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The Farey Sum of Pythagorean and Eisenstein Triples

Mircea Crasmareanu*

Abstract

A composition law, inspired by the Farey addition, is introduced on the set of Pythagorean triples. We study some of its properties as well as two symmetric matrices naturally associated to a given Pythagorean triple. Several examples are discussed, some of them involving the degenerated Pythagorean triple $(1, 0, 1)$. The case of Eisenstein triples is also presented.

Keywords: *Farey sum, Pythagorean (Eisenstein) triple, Symmetric matrix*

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****Corresponding author***

1. The Farey composition law on Pythagorean triples

Fix the set $\mathbb{N}^2(<) := \{(p, q) \in \mathbb{N}^* \times \mathbb{N}^*; p < q\}$ and the map:

$$P : \mathbb{N}^2(<) \to (\mathbb{N}^*)^3, \quad P(p, q) := (q^2 - p^2, 2pq, p^2 + q^2).$$

It is well-known that P provides a parametrization (up to a strictly positive multiplicative factor) of the set of Pythagorean triples $PT := \{(a, b, c) \in (\mathbb{N}^*)^3; 2|b, a^2 + b^2 = c^2\}$. If, in addition $\gcd(p, q) = 1$ with $2 \nmid (q - p)$ then (a, b, c) is a primitive (i.e. $\gcd(a, b) = 1$) Pythagorean triple.

The aim of this short note is to study the transport of a natural sum from $\mathbb{N}^2(<)$ to PT. Namely, defining $(p, q) \oplus (p', q') := (p + p', q + q')$ it follows the pair $(PT, \oplus)$ with:

$$(a, b, c,) \oplus (a', b', c') := (a'', b'', c'') = ((q + q')^2 - (p + p')^2, 2(p + p')(q + q'), (p + p')^2 + (q + q')^2).$$

More precisely, we have:

$$a'' := a + a' + 2(qq' - pp'), \quad b'' := b + b' + 2(pq' + qp'), \quad c'' := c + c' + 2(qq' + pp'). \tag{1.1}$$

Remark 1.1. If the initial pair (p, q) from $\mathbb{N}^2(<)$ is considered as the ratio $\frac{p}{q} \in (0, 1)$ then the sum:

$$\frac{p}{q} \oplus \frac{p'}{q'} := \frac{p + p'}{q + q'}$$

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is called *the mediant* in [1] due to the double inequality:

$$\frac{p}{q} < \frac{p+p'}{q+q'} < \frac{p'}{q'}.$$

But we prefer to use the name of *Farey sum* after [2, p. 209] although, obviously, the initial sum on $\mathbb{N}^2(<)$ is the restriction of the additive law of the real 2-dimensional linear space $\mathbb{R}^2$; another source for the applications of the Farey sequences in hyperbolic dynamics is [3]. Our choice for this name is also inspired by the very nice picture of page 23 from the book [4] illustrating a relationship between the circular Farey diagram and the Pythagorean triples. We point out that a group structure on the subset of primitive Pythagorean triples is considered in [5]. □

Properties 1.1. 1) The composition law $\oplus$ on PT is commutative but without a neutral element.
2) The *height* of the Pythagorean triple $pt := (a, b, c)$ is $h(pt) := c - b = (q-p)^2$. For our triple of Pythagorean triples it follows:

$$h((pt)'') = h((pt)') + h(pt) - 2(q-p)(q'-p') < h((pt)') + h(pt).$$

3) The usual CBS inequality provides an upper bound for the resulting Pythagorean triple in terms of the given $(a, b, c,), (a', b', c') \in PT$:

$$a'' < a + a' + 2\sqrt{cc'}, \quad b'' < b + b' + 2\sqrt{cc'}, \quad \sqrt{c''} \leq \sqrt{c} + \sqrt{c'} \tag{1.2}$$

with equality in the last relation if and only if $c = c'$ which, in turn, yields $c'' = 4c = 4c'$ as consequence of the relation:

$$(a, b, c) \oplus (a, b, c) = 4(a, b, c).$$

□

Example 1.1. 1) Since $(1, 3) \oplus (2, 3) = (3, 6)$ we have $2(4, 3, 5) \oplus (5, 12, 13) = 9(3, 4, 5)$.
2) The sum $(1, 2) \oplus (1, 3) = (2, 5)$ gives $(3, 4, 5) \oplus 2(4, 3, 5) = (21, 20, 29)$.
3) The restriction of the complex multiplication to the unit circle S^1 gives a group multiplication on the set of *all* Pythagorean triples:

