

On Semi-Riemannian Manifolds Satisfying Some Generalized Einstein Metric Conditions

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ABSTRACT

The derivation-commutator $R \cdot C - C \cdot R$ of a semi-Riemannian manifold (M, g) , $\dim M \geq 4$, formed by its Riemann-Christoffel curvature tensor R and the Weyl conformal curvature tensor C , under some assumptions, can be expressed as a linear combination of $(0, 6)$ -Tachibana tensors $Q(A, T)$, where A is a symmetric $(0, 2)$ -tensor and T a generalized curvature tensor. These conditions form a family of generalized Einstein metric conditions. In this survey paper we present recent results on manifolds and submanifolds, and in particular hypersurfaces, satisfying such conditions.

Keywords: Warped product manifold, Einstein, quasi-Einstein, 2-quasi-Einstein and partially Einstein manifold, generalized Einstein metric condition, pseudosymmetry type curvature condition, hypersurface, Chen ideal submanifold.

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1. Introduction

Let (M, g) be a semi-Riemannian manifold. We denote by g, ∇, R, S, κ and C , the metric tensor, the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of (M, g) , respectively. Now we can define the $(0, 2)$ -tensors S^2 and S^3 , the $(0, 4)$ -tensors $R \cdot S, C \cdot S$ and $Q(A, B)$, and the $(0, 6)$ -tensors $R \cdot R, R \cdot C, C \cdot R, C \cdot C$ and $Q(A, T)$, where A and B are symmetric $(0, 2)$ -tensors and T a generalized curvature tensor. Thus, in particular, we have the derivation-commutator $R \cdot C - C \cdot R$ or $C \cdot R - R \cdot C$ [68, Section 1]. Furthermore for A and B we define their Kulkarni-Nomizu product $A \wedge B$. For precise definitions of the symbols used, we refer to Section 2 of this paper, as well as to [51, Section 1], [55, Section 1], [56, Chapter 6] and [68, Sections 1 and 2].

A semi-Riemannian manifold (M, g) , $\dim M = n \geq 2$, is said to be an *Einstein manifold* [3], or an *Einstein space*, if at every point of M its Ricci tensor S is proportional to g , i.e.,

$$S = \frac{\kappa}{n}g \quad (1.1)$$

on M , assuming that the scalar curvature κ is constant when $n = 2$. According to [3, p. 432] this condition is called the *Einstein metric condition*. Einstein manifolds form a natural subclass of several classes of semi-Riemannian manifolds which are determined by curvature conditions imposed on their Ricci tensor [3, Table, pp. 432-433]. These conditions are called *generalized Einstein metric conditions* [3, Chapter XVI].

Semi-Riemannian manifolds of dimension ≥ 4 and in particular, hypersurfaces in spaces of constant curvature or Chen ideal submanifolds in Euclidean spaces, satisfying curvature conditions of the form

(*) the derivation-commutator $R \cdot C - C \cdot R$ and a finite sum of the Tachibana tensors of the form $Q(A, T)$ are linearly dependent,

where $A = g, A = S,$ or $A = S^2,$ and $T = R, T = C, T = g \wedge g, T = g \wedge S, T = g \wedge S^2, T = S \wedge S, T = S \wedge S^2,$ or $T = S^2 \wedge S^2,$ were studied in several papers, see, e.g., [2, 16, 48, 49, 51, 54, 55, 59, 63, 68]. Conditions of this form are also generalized Einstein metric conditions, see, e.g., [43, Section 6] or [68, Section 1]. We refer to [43] for a survey of results on manifolds satisfying conditions of the form (*).

Precisely, in that paper results obtained to 2010 are presented. In the next years studies on manifolds and submanifolds, and in particular hypersurfaces, satisfying such conditions are continued. As it was already mentioned in Abstract, in this paper we present a survey of recent results on that subject.

On every semi-Riemannian Einstein manifold $(M, g), \dim M = n \geq 4,$ the following identity is satisfied [63, Theorem 3.1] (see also [68, p. 107])

$$R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n} Q(g, C).$$

This, by (1.1) and $\frac{\kappa}{(n-1)n} = \frac{\kappa}{n-1} - \frac{\kappa}{n},$ turns into

$$C \cdot R - R \cdot C = Q(S, C) - \frac{\kappa}{n-1} Q(g, C). \tag{1.2}$$

We mention that there are non-Einstein and non-conformally flat semi-Riemannian manifolds satisfying (1.2). Namely, every Roter space satisfies (1.2) (see Section 4).

