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On a Sequence of Slant Submanifolds in Almost Product Riemannian Setting

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ABSTRACT

We prove that the property of being pointwise slant is transitive on a class of proper pointwise slant submanifolds of almost product Riemannian manifolds, and we illustrate this fact with an example. For a given almost product Riemannian manifold, we consider a sequence of pointwise slant immersed submanifolds and we explicitly determine the relation between the slant functions. Moreover, we state this result in a more general case.

Keywords: *Pointwise slant submanifold, almost product Riemannian manifold, anti-invariant submanifold.*
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1. Preliminaries

Recently, in [1], we proved a novel property, more precisely, that a pointwise slant submanifold M_3 of a manifold M_2 which is pointwise slant in an almost Hermitian manifold (M_1, g, φ_1), is also pointwise slant in M_1 (considering on the submanifolds the naturally induced almost complex structures), and we determined the relation between the slant functions.

The aim of the present paper is to show that this result is also valid in the almost product Riemannian framework, and even in a more general case. Firstly, we will briefly recall the definitions of almost product Riemannian structures and slant submanifolds in this context, and we will construct an induced structure on a submanifold of a Riemannian manifold endowed with a symmetric $(1,1)$-tensor field $\neq \pm I$. In the entire paper, we will consider only submanifolds defined by injective immersions. Also, we will use the same notation for a metric on a manifold as for the induced metric on a submanifold.

Let M be a smooth manifold. A $(1,1)$-tensor field $\varphi \neq \pm I$ on M is called an *almost product structure* if $\varphi^2 = I$, where I is the identity on the set of smooth sections $\Gamma(TM)$ of M. A pair of a Riemannian metric g and an almost product structure φ on M is called an *almost product Riemannian structure* if φ is g-symmetric, i.e., $g(\varphi X, Y) = g(X, \varphi Y)$ for any $X, Y \in \Gamma(TM)$. In this case, the triple (M, g, φ) is called an *almost product Riemannian manifold*.

The notion of *pointwise slant submanifold* was firstly considered by Etayo [3] and Chen [2] in almost Hermitian geometry. In the same way, one can talk about pointwise slant submanifolds of any Riemannian manifold endowed with a symmetric (or skew-symmetric) $(1,1)$-tensor field $\neq \pm I$. We shall state here the definition for a general case.

We consider M_2 an immersed submanifold of a Riemannian manifold (M_1, g) endowed with a g-symmetric $(1,1)$-tensor field $\varphi_1 \neq \pm I$. For any $a \in M_2$, we have the orthogonal splitting $T_aM_1 = T_aM_2 \oplus T_a^{\perp}M_2$, and for any $w \in T_aM_2$, we denote $\varphi_1 w = T_1 w + N_1 w$, where $T_1 w$ and $N_1 w$ is the tangential and the normal component of $\varphi_1 w$, respectively. Then, M_2 is called a *pointwise slant submanifold* of (M_1, g, φ_1) if, for any $X \in \Gamma(TM_2)$ and $a \in M_2$ with $X_a \neq 0$, $\varphi_1 X_a \neq 0$ and the angle between $\varphi_1 X_a$ and T_aM_2 is not zero and does not depend on X_a, but only on a. Moreover, if the slant function τ satisfies $\tau(a) \neq \frac{\pi}{2}$ for any $a \in M_2$, then M_2 is called a *proper pointwise slant submanifold*. If the angle between $\varphi_1 X_a$ and T_aM_2 does not depend on X_a, neither on a, then M_2

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is called a *slant submanifold* of (M_1, g, φ_1) with the slant angle τ (respectively, a *proper slant submanifold* if $\tau \neq \frac{\pi}{2}$, and an *anti-invariant submanifold* if $\tau = \frac{\pi}{2}$).

We shall further consider a pointwise slant submanifold of a Riemannian manifold (M_1, g) endowed with a g-symmetric $(1,1)$-tensor field $\varphi_1 \neq \pm I$ satisfying $\varphi_1^2 = q_1 I$ for q_1 a positive smooth function on M_1. In this case, we have $T_1^2 = q_1 \cos^2 \tau \cdot I$.