$$(a, b, c) \odot (a', b', c') = (aa' - bb', ab' + a'b, cc'), \quad (a, b, c) \odot (a, b, c) = (a^2 - b^2, 2ab, c^2 = a^2 + b^2)$$

having as neutral element the degenerate Pythagorean triple $(1, 0, 1)$ which can be considered as the image through the map P of the pair $(\tilde{p}, \tilde{q}) = (0, 1)$. For our sum we have:

$$(a, b, c) \oplus (1, 0, 1) = (a + 2q + 1, b + 2p, c + 2q + 1), \quad (3, 4, 5) \oplus (1, 0, 1) = 2(4, 3, 5).$$

4) Fix $k \in \mathbb{N}^*$ and a triangle Δ. Then we call Δ as being a *k-triangle* if its area $\mathcal{A}$ is k times its semi-parameter $s = \frac{1}{2}(a + b + c)$. Let us find the k-rectangular triangles for a prime number k. From $ab = k(a + b + c)$ it results:

$$p(q - p) = k$$

with only two solutions:

$$\begin{cases} (p = 1, q = k+1), & (pt)_1 = (k(k+2), 2(k+1), k^2 + 2k + 2), \quad \mathcal{A}_1 = k(k+1)(k+2), \\ (p = k, q = k+1), & (pt)_2 = (2k+1, 2k(k+1), 2k^2 + 2k + 1), \quad \mathcal{A}_2 = k(k+1)(2k+1) \end{cases}$$

Hence, their Farey sum is:

$$(pt)_1 \oplus (pt)_2 = (k+1)^2(3, 4, 5).$$

Also concerning the area there exist pairs of Pythagorean triples sharing it; for example the area $\mathcal{A} = 210$ is provided by:

$$(p_1 = 2, q_1 = 5) \quad (pt)_1 = (21, 20, 29), \quad (p_2 = 1, q_2 = 6), \quad (pt)_2 = (35, 12, 37)$$

and their Farey sum is:

$$(p = 3, q = 11), \quad pt = 2(56, 33, 65).$$

5) Let $(F_n)_{n \in \mathbb{N}}$ be the Fibonacci sequence and let $p = p_n := F_{n+1} < q = q_n := F_{n+2}$. It results the *$n$-Fibonacci-Pythagorean triple* $(Fpt)_n = (a_n, b_n, c_n)$:

$$a_n = F_n F_{n+3}, \quad b_n = 2F_{n+1}F_{n+2}, \quad c_n = F_{n+1}^2 + F_{n+2}^2$$

for which we have the Farey sum of Fibonacci type:

$$(Fpt)_n \oplus (Fpt)_{n+1} = (Fpt)_{n+2}.$$

6) Fix c a hypotenuse which as natural number has only two representations as sum of different squares; for example $65 = 1^2 + 8^2 = 4^2 + 7^2$ or $145 = 1^2 + 12^2 = 8^2 + 9^2$. Then we call the corresponding Pythagorean triples (a_1, b_1, c), (a_2, b_2, c) as being $hypotenuse - related$ and we can perform their Farey sum. For our examples above we have:

$$\begin{cases} (p_1 = 1, q_1 = 8) \oplus (p_2 = 4, q_2 = 7) = (p = 5, q = 15), (63, 16, 65) \oplus (33, 56, 65) = 50(4, 3, 5), \\ (p_1 = 1, q_1 = 12) \oplus (p_2 = 8, q_2 = 9) = (p = 9, q = 21), (143, 24, 145) \oplus (17, 144, 145) = 18(20, 21, 29). \end{cases}$$

The class of these c is provided by the expression $c = p_1^{a_1} p_2^{a_2}$ with $p_1 < p_2$ prime numbers of the form $4k + 1$; recall also that any prime number of the form $4k + 1$ is a sum of two squares. Related to this discussion we recall that a positive integer k is a sum of two triangular numbers:

$$k = \frac{u(u+1)}{2} + \frac{v(v+1)}{2} \tag{1.3}$$

if and only if $4k + 1$ is a sum of squares; namely (1.3) implies $4k + 1 = (v - u)^2 + (u + v + 1)^2$. Hence this k with $u < v$ provides the Pythagorean triple:

$$(p = v - u < q = u + v + 1), \quad a = (2u + 1)(2v + 1), \quad b = 2(v - u)(u + v + 1), \quad c = 4k + 1. \tag{1.4}$$