Let (M, g) be a semi-Riemannian manifold of dimension $n \geq 3.$ We define the subsets \mathcal{U}_R and \mathcal{U}_S of M by $\mathcal{U}_R = \{x \in M \mid R - \frac{\kappa}{2(n-1)n} g \wedge g \neq 0 \text{ at } x\}$ and $\mathcal{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\},$ respectively. If $n \geq 4$ then we define the set $\mathcal{U}_C \subset M$ as the set of all points of (M, g) at which $C \neq 0.$ We note that on any semi-Riemannian manifold $(M, g), \dim M = n \geq 4,$ we have (see, e.g., [40])

$$\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R. \tag{1.3}$$

In Section 3 we present a survey on semi-Riemannian manifolds satisfying curvature conditions known as pseudosymmetry type curvature conditions. In particular, curvature conditions of the form (*) are conditions of this kind.

An extension of the class of Einstein manifolds also form quasi-Einstein, 2-quasi-Einstein and partially Einstein manifolds. These manifolds satisfy some pseudosymmetry type curvature conditions (see Section 4).

In Section 5 we consider warped product manifolds $\overline{M} \times_F \tilde{N}$ with a 2-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g}),$ a warping function F and an $(n-2)$ -dimensional semi-Riemannian manifold $(\tilde{N}, \tilde{g}),$ $n \geq 4,$ assuming that (\tilde{N}, \tilde{g}) is a space of constant curvature when $n \geq 5.$ Such manifolds satisfy some pseudosymmetry type curvature conditions (see Theorems 5.1 and 5.2). We mention that the warped product manifolds $\overline{M} \times_F \tilde{N},$ with a 2-dimensional Riemannian manifold $(\overline{M}, \overline{g})$ and an $(n-2)$ -dimensional unit sphere $\mathbb{S}^{n-2}, n \geq 4,$ where the warping function F is a solution of some second order quasilinear elliptic partial differential equation in the plane are related to Chen ideal submanifolds (see Section 11 for details).

Section 6 contains results on semi-Riemannian manifolds (M, g) of dimension ≥ 4 satisfying the following generalized Einstein metric condition on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$

$$R \cdot C - C \cdot R = LQ(S, C), \tag{1.4}$$

where L is some function on this set.

Let $N_s^{n+1}(c), n \geq 4,$ be a semi-Riemannian space of constant curvature with signature $(s, n+1-s),$ where $c = \frac{\tilde{\kappa}}{n(n+1)}$ and $\tilde{\kappa}$ is its scalar curvature. Let M be a connected hypersurface isometrically immersed in $N_s^{n+1}(c).$ We denote by $\mathcal{U}_H \subset M$ the set of all points at which the tensor H^2 is not a linear combination of the metric tensor g and the second fundamental tensor H of $M.$ We can verify that $\mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C \subset M$ (see, e.g., [82, p. 137]). We note that

$$H^2 = \alpha H + \beta g \tag{1.5}$$

on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H,$ where α and β are some functions defined on this set.

As it was stated in [47, Corollary 4.1], for a hypersurface M in $N_s^{n+1}(c), n \geq 4,$ if at every point of $\mathcal{U}_H \subset M$ one of the tensors $R \cdot C, C \cdot R$ or $R \cdot C - C \cdot R$ is a linear combination of the tensor $R \cdot R$ and a finite sum of

the Tachibana tensors of the form $Q(A, T)$, where A is a symmetric $(0, 2)$ -tensor and T a generalized curvature tensor, then the following equation is satisfied on \mathcal{U}_H

$$H^3 = \text{tr}(H)H^2 + \psi H + \rho g, \tag{1.6}$$

where ψ and ρ are some functions on \mathcal{U}_H . We also mention that if the condition

$$R \cdot C - C \cdot R = Q(g, T),$$

where T is a generalized curvature tensor, holds on the set \mathcal{U}_H of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, then the tensor T is a linear combination of the tensors: $R, g \wedge g, g \wedge S, S \wedge S$ and $g \wedge S^2$ [54, Theorem 5.2 (iii)].