By a direct computation, we obtain

Lemma 1.1. *If M_2 is a proper pointwise slant submanifold, with a slant function τ, of a Riemannian manifold (M_1, g) endowed with a g-symmetric $(1,1)$-tensor field $\varphi_1 \neq \pm I$ satisfying $\varphi_1^2 = q_1 I$ with q_1 a positive smooth function on M_1, then, for any positive smooth function q_2 on M_2, the structure $\varphi_2 := \sqrt{\frac{q_2}{q_1}} \frac{1}{\cos \tau} \cdot T_1$ is g-symmetric, and it satisfies $\varphi_2^2 = q_2 I$.*

In particular, for $q_1 = q_2 = 1$, we have $\varphi_2^2 = I$, and we will say that $\varphi_2 := \frac{1}{\cos \tau} \cdot T_1$ is the *naturally induced almost product structure* on M_2.

2. The transitivity property

We will prove that the property of being a pointwise slant submanifold is transitive on a class of proper pointwise slant submanifolds of almost product Riemannian manifolds, and we generalize this result.

For $i \in \mathbb{N}^*$ and M_{i+1} an immersed submanifold of an almost product Riemannian manifold (M_i, g, φ_i), we have the orthogonal decomposition $T_a M_i = T_a M_{i+1} \oplus T_a^{\perp_i} M_{i+1}$ for any $a \in M_{i+1}$, where $T_a^{\perp_i} M_{i+1}$ stands for the orthogonal complement of $T_a M_{i+1}$ in $T_a M_i$, and for any $w \in T_a M_{i+1}$, we denote $\varphi_i w = T_i w + N_i w$, where $T_i w$ and $N_i w$ is the tangential and the normal component of $\varphi_i w$, respectively.

Theorem 2.1. *Let (M_1, g, φ_1) be an almost product Riemannian manifold. Let $k \geq 2$, and let M_{i+1} be a pointwise slant submanifold of the almost product Riemannian manifold (M_i, g, φ_i), with the slant function τ_i, $i \in \{1, \ldots, k\}$, such that $\tau_i(a) \neq \frac{\pi}{2}$ for any $a \in M_{i+1}$, $i \in \{1, \ldots, k-1\}$, where $\varphi_i := \frac{1}{\cos \tau_{i-1}} \cdot T_{i-1}$, $i \in \{2, \ldots, k\}$. Then, M_{k+1} is a pointwise slant submanifold of (M_1, g, φ_1), with the slant function*

$$\tilde{\tau}_1 = \arccos \Big(\prod_{i=1}^{k} \cos \tau_i \Big).$$

Proof. For any $a \in M_{k+1}$ and for any $w \in T_a M_{k+1}$, we have $\varphi_k w = T_k w + N_k w$. Since M_{i+1} is an immersed submanifold of M_i for any $i \in \{1, \ldots, k\}$, M_{k+1} is also an immersed submanifold of (M_1, g, φ_1), and we denote $\varphi_1 w = \tilde{T}_1 w + \tilde{N}_1 w$ with $\tilde{T}_1 w \in T_a M_{k+1}$ and $\tilde{N}_1 w \in T_a^{\perp_1} M_{k+1}$, where $T_a^{\perp_1} M_{k+1}$ stands for the orthogonal complement of $T_a M_{k+1}$ in $T_a M_1$. Then, we have

$$\begin{aligned}
\tilde{T}_1 w + \tilde{N}_1 w &= \varphi_1 w = T_1 w + N_1 w \\
&= \cos \tau_1(a) \cdot \varphi_2 w + N_1 w \\
&= \cos \tau_1(a)(T_2 w + N_2 w) + N_1 w \\
&\cdots \\
&= \prod_{i=1}^{k-1} \cos \tau_i(a) \cdot T_k w + N_1 w + \sum_{i=2}^{k} \Big(\prod_{j=1}^{i-1} \cos \tau_j(a) \cdot N_i w \Big),
\end{aligned}$$