As example, $c = 65$ is provided by $k = 16$ which is generated by two triangular numbers:

$$u_1 = 3 < v_2 = 4, \quad (a_1, b_1, c) = (63, 16, 65), \quad u_2 = 1 < v_2 = 5, \quad (a_2, b_2, c) = (33, 56, 65).$$

7) Fix $2N$ an even number and ask the given triangle has the perimeter $2s = 2N$. It follows the quadratic Diophantine equation:

$$q(p + q) = N$$

which for some value of N has only two solutions; namely $N \in \{120, 180, 240, 252, 336, ...\}$. Then we call the corresponding Pythagorean triples (a_1, b_1, c), (a_2, b_2, c) as being $perimeter - related$ and we can perform their Farey sum. For the example of $N = 120$ we have $(p_1 = 2 < q_1 = 10)$ and $(p_1 = 7 < p_2 = 8)$ and then:

$$2^3(12, 5, 13) \oplus (15, 112, 113) = 3^4(3, 4, 5).$$

Returning to the last inequality (1.2) the right-hand-side of it can be interpreted in terms of a quasi-arithmetic mean. Fix an open real interval I and $M : I \times I \to I$ a *mean* i.e. for any pair $(x, y) \in I \times I$ we have the double inequality:

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}.$$

Recall also that M is called *quasi-arithmetic* if there exists a continuous and strictly monotonic function $f : I \to \mathbb{R}$ such that:

$$M(x, y) = M_f(x, y) := f^{-1}\left(\frac{f(x) + f(y)}{2}\right).$$

Hence, with $I = \mathbb{R}_+^* := (0, +\infty)$ the last inequality (1.2) reads:

$$c'' \leq 4M_{\sqrt{\cdot}}(c, c').$$

2. Two symmetric matrices associated to a given Pythagorean triple

In the following we provide a matrix formalism associated to a given Pythagorean triple. Namely, the relations (1.1) can be put into the form:

$$\begin{pmatrix} a'' \\ b'' \\ c'' \end{pmatrix} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} + 2\Gamma \cdot \begin{pmatrix} p' \\ q' \end{pmatrix}, \quad \Gamma := \begin{pmatrix} -p & q \\ q & p \\ p & q \end{pmatrix} \in M_{3,2}(\mathbb{Z}^*).$$

The matrix Γ and its transpose Γ^t provides two new matrices.

I) a symmetric 2×2 one:

$$A := \Gamma^t \cdot \Gamma = \begin{pmatrix} 2p^2+q^2 & pq \\ pq & p^2+2q^2 \end{pmatrix} = \begin{pmatrix} c+p^2 & \frac{b}{2} \\ \frac{b}{2} & c+q^2 \end{pmatrix} \in Sym(2, \mathbb{N}^*), \det A = 2c^2 > 0.$$

Allowing the pair (p, q) to be a point in the Euclidean plane $\mathbb{R}^2$ then the map P is a regular parametrization from $\mathbb{R}^2 \setminus \{(0,0)\}$ of the cone $C : x^2 + y^2 - z^2 = 0$ (which can be called *the Pythagorean cone*) and hence A is exactly the first fundamental form of this quadric in $\mathbb{R}^3$. Its coefficients of the fundamental forms are in the Gauss notation:

$$\begin{cases} E = 4(2p^2+q^2), \quad F = 4pq (= 2b), \quad G = 4(p^2+2q^2) \\ L = \frac{2\sqrt{2}p^2}{p^2+q^2} \leq 2\sqrt{2}, \quad M = \frac{\sqrt{2}pq}{p^2+q^2} \left(= \frac{\sqrt{2}}{2}\sin B < \frac{\sqrt{2}}{2}\right), \quad N = \frac{2\sqrt{2}q^2}{p^2+q^2} \leq 2\sqrt{2}. \end{cases}$$

Returning to the matrix A, recall after [6] that any symmetric 2×2 matrix has two Hermitian parameters, one real being half of its trace, and one complex, called *Hopf invariant*, which for our A is:

$$H(A) = \frac{p^2 - q^2}{2} - (pq)i = \frac{1}{2}(p - iq)^2 \in \mathbb{C}^*.$$

Let us remark that if p and q share the same parity (which means that (a, b, c) is not a primitive Pythagorean triple since 2 divides also a) then $H(A)$ is a Gaussian integer. Recall also that a proof of the fact that the map P is a parametrization of the set PT is based exactly on the complex number $(p + iq)^2 = 2\overline{H(A)}$ since $c = |2H(A)|$. The eigenvalues and associated eigenvectors of the matrix A are:

$$\lambda_1 = c < \lambda_2 = 2c, \quad \bar{v}_1 = (-q, p) = -q + ip = i \cdot \sqrt{2\overline{H(A)}}, \quad \bar{v}_2 = (p, q) = p + iq = \sqrt{2\overline{H(A)}}.$$