In Section 7 we present results on 2-quasi-umbilical hypersurfaces in $N_s^{n+1}(c)$, $n \geq 4$, contained in [53]. In particular (see Theorem 7.3), we present curvature properties of a minimal 2-quasi-umbilical hypersurface M in an Euclidean space \mathbb{E}^{n+1} , $n \geq 4$, having exactly three distinct principal curvatures λ_1, λ_2 and λ_3 satisfying at every point of M : $\lambda_1 = 0, \lambda_2 = -(n - 2)\lambda$ and $\lambda_3 = \lambda_4 = \dots = \lambda_n = \lambda \neq 0$, where λ is a non-zero function on M . It is easy to check that $\text{tr}(H) = 0$ and

$$H^3 = \phi H^2 + \psi H \tag{1.7}$$

on $\mathcal{U}_H \subset M$, where $\phi = -(n - 3)\lambda$ and $\psi = (n - 2)\lambda^2$. Because $\text{tr}(H) \neq \phi$ at every point of \mathcal{U}_H , (1.6) is not satisfied on \mathcal{U}_H . Now, in view of the presented above result (i.e., [47, Corollary 4.1]), we conclude that the derivation-commutator $R \cdot C - C \cdot R$ of the considered hypersurface M cannot be expressed on \mathcal{U}_H as a linear combination of certain Tachibana tensors of the form $Q(A, T)$, where A is a symmetric $(0, 2)$ -tensor and T a generalized curvature tensor. The tensor $R \cdot C - C \cdot R$ of that hypersurface we can express, for instance, by (7.7). In this section we also present results on type number two hypersurfaces in $N_s^{n+1}(c)$, $n \geq 3$.

Results on hypersurfaces in $N_s^{n+1}(c)$, $n \geq 4$, satisfying some special generalized Einstein metric conditions of the form (*) are given in Sections 8, 9 and 10. In Section 8 we present results on hypersurfaces M in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.6) on $\mathcal{U}_H \subset M$, for some functions ψ and ρ on \mathcal{U}_H . Among other things it was stated that some generalized Einstein metric condition of the form (*) is satisfied on \mathcal{U}_H (see Theorem 8.3). Sections 9 and 10 contain results on quasi-Einstein and non-quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ the conditions: (1.4) and

$$R \cdot C - C \cdot R = L_1 Q(S, C) + L_2 Q(g, C), \tag{1.8}$$

where L, L_1 and L_2 are some functions defined on this set.

In Section 11 we present results on non-Einstein and non-conformally flat Chen ideal submanifolds satisfying some generalized Einstein metric conditions of the form (*). For instance, Theorems 11.5, 11.6 and 11.7 (see Theorems 5, 6 and 7 of [68]) contain results on Chen ideal submanifolds M in \mathbb{E}^{n+m} , $n \geq 4, m \geq 1$, satisfying the following conditions of the form (*): (1.8),

$$R \cdot C - C \cdot R = L_3 Q(g, R) + L_4 Q(S, R), \tag{1.9}$$

$$R \cdot C - C \cdot R = L_5 Q(g, g \wedge S) + L_6 Q(S, g \wedge S), \tag{1.10}$$

respectively, where L_3, L_4, L_5 and L_6 are some functions defined on M .

2. Preliminaries.

Throughout this paper, all manifolds are assumed to be connected paracompact manifolds of class C^∞ . Let (M, g) , $\dim M = n \geq 3$, be a semi-Riemannian manifold, and let ∇ be its Levi-Civita connection and $\mathfrak{X}(M)$ the Lie algebra of vector fields on M . We define on M the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X, Y)$ of $\mathfrak{X}(M)$ by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \quad \mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

respectively, where $X, Y, Z \in \mathfrak{X}(M)$ and A is a symmetric $(0, 2)$ -tensor on M . The Ricci tensor S , the Ricci operator \mathcal{S} and the scalar curvature κ of (M, g) are defined by

$$S(X, Y) = \text{tr}\{Z \rightarrow \mathcal{R}(Z, X)Y\}, \quad g(\mathcal{S}X, Y) = S(X, Y), \quad \kappa = \text{tr } S,$$

respectively. The endomorphism $\mathcal{C}(X, Y)$ is defined by

$$\mathcal{C}(X, Y)Z = \mathcal{R}(X, Y)Z - \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1}X \wedge_g Y)Z.$$