$$\cos^2 \tilde{\tau}_1(a) \cdot \|w\|^2 = \|\tilde{T}_1 w\|^2 = \Big(\prod_{i=1}^{k-1} \cos \tau_i(a) \Big)^2 \|T_k w\|^2 = \Big(\prod_{i=1}^{k} \cos \tau_i(a) \Big)^2 \|w\|^2,$$

hence the conclusion. □

Remark 2.1. We notice that the slant function increases, at each step. Also, this construction will stop when we consider M_{k+1} an anti-invariant submanifold of M_k, since we can not define in the same way the naturally induced structure anymore. In this case (for $\tau_k = \frac{\pi}{2}$), M_{k+1} is an anti-invariant submanifold of (M_1, g, φ_1).

In particular, for $k = 2$, we obtain

Corollary 2.1. *Let M_2 be a proper pointwise slant submanifold of an almost product Riemannian manifold (M_1, g, φ_1), with the slant function τ_1, let $\varphi_2 := \frac{1}{\cos\tau_1} \cdot T_1$, and let M_3 be a pointwise slant submanifold of the almost product Riemannian manifold (M_2, g, φ_2), with the slant function τ_2. Then, M_3 is a pointwise slant submanifold of (M_1, g, φ_1), with the slant function $\tilde{\tau}_1 = \arccos\Big(\cos\tau_1\cos\tau_2\Big)$.*

Example 2.1. Excepting the $(1,1)$-tensor field on the ambient manifold, we shall consider the same manifold, the same metric, and the same immersed submanifolds like in [1] to illustrate this result in the almost product Riemannian case.

Let M_1 be the eight-dimensional almost product Riemannian manifold $\mathbb{R}^8$ with the Euclidean metric $\langle\cdot,\cdot\rangle$ and with the almost product structure φ_1 defined by

$$\varphi_1\Big(\frac{\partial}{\partial x_i}\Big) = \frac{\partial}{\partial y_i}, \;\; \varphi_1\Big(\frac{\partial}{\partial y_i}\Big) = \frac{\partial}{\partial x_i}$$

for $i \in \{1,\dots,4\}$, where we denoted by $(x_1, y_1, \dots, x_4, y_4)$ the canonical coordinates in $\mathbb{R}^8$, let M_2 be an immersed submanifold of M_1, defined by

$$H_1 : \mathbb{R}^4 \to \mathbb{R}^8,$$

$$H_1(u_1,u_2,u_3,u_4) := \Big(u_1+u_2, u_1-u_2, u_3+u_4, u_3-u_4, u_1, u_2, u_3, u_4\Big),$$

let M_3 be an immersed submanifold of $\mathbb{R}^4$, defined by

$$H_2 : \mathbb{R}^2 \to \mathbb{R}^4, \;\; H_2(u_1,u_2) := \Big(u_1, u_1+u_2, u_1-u_2, u_2\Big),$$

and let $\tilde{M}_3$ be the submanifold of M_2 defined by $\tilde{H}_1 := H_1 \circ H_2 : \mathbb{R}^2 \to \mathbb{R}^8$:

$$\tilde{H}_1(u_1,u_2) := \Big(2u_1+u_2, -u_2, u_1, u_1-2u_2, u_1, u_1+u_2, u_1-u_2, u_2\Big).$$

Then $\tilde{M}_3$ is an immersed submanifold of M_1, with the immersion given by $\tilde{H}_1 : \mathbb{R}^2 \to \mathbb{R}^8$.

Following similar computations like for the almost Hermitian case (see [1]), we obtain that M_2 is a slant submanifold of $(M_1, \langle\cdot,\cdot\rangle, \varphi_1)$, with the slant angle $\tau_1 = \arccos\frac{1}{3}$, $\tilde{M}_3$ is a slant submanifold of $(M_2, \langle\cdot,\cdot\rangle, \varphi_2)$, with the slant angle $\tau_2 = \arccos\frac{2}{3}$, and $\tilde{M}_3$ is a slant submanifold of $(M_1, \langle\cdot,\cdot\rangle, \varphi_1)$, with the slant angle $\tilde{\tau}_1 = \arccos\frac{2}{9}$; they satisfy $\cos\tilde{\tau}_1 = \cos\tau_1\cos\tau_2$.

