\phi\widehat{V}) + \nabla^{\vec{f}}_{\phi\widehat{U}}\vec{f}_*(\phi\widehat{V}) \\ =& \quad -\vec{f}_*(\mathcal{A}_{\zeta\widehat{U}}\delta\widehat{V} + \mathcal{V}\nabla_{\zeta\widehat{U}}\delta\widehat{V} + \mathcal{T}_{\delta\widehat{U}}\zeta\widehat{V} + \mathcal{H}\nabla_{\delta\widehat{U}}\zeta\widehat{V}) \\ & + (\nabla\vec{f}_*)(\zeta\widehat{U}, \zeta\widehat{V}) - \nabla^{\vec{f}}_{\zeta\widehat{U}}\vec{f}_*\zeta\widehat{V} + \nabla^{\vec{f}}_{\phi\widehat{U}}\vec{f}_*\phi\widehat{V} \\ & + \vec{f}_*(\phi\nabla_{\delta\widehat{U}}\phi\delta\widehat{V} - \eta(\nabla_{\delta\widehat{U}}\delta\widehat{V})\xi) \end{aligned}$$

On using equations (3.2), (3.5) with Lemma 2.1 and Lemma 3.2, the above equation finally takes the form

$$\begin{aligned} \vec{f}_*(\nabla_{\widehat{U}}\widehat{V}) =& -\cos^2\theta_1\vec{f}_*(P\mathcal{T}_{\delta\widehat{U}}\widehat{V} + Q\mathcal{T}_{\delta\widehat{U}}\widehat{V} + \delta\mathcal{V}\nabla_{\delta\widehat{U}}\widehat{V} + \zeta\mathcal{V}\nabla_{\delta\widehat{U}}\widehat{V}) \\ & + \vec{f}_*(\delta\mathcal{T}_{\delta\widehat{U}}\zeta\delta\widehat{V} + \zeta\mathcal{T}_{\delta\widehat{U}}\zeta\delta\widehat{V} + P\mathcal{H}\nabla_{\delta\widehat{U}}\zeta\delta\widehat{V} + Q\mathcal{H}\nabla_{\delta\widehat{U}}\zeta\delta\widehat{V}) \\ & - \vec{f}_*(\mathcal{A}_{\zeta\widehat{U}}\delta\widehat{V} + \mathcal{V}\nabla_{\zeta\widehat{U}}\delta\widehat{V} + \mathcal{T}_{\delta\widehat{U}}\zeta\widehat{V} + \mathcal{H}\nabla_{\delta\widehat{U}}\zeta\widehat{V}) \\ & + \zeta\widehat{U}(\ln\lambda)\vec{f}_*\zeta\widehat{V} + \zeta\widehat{V}(\ln\lambda)\vec{f}_*\zeta\widehat{U} - g_1(\zeta\widehat{U}, \zeta\widehat{V})\vec{f}_*(grad\ln\lambda) \\ & + g_1(\delta\widehat{U}, \delta\widehat{V})\vec{f}_*\xi - \nabla^{\vec{f}}_{\zeta\widehat{U}}\vec{f}_*\zeta\widehat{V} + \nabla^{\vec{f}}_{\phi\widehat{U}}\vec{f}_*\phi\widehat{V}. \end{aligned}$$

from which we get the desired result. □

Theorem 6.2. *Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion from Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2) with slant angles θ_1 and θ_2. Suppose that $\vec{f}$ is $\mathfrak{D}^{\theta_2}$-ϕ-pluriharmonic. Then $\mathfrak{D}^{\theta_2}$ defines totally geodesic foliation M_1 if and only if*

$$\begin{aligned} & \vec{f}_*(\zeta\mathcal{T}_{\delta\widehat{Z}}\zeta\delta\widehat{W} + Q\mathcal{H}\nabla_{\delta\widehat{Z}}\zeta\delta\widehat{W}) - \vec{f}_*(\mathcal{A}_{\zeta\widehat{Z}}\delta\widehat{W} + \mathcal{H}\nabla_{\delta\widehat{Z}}\zeta\widehat{W}) \\ & = \cos^2\theta_2\vec{f}_*(Q\mathcal{T}_{\delta\widehat{Z}}\widehat{W} + \zeta\mathcal{V}\nabla_{\delta\widehat{Z}}\widehat{W}) - \nabla^{\vec{f}}_{\phi\widehat{U}}\vec{f}_*\phi\widehat{V} - \cos^2\theta_2 g_1(\widehat{Z}, \widehat{W})\vec{f}_*\xi + \nabla^{\vec{f}}_{\zeta\widehat{Z}}\vec{f}_*\zeta\widehat{W} \\ & \quad - \zeta\widehat{Z}(\ln\lambda)\vec{f}_*\zeta\widehat{W} - \zeta\widehat{W}(\ln\lambda)\vec{f}_*\zeta\widehat{Z} + \sin^2\theta_2 g_1(\widehat{Z}, \widehat{W})\vec{f}_*(grad\ln\lambda) \end{aligned}$$

for any $\widehat{Z}, \widehat{W} \in \Gamma(\mathfrak{D}^{\theta_2})$.