Now the Riemann-Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M, g) are defined by

$$R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4), \quad C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4),$$

respectively, where $X_1, X_2, \dots \in \mathfrak{X}(M)$. For a symmetric $(0, 2)$ -tensor A we denote by \mathcal{A} the endomorphism related to A by $g(\mathcal{A}X, Y) = A(X, Y)$, $X, Y \in \mathfrak{X}(M)$. The $(0, 2)$ -tensors A^p , $p = 2, 3, \dots$, are defined by $A^p(X, Y) = A^{p-1}(\mathcal{A}X, Y)$, $X, Y \in \mathfrak{X}(M)$, assuming that $A^1 = A$. In this way, for $A = S$ and $\mathcal{A} = S$ we get the tensors S^p , $p = 2, 3, \dots$, assuming that $S^1 = S$.

Let \mathcal{B} be a tensor field sending any $X, Y \in \mathfrak{X}(M)$ to a skew-symmetric endomorphism $\mathcal{B}(X, Y)$, and let B be the $(0, 4)$ -tensor associated with \mathcal{B} by

$$B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4), \tag{2.1}$$

where $X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$. The tensor B is said to be a *generalized curvature tensor* if the following two conditions are fulfilled:

$$\begin{aligned} B(X_1, X_2, X_3, X_4) &= B(X_3, X_4, X_1, X_2), \\ B(X_1, X_2, X_3, X_4) + B(X_2, X_3, X_1, X_4) + B(X_3, X_1, X_2, X_4) &= 0, \end{aligned}$$

where $X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$. For \mathcal{B} as above, let B be again defined by (2.1). We extend the endomorphism $\mathcal{B}(X, Y)$, $X, Y \in \mathfrak{X}(M)$, to a derivation $\mathcal{B}(X, Y) \cdot$ of the algebra of tensor fields on M , assuming that it commutes with contractions and $\mathcal{B}(X, Y) \cdot f = 0$ for any smooth function f on M . Now for a $(0, k)$ -tensor field T , $k \geq 1$, we can define the $(0, k+2)$ -tensor $B \cdot T$ by

$$\begin{aligned} (B \cdot T)(X_1, \dots, X_k, X, Y) &= (\mathcal{B}(X, Y) \cdot T)(X_1, \dots, X_k) \\ &= -T(\mathcal{B}(X, Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, \mathcal{B}(X, Y)X_k), \end{aligned}$$

where $X, Y, X_1, \dots, X_k \in \mathfrak{X}(M)$. If A is a symmetric $(0, 2)$ -tensor then we define the $(0, k+2)$ -tensor $Q(A, T)$ by

$$\begin{aligned} Q(A, T)(X_1, \dots, X_k, X, Y) &= (X \wedge_A Y \cdot T)(X_1, \dots, X_k) \\ &= -T((X \wedge_A Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, (X \wedge_A Y)X_k). \end{aligned}$$

In this manner we obtain the $(0, 6)$ -tensors $B \cdot B$ and $Q(A, B)$. Substituting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$ we get the tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $R \cdot S$, $Q(g, R)$, $Q(S, R)$, $Q(g, C)$, $Q(S, C)$, $Q(g, g \wedge S)$, $Q(g, S \wedge S)$ and $Q(g, S)$, $Q(g, S^2)$, $Q(S, S^2)$.

For a symmetric $(0, 2)$ -tensor E and a $(0, k)$ -tensor T , $k \geq 2$, we define their *Kulkarni-Nomizu tensor* $E \wedge T$ by (see, e.g., [40, Section 2])

$$\begin{aligned} (E \wedge T)(X_1, X_2, X_3, X_4; Y_3, \dots, Y_k) &= E(X_1, X_4)T(X_2, X_3, Y_3, \dots, Y_k) + E(X_2, X_3)T(X_1, X_4, Y_3, \dots, Y_k) \\ &\quad - E(X_1, X_3)T(X_2, X_4, Y_3, \dots, Y_k) - E(X_2, X_4)T(X_1, X_3, Y_3, \dots, Y_k). \end{aligned}$$