From Corollary 2.1, we conclude

Proposition 2.1. *The property of a submanifold to be pointwise slant (with respect to the naturally induced almost product structure) is transitive on the class of proper pointwise slant submanifolds of almost product Riemannian manifolds.*

Theorem 2.1 can be extended to manifolds endowed with more general $(1,1)$-tensor fields than the almost product ones, namely those which satisfy $\varphi^2 = qI$ with q a positive smooth function.

Theorem 2.2. *Let (M_1, g, φ_1) be a Riemannian manifold endowed with a g-symmetric $(1,1)$-tensor field $\varphi_1 \neq \pm I$ which satisfies $\varphi_1^2 = q_1 I$ with q_1 a positive smooth function on M_1. Let $k \geq 2$, and let M_{i+1} be a pointwise slant submanifold of (M_i, g, φ_i), with the slant function τ_i, $i \in \{1,\dots,k\}$, such that $\tau_i(a) \neq \frac{\pi}{2}$ for any $a \in M_{i+1}$, $i \in \{1,\dots,k-1\}$, where $\varphi_i := \sqrt{\frac{q_i}{q_{i-1}}}\frac{1}{\cos\tau_{i-1}} \cdot T_{i-1}$ with q_i a positive smooth function on M_i, $i \in \{2,\dots,k\}$. Then, M_{k+1} is a pointwise slant submanifold of (M_1, g, φ_1), with the slant function*

$$\tilde{\tau}_1 = \arccos\Big(\prod_{i=1}^{k}\cos\tau_i\Big).$$

Proof. In this case, $\varphi_i^2 = q_i I$, $i \in \{1,\dots,k\}$. Using similar notations like in Theorem 2.1, we have

$$T_i^2 = q_i\cos^2\tau_i \cdot I, i \in \{1,\dots,k\}, \;\; \tilde{T}_1^2 = q_1\cos^2\tilde{\tau}_1 \cdot I,$$

and, for any $w \in T_aM_{k+1}$, we get

$$\begin{aligned}
\tilde{T}_1 w + \tilde{N}_1 w = \varphi_1 w &= T_1 w + N_1 w \\
&= \sqrt{\frac{q_1(a)}{q_2(a)}} \cos \tau_1(a) \cdot \varphi_2 w + N_1 w \\
&= \sqrt{\frac{q_1(a)}{q_2(a)}} \cos \tau_1(a)(T_2 w + N_2 w) + N_1 w \\
&\dots \\
&= \sqrt{\frac{q_1(a)}{q_k(a)}} \Big(\prod_{i=1}^{k-1} \cos \tau_i(a) \Big) T_k w + N_1 w + \sum_{i=2}^{k} \Big[\sqrt{\frac{q_1(a)}{q_k(a)}} \Big(\prod_{i=1}^{k-1} \cos \tau_i(a) \Big) N_i w \Big],
\end{aligned}$$

$$q_1(a) \cos^2 \tilde{\tau}_1(a) \cdot \|w\|^2 = \|\tilde{T}_1 w\|^2 = \frac{q_1(a)}{q_k(a)} \Big(\prod_{i=1}^{k-1} \cos \tau_i(a) \Big)^2 \|T_k w\|^2 = q_1(a) \Big(\prod_{i=1}^{k} \cos \tau_i(a) \Big)^2 \|w\|^2,$$

hence the conclusion. □

Remark 2.2. We showed that the slant function $\tilde{\tau}_1$ does not depend on the chosen structure φ_i on M_i if we chose it to be homothetically to T_{i-1}, $i \in \{2, \dots, k\}$.

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Author's contributions

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