Proof. The proof of the theorem is similar to the proof of Theorem 6.1. □

Theorem 6.3. *Let $\vec{f}$ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion from Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2) with slant angles θ_1 and θ_2. Suppose that $\vec{f}$ is $((ker\vec{f}_*)^{\perp} - ker\vec{f}_*)$-$\phi$-pluriharmonic. Then the following assertion are equivalent.*

(i) The horizontal distribution $(ker\vec{f}_*)^{\perp}$ defines totally geodesic foliation on M_1.

(ii) $$\begin{aligned} & (cos^2\theta_1)\vec{f}_*\{Q\mathcal{T}_{P\widehat{X}}\mathfrak{B}\widehat{U} + \zeta\mathcal{V}\nabla_{P\widehat{X}}\mathfrak{B}\widehat{U} + Q\mathcal{A}_{Q\widehat{X}}\mathfrak{B}\widehat{U} + \zeta\mathcal{V}\nabla_{Q\widehat{X}}\mathfrak{B}\widehat{U}\} \\ & + (cos^2\theta_2)\vec{f}_*\{Q\mathcal{T}_{P\widehat{X}}\mathfrak{C}\widehat{U} + \zeta\mathcal{V}\nabla_{P\widehat{X}}\mathfrak{C}\widehat{U} + Q\mathcal{A}_{Q\widehat{X}}\mathfrak{C}\widehat{U} + \zeta\mathcal{V}\nabla_{Q\widehat{X}}\mathfrak{C}\widehat{U}\} \\ & - \vec{f}_*\{\zeta\mathcal{A}_{Q\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U} + \zeta\mathcal{A}_{Q\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U} - \mathcal{H}\nabla_{P\widehat{X}}\zeta\widehat{U}\} \\ & = -\vec{f}_*\{Q\mathcal{T}_{P\widehat{X}}\mathfrak{A}\widehat{U} + \zeta\mathcal{V}\nabla_{P\widehat{X}}\mathfrak{A}\widehat{U} + Q\mathcal{A}_{Q\widehat{X}}\delta\mathfrak{A}\widehat{U} + \zeta\mathcal{V}\nabla_{Q\widehat{X}}\mathfrak{A}\widehat{U}\} \\ & + \vec{f}_*\{\zeta\mathcal{T}_{P\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U} + Q\mathcal{H}\nabla_{P\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U} + \zeta\mathcal{T}_{P\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U} + Q\mathcal{H}\nabla_{P\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U}\} + Q\mathcal{H}\nabla_{Q\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U} \\ & + Q\mathcal{H}\nabla_{Q\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U} - \vec{f}_*(\nabla_{\widehat{X}}\widehat{U}) + \nabla^{\vec{f}}_{\phi\widehat{X}}\vec{f}_*\zeta\widehat{U} + g_1(P\widehat{X}\delta\widehat{U})\vec{f}_*\xi. \end{aligned}$$

for any $\widehat{X} \in \Gamma(ker\vec{f}_*)^{\perp}$ and $\widehat{U} \in \Gamma(ker\vec{f}_*)$

Proof. For any $\widehat{X} \in \Gamma(ker\vec{f}_*)^{\perp}$ and $\widehat{U} \in \Gamma(ker\vec{f}_*)$, since $\vec{f}$ is $((ker\vec{f}_*)^{\perp} - ker\vec{f}_*)$-$\phi$-pluriharmonic, then by using (2.15), (3.2) and (3.5), we get

$$\vec{f}_*(\nabla_{Q\widehat{X}}\zeta\widehat{U}) = -\vec{f}_*(\nabla_{P\widehat{X}}\delta\widehat{U} + \nabla_{P\widehat{X}}\zeta\widehat{U} + \nabla_{Q\widehat{X}}\delta\widehat{U}) - \vec{f}_*(\nabla_{\widehat{X}}\widehat{U}) + \nabla^{\vec{f}}_{\phi\widehat{X}}\vec{f}_*\zeta\widehat{U}.$$