So, the invertible matrix making A a diagonal one is:

$$\begin{cases} S = \begin{pmatrix} -q & p \\ p & q \end{pmatrix} \in GL(2, \mathbb{Z}) \cap Sym(2), \quad S^{-1} = \frac{1}{c}S \in GL(2, \mathbb{Q}) \cap Sym(2), \\ S^{-1} \cdot A \cdot S = diag(c, 2c), \quad H(S) = -q - pi, \quad \det S = -c < 0. \end{cases}$$

For example:

$$A(1,0,1) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad S(1,0,1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall that a matrix $U \in GL(n, \mathbb{R})$ can be consider as corresponding to a mathematical game $G(U)$ of two persons, both having n strategies; then *the value* of this game is ([7, p. 449]) $v(G(U)) = \frac{1}{s(U^{-1})}$ where $s(U^{-1})$ means the sum of all elements of U^{-1}. For the matrix A the value of its corresponding game is:

$$v(G(A)) = \frac{2c^2}{3c - b} < c, \quad v(G(p = 1, q = 2)) = \frac{50}{11}.$$

□

II) a symmetric 3×3 one:

$$\begin{cases} B := \Gamma \cdot \Gamma^t = \begin{pmatrix} c & 0 & a \\ 0 & c & b \\ a & b & c \end{pmatrix} \in Sym(3, \mathbb{N}^*), \\ \frac{1}{c}B = \begin{pmatrix} 1 & 0 & \sin(\angle A) \\ 0 & 1 & \sin(\angle B) = \cos(\angle A) \\ \sin(\angle A) & \sin(\angle B) = \cos(\angle A) & 1 \end{pmatrix} \in Sym(3) = Sym(3, \mathbb{R}). \end{cases} \tag{2.1}$$

Again, its eigenvalues and associated eigenvectors are:

$$\lambda_1 = 0 < \lambda_2 = c < \lambda_3 = 2c, \quad \bar{v}_1 = (-a, -b, c), \quad \bar{v}_2 = (-b, a, 0), \quad \bar{v}_3 = (a, b, c).$$

Hence, the invertible matrix making the matrix B a diagonal one is:

$$\begin{cases} S = \begin{pmatrix} -a & -b & a \\ -b & a & b \\ c & 0 & c \end{pmatrix} \in GL(3,\mathbb{Z}) \quad \det S = -2c^3 < 0, \\ S^{-1} = \frac{1}{2c^2}\begin{pmatrix} -a & -b & c \\ -2b & 2a & 0 \\ a & b & c \end{pmatrix} \in GL(3,\mathbb{Q}), \quad S^{-1}\cdot B \cdot S = diag(0, c, 2c). \end{cases}$$

Recall also that a matrix from $Sym(3)$ represents geometrically a conic, see for example [8]. The conic associated to the matrix B reduces to the double point $\left(-\frac{a}{c}, -\frac{b}{c}\right) \in S^1$. For example:

$$B(1,0,1) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad S(1,0,1) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let us remark that the second matrix from the relation (2.1) yields the function:

$$f : \left(0, \frac{\pi}{2}\right) \to Sym(3) \setminus GL(3,\mathbb{R}), \quad f(t) := \begin{pmatrix} 1 & 0 & \sin t \\ 0 & 1 & \cos t \\ \sin t & \cos t & 1 \end{pmatrix}$$

as restriction to (the first quadrant of) the unit circle S^1 of the map $F : \mathbb{R}^2 \to Sym(3)$:

$$\begin{cases} F(x,y) := \begin{pmatrix} x^2+y^2 & 0 & y \\ 0 & x^2+y^2 & x \\ y & x & x^2+y^2 \end{pmatrix}, \quad \det F(x,y) = (x^2+y^2)^2(x^2+y^2-1), \\ F|_{\mathbb{C}^*} : (x,y) = r(\cos\varphi, \sin\varphi), F(r,\varphi) := r\begin{pmatrix} r & 0 & \sin\varphi \\ 0 & r & \cos\varphi \\ \sin\varphi & \cos\varphi & r \end{pmatrix}, \det F(r,\varphi) = r^4(r^2-1). \end{cases}$$