It is obvious that the following tensors are generalized curvature tensors: R , C and $E \wedge F$, where E and F are symmetric $(0, 2)$ -tensors. We have

$$C = R - \frac{1}{n-2}g \wedge S + \frac{\kappa}{2(n-2)(n-1)}g \wedge g \tag{2.2}$$

and (see, e.g., [40, Lemma 2.2(i)])

$$(a) \quad Q(E, E \wedge F) = -\frac{1}{2}Q(F, E \wedge E), \quad (b) \quad E \wedge Q(E, F) = -\frac{1}{2}Q(F, E \wedge E). \tag{2.3}$$

By an application of (2.3)(a) we obtain on M the identities

$$Q(g, g \wedge S) = -\frac{1}{2}Q(S, g \wedge g), \quad Q(S, g \wedge S) = -\frac{1}{2}Q(g, S \wedge S). \tag{2.4}$$

Further, by making use of (2.2) and (2.4), we get immediately

$$\begin{aligned} Q(g, C) &= Q(g, R) - \frac{1}{n-2}Q(g, g \wedge S) + \frac{\kappa}{2(n-2)(n-1)}Q(g, g \wedge g) \\ &= Q(g, R) - \frac{1}{n-2}Q(g, g \wedge S), \\ Q(S, C) &= Q(S, R) - \frac{1}{n-2}Q(S, g \wedge S) + \frac{\kappa}{2(n-2)(n-1)}Q(S, g \wedge g) \\ &= Q(S, R) + \frac{1}{2(n-2)}Q(g, S \wedge S) - \frac{\kappa}{(n-2)(n-1)}Q(g, g \wedge S). \end{aligned}$$

Let E_1, E_2 and F be symmetric $(0, 2)$ -tensors. We have (see, e.g., [16, Lemma 2.1(i)] and references therein)

$$E_1 \wedge Q(E_2, F) + E_2 \wedge Q(E_1, F) + Q(F, E_1 \wedge E_2) = 0. \tag{2.5}$$

From (2.5) we get easily (see also [40, Lemma 2.2(iii)] and references therein)

$$Q(F, E \wedge E_1) + Q(E, E_1 \wedge F) + Q(E_1, F \wedge E) = 0.$$

We denote by $g_{ij}, g^{ij}, R_{hijk}, S_{ij}, C_{hijk}, (R \cdot S)_{hklm}, Q(g, S)_{hklm}, Q(g, R)_{hijklm}, Q(S, R)_{hijklm}, (R \cdot C)_{hijklm}$ and $(C \cdot R)_{hijklm}$, the local components of the tensors $g, g^{-1}, R, S, C, R \cdot S, Q(g, S), Q(g, R), Q(S, R), R \cdot C$ and $C \cdot R$, respectively. On every semi-Riemannian manifold $(M, g), n \geq 4$, the following identity is satisfied (see, e.g., [43, Section 2])

$$(n-2)(R \cdot C - C \cdot R) = Q(S, R) - \frac{\kappa}{n-1}Q(g, R) - g \wedge (R \cdot S) + P, \tag{2.6}$$

where the $(0, 6)$ -tensor P is defined by

$$\begin{aligned} P(X_1, X_2, X_3, X_4; X, Y) &= g(X, X_1)R(SY, X_2, X_3, X_4) - g(Y, X_1)R(SX, X_2, X_3, X_4) \\ &\quad + g(X, X_2)R(X_1, SY, X_3, X_4) - g(Y, X_2)R(X_1, SX, X_3, X_4) \\ &\quad + g(X, X_3)R(X_1, X_2, SY, X_4) - g(Y, X_3)R(X_1, X_2, SX, X_4) \\ &\quad + g(X, X_4)R(X_1, X_2, X_3, SY) - g(Y, X_4)R(X_1, X_2, X_3, SX), \end{aligned}$$

where $X_1, X_2, X_3, X_4, X, Y \in \mathfrak{X}(M)$. The local expression of (2.6) reads

$$\begin{aligned} (n-2)(R \cdot C - C \cdot R)_{hijklm} &= Q(S, R)_{hijklm} - \frac{\kappa}{n-1}Q(g, R)_{hijklm} \\ &\quad + g_{hl}A_{mijk} - g_{hm}A_{lij k} - g_{il}A_{mhjk} + g_{im}A_{lhjk} \\ &\quad + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi} \\ &\quad - g_{ij}(R \cdot S)_{hklm} - g_{hk}(R \cdot S)_{ijlm} + g_{ik}(R \cdot S)_{hjlm} + g_{hj}(R \cdot S)_{iklm}, \end{aligned}$$

where $A_{mijk} = g^{rs}S_{mr}R_{sijk}$ and

$$\begin{aligned} (R \cdot S)_{hklm} &= g^{rs}(S_{hr}R_{sklm} + S_{kr}R_{shlm}), \\ Q(g, S)_{hklm} &= g_{hl}S_{km} + g_{kl}S_{hm} - g_{hm}S_{kl} - g_{km}S_{hl}, \\ Q(g, R)_{hijklm} &= g_{hl}R_{mijk} + g_{il}R_{hmjk} + g_{jl}R_{himk} + g_{kl}R_{hijm} - g_{hm}R_{lij k} - g_{im}R_{hljk} - g_{jm}R_{hil k} - g_{km}R_{hijl}, \\ Q(S, R)_{hijklm} &= S_{hl}R_{mijk} + S_{il}R_{hmjk} + S_{jl}R_{himk} + S_{kl}R_{hijm} - S_{hm}R_{lij k} - S_{im}R_{hljk} - S_{jm}R_{hil k} - S_{km}R_{hijl}, \\ (R \cdot C)_{hijklm} &= g^{rs}(C_{rij k}R_{shlm} + C_{hrjk}R_{sil m} + C_{hir k}R_{sjlm} + C_{hij r}R_{sklm}), \\ (C \cdot R)_{hijklm} &= g^{rs}(R_{rij k}C_{shlm} + R_{hrjk}C_{sil m} + R_{hir k}C_{sjlm} + R_{hij r}C_{sklm}). \end{aligned}$$

Let A be a symmetric $(0, 2)$ -tensor on a semi-Riemannian manifold $(M, g), \dim M = n \geq 3$. Let A_{ij} be the local components of the tensor A . Further, let A^2 and A^3 be the $(0, 2)$ -tensors with the local components $A^2_{ij} = A_{ir}g^{rs}A_{sj}$ and $A^3_{ij} = A^2_{ir}g^{rs}A_{sj}$, respectively. We have

$$\text{tr}_g(A) = \text{tr}(A) = g^{rs}A_{rs}, \quad \text{tr}_g(A^2) = \text{tr}(A^2) = g^{rs}A^2_{rs}, \quad \text{tr}_g(A^3) = \text{tr}(A^3) = g^{rs}A^3_{rs}.$$

We denote by \mathcal{U}_A the set of points of M at which $A \neq \frac{\text{tr}(A)}{n}g$.

Proposition 2.1. Let A be a symmetric $(0, 2)$ -tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, such that $\text{rank} A = 2$ on $\mathcal{U}_A \subset M$. Then on this set we have [36, Lemma 2.1; eqs. (2.6) and (2.10)]

$$A^3 = \text{tr}(A)A^2 + \frac{\text{tr}(A^2) - (\text{tr}(A))^2}{2}A \quad (2.7)$$

$$A \wedge A^2 = \frac{\text{tr}(A)}{2}A \wedge A. \quad (2.8)$$

Moreover, the following identity is satisfied on \mathcal{U}_A

$$Q(A^2, \frac{1}{2}A \wedge A) = -Q(A, A \wedge A^2) = -\frac{\text{tr}(A)}{2}Q(A, A \wedge A) = 0. \quad (2.9)$$

Proposition 2.2. Let A be a symmetric $(0, 2)$ -tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$. (i) ([44, Lemma 2.1], see also [42, Proposition 2.1 (i)]) If the following condition is satisfied on $\mathcal{U}_A \subset M$

$$\text{rank}(A - \alpha g) = 1$$

then

$$g \wedge A^2 + \frac{n-2}{2}A \wedge A - \text{tr}(A)g \wedge A + \frac{(\text{tr}(A))^2 - \text{tr}(A^2)}{2(n-1)}g \wedge g = 0 \quad (2.10)$$

and

$$A^2 - \frac{\text{tr}(A^2)}{n} = (\text{tr}(A) - (n-2)\alpha) \left(A - \frac{\text{tr}(A)}{n}g \right)$$

on \mathcal{U