Taking account the fact from (2.1) and (2.10), we have

$$\vec{f}_*(\nabla_{Q\widehat{X}}\zeta\widehat{U}) = -\vec{f}_*(\mathcal{T}_{P\widehat{X}}\zeta\widehat{U} + \mathcal{H}\nabla_{P\widehat{X}}\zeta\widehat{U}) - \vec{f}_*(\nabla_{\widehat{X}}\widehat{U}) + \nabla^{\vec{f}}_{\phi\widehat{X}}\vec{f}_*\zeta\widehat{U}$$
$$+ \vec{f}_*(\phi\nabla_{P\widehat{X}}\phi\delta\widehat{U}) + g_1(P\widehat{X}, \delta\widehat{U})\vec{f}_*\xi + \vec{f}_*(\phi\nabla_{Q\widehat{X}}\phi\delta\widehat{U}).$$

Now on using decomposition (3.1), Lemma 3.2, Lemma 3.3 with equations (3.2), we may yields

$$\vec{f}_*(\nabla_{Q\widehat{X}}\zeta\widehat{U}) = \vec{f}_*\{-\phi\nabla_{P\widehat{X}}\mathfrak{A}\widehat{U} - \cos^2\theta_1\phi\nabla_{P\widehat{X}}\mathfrak{B}\widehat{U} - \cos^2\theta_2\phi\nabla_{P\widehat{X}}\mathfrak{C}\widehat{U} + g_1(P\widehat{X}, \delta\widehat{U})\vec{f}_*\xi$$
$$+ \vec{f}_*\{-\phi\nabla_{Q\widehat{X}}\mathfrak{A}\widehat{U} - \cos^2\theta_1\phi\nabla_{Q\widehat{X}}\mathfrak{B}\widehat{U} - \cos^2\theta_2\phi\nabla_{Q\widehat{X}}\mathfrak{C}\widehat{U}$$
$$+ \vec{f}_*\{\phi\nabla_{P\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U} + \phi\nabla_{P\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U} + \phi\nabla_{Q\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U} + \phi\nabla_{Q\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U}\}$$
$$- \vec{f}_*(\mathcal{H}\nabla_{P\widehat{X}}\zeta\widehat{U}) - \vec{f}_*(\nabla_{\widehat{X}}\widehat{U}) + \nabla^{\vec{f}}_{\phi\widehat{X}}\vec{f}_*\zeta\widehat{U}.$$

From equations (2.9)-(2.12) and after simple calculation, we may write

$$- cos^2\theta_1(Q\mathcal{T}_{P\widehat{X}}\mathfrak{B}\widehat{U} + \zeta\mathcal{V}\nabla_{P\widehat{X}}\mathfrak{B}\widehat{U} + Q\mathcal{A}_{Q\widehat{X}}\mathfrak{B}\widehat{U} + \zeta\mathcal{V}\nabla_{Q\widehat{X}}\mathfrak{B}\widehat{U})$$
$$- cos^2\theta_2(Q\mathcal{T}_{P\widehat{X}}\mathfrak{C}\widehat{U} + \zeta\mathcal{V}\nabla_{P\widehat{X}}\mathfrak{C}\widehat{U} + Q\mathcal{A}_{Q\widehat{X}}\mathfrak{C}\widehat{U} + \zeta\mathcal{V}\nabla_{Q\widehat{X}}\mathfrak{C}\widehat{U})$$
$$+ \vec{f}_*\{\zeta\mathcal{A}_{Q\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U} + \zeta\mathcal{A}_{Q\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U} - \mathcal{H}\nabla_{P\widehat{X}}\zeta\widehat{U}\}$$
$$- \vec{f}_*\{Q\mathcal{T}_{P\widehat{X}}\mathfrak{A}\widehat{U} + \zeta\mathcal{V}\nabla_{P\widehat{X}}\mathfrak{A}\widehat{U} + Q\mathcal{A}_{Q\widehat{X}}\mathfrak{A}\widehat{U} + \zeta\mathcal{V}\nabla_{Q\widehat{X}}\mathfrak{A}\widehat{U}\}$$
$$+ \vec{f}_*\{\zeta\mathcal{T}_{P\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U} + Q\mathcal{H}\nabla_{P\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U} + \zeta\mathcal{T}_{P\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U} + Q\mathcal{H}\nabla_{P\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U}\}$$
$$+ \vec{f}_*(Q\mathcal{H}\nabla_{Q\widehat{X}}\zeta\delta\mathfrak{B}\widehat{U} + Q\mathcal{H}\nabla_{Q\widehat{X}}\zeta\delta\mathfrak{C}\widehat{U}) - \vec{f}_*(\nabla_{\widehat{X}}\widehat{U}) + \nabla^{\vec{f}}_{\phi\widehat{X}}\vec{f}_*\zeta\widehat{U} + g_1(P\widehat{X}, \delta\widehat{U})\vec{f}_*\xi$$

□

which completes the proof of theorem.

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Author's contributions

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