The matrix $S \in GL(3,\mathbb{R})$ making diagonal the symmetric matrix $f(t)$ is:

$$\begin{cases} S(t) := \frac{1}{2}\begin{pmatrix} -\sin t & -\cos t & 1 \\ -\sin 2t & 2\sin^2 t & 0 \\ \sin t & \cos t & 1 \end{pmatrix}, \quad S^{-1}(t) := \begin{pmatrix} -\sin t & -\frac{\cos t}{\sin t} & \sin t \\ -\cos t & 1 & \cos t \\ 1 & 0 & 1 \end{pmatrix}, \\ S(t)\cdot f(t) \cdot S^{-1}(t) = diag(0,1,2) \end{cases}$$

while the matrix $S \in GL(3,\mathbb{R})$ making diagonal the symmetric matrix $F|_{\mathbb{C}^*}$ is:

$$\begin{cases} S(x,y) := \frac{1}{2r^2}\begin{pmatrix} -yr & -xr & r^2 \\ -2xy & 2y^2 & 0 \\ yr & xr & r^2 \end{pmatrix}, \quad S^{-1}(x,y) := \begin{pmatrix} -\frac{y}{r} & -\frac{x}{y} & \frac{y}{r} \\ -\frac{x}{r} & 1 & \frac{x}{r} \\ 1 & 0 & 1 \end{pmatrix}, \\ S(t)\cdot F(r,\varphi)\cdot S^{-1}(t) = diag(r^2 - r, r^2, r^2 + r). \end{cases}$$

From a differentiable point of view F is an immersion of $\mathbb{R}^2$ into $\mathbb{R}^6 = Sym(3)$ since the rank of the Jacobian matrix of F is 2. With the notation $u = x^2 + y^2$ the equation $\det F = 1$, i.e. $F(x,y) \in SL(3,\mathbb{R})$, means the cubic equation:

$$u^3 - u^2 - 1 = \left(u - \frac{1}{3}\right)^3 - \frac{1}{3}\left(u - \frac{1}{3}\right) - \frac{29}{27} = 0$$

which admits only one real (and positive) solution $u_1 \simeq 1.4656$. Naturally, we can associate the cubic (in fact elliptic) plane curve:

$$\mathcal{C} : v^2 = u^3 - u^2 - 1$$

whose details can be found on: https://www.lmfdb.org/EllipticCurve/Q/496/e/1. □

Returning to the case of 2×2 matrices let us remark that the first part of relations (1.4) gives an affine map:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} p \\ q \end{pmatrix} := C \cdot \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C := \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The eigenvalues of C and the square matrix S making C a diagonal one are:

$$\lambda_1 = -\sqrt{2} < \lambda_2 = \sqrt{2}, \quad S = \begin{pmatrix} -1-\sqrt{2} & \sqrt{2}-1 \\ 1 & 1 \end{pmatrix}, \quad S^{-1} = \frac{1}{4}\begin{pmatrix} -\sqrt{2} & 2-\sqrt{2} \\ \sqrt{2} & 2+\sqrt{2} \end{pmatrix}$$

with $S^{-1}CS = diag(-\sqrt{2}, \sqrt{2})$.

We finish this section by introducing a composition law on $\mathbb{R}^*_+ = (0, +\infty)$, inspired by the equality case discussed in the Property 1.2.3):

$$x \oplus_F y := (\sqrt{x} + \sqrt{y})^2.$$

Apart from commutativity and $x \oplus_F x = 4x$ we note the property $\cos^2 t \oplus_F \sin^2 t = 1 + \sin 2t$.

3. The Farey sum of a class of Eisenstein triples

For the sake of completeness we present now the case of Eisenstein triples. Recall that an Eisenstein triangle has an angle of $60°$. By supposing this angle to be $\angle C$ it results:

$$a^2 - ab + b^2 = c^2$$

and hence, a *Eisenstein triple* is a triple of positive integers satisfying this Diophantine equation; then $\min\{a, b\} \leq c \leq \max\{a, b\}$. We point out that recently, the Eisenstein triples are used in [9] to characterize the bijective digitized rotations on the hexagonal grid. Contrary to the Pythagorean case we have only a *partial* parametrization:

$$a = a(p, q) := q^2 - p^2, \quad b = b(p, q) := 2pq - p^2, \quad c = c(p, q) := p^2 + q^2 - pq = (q - p)^2 + pq \tag{3.1}$$

and the limit case $p = q$ gives the degenerate Eisenstein triple $p^2(0, 1, 1)$. Then we can define a Farey sum on the class of (3.1) (p, q)-*Eisenstein triples*:

$$(a, b, c,) \oplus (a', b', c') := (a'', b'', c'') =$$

$$= ((q + q')^2 - (p + p')^2, 2(p + p')(q + q') - (p + p')^2, (p + p')^2 + (q + q')^2 - (p + p')(q + q')). \tag{3.2}$$

Example 3.1. 1) Since $(p = 1, q = 2)$ yields the equilateral triangle $3(1, 1, 1)$ and $(p = 1, q = 3)$ gives the Eisenstein triple $(8, 5, 7)$ we have:

$$(3, 3, 3) \oplus (8, 5, 7) = (21, 16, 19), \quad (p'' = 2, q'' = 5).$$

2) Again $(a, b, c) \oplus (a, b, c) = 4(a, b, c)$ and $(a, b, c) \oplus (0, 1, 1) = (a + 2(q - p), b + 2q + 1, c + p + q + 1)$ with the example $3(1, 1, 1) \oplus (0, 1, 1) = (5, 8, 7)$.

The matrix expression of the Farey sum (3.2) is:

$$\begin{pmatrix} a'' \\ b'' \\ c'' \end{pmatrix} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} + \Gamma \cdot \begin{pmatrix} p' \\ q' \end{pmatrix}, \quad \Gamma := \begin{pmatrix} -2p & 2q \\ 2(q-p) & 2p \\ 2p-q & 2q-p \end{pmatrix} \in M_{3,2}(\mathbb{Z}).$$

The associated symmetric matrices are:

I)

$$A := \Gamma^t \cdot \Gamma = \begin{pmatrix} 12(p^2 - pq) + 5q^2 & -6p^2 + 5pq - 2q^2 \\ -6p^2 + 5pq - 2q^2 & 5p^2 + 4(2q^2 - pq) \end{pmatrix} \in Sym(2, \mathbb{Z})$$

with:

$$TrA = 17p^2 - 16pq + 13q^2, \quad \det A = 12(2p^4 - 4p^3q + 10p^2q^2 - 8pq^3 + 3q^4).$$

II)

$$B := \Gamma \cdot \Gamma^t = \begin{pmatrix} 4(p^2 + q^2) & 4p^2 & -4p^2 + 6pq - 2q^2 \\ 4p^2 & 4(2p^2 - 2pq + q^2) & -6p^2 + 10pq - 2q^2 \\ -4p^2 + 6pq - 2q^2 & -6p^2 + 10pq - 2q^2 & 5p^2 - 8pq + 5q^2 \end{pmatrix} \in Sym(3, \mathbb{Z})$$

with:

$$TrB = 17p^2 - 16pq + 13q^2, \quad \det B = 48q(2p^5 - 6p^4q + 10p^3q^2 - 7p^2q^3 + q^5).$$

To the equilateral triangle $3(1,1,1)$ corresponds the matrices:

I)

$$A(p=1,q=2)=\begin{pmatrix} 8 & -4 \\ -4 & 29 \end{pmatrix}\in Sym(2,\mathbb{Z}), \quad \lambda_1=\frac{37-\sqrt{505}}{2}<\lambda_2=\frac{37+\sqrt{505}}{2}$$

with $TrA=37$, $\det A=6^3$, Hopf invariant $H(A)=-\frac{21}{2}+4i$ and:

$$S=\frac{1}{8}\begin{pmatrix} 21+\sqrt{505} & 21-\sqrt{505} \\ 8 & 8 \end{pmatrix}\in GL(2,\mathbb{R}), \quad S^{-1}=\frac{1}{2\sqrt{505}}\begin{pmatrix} 8 & \sqrt{505}-21 \\ -8 & \sqrt{505}+21 \end{pmatrix}.$$

II)

$$\begin{cases} B(p=1,q=2)=\begin{pmatrix} 20 & 4 & 0 \\ 4 & 8 & 6 \\ 0 & 6 & 9 \end{pmatrix}\in Sym(3,\mathbb{Z}), \\ \lambda_1\simeq 1.98<\lambda_2\simeq 13.51<\lambda_3\simeq 21.50, \quad \det B=576=24^2. \end{cases}$$

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Affiliations

MIRCEA CRASMAREANU
ADDRESS: Faculty of Mathematics, University "Al. I. Cuza", Iasi, 700506, Romania
http://www.math.uaic.ro/∼mcrasm .
E-MAIL: mcrasm@uaic.ro
ORCID ID: 0000-0002-5230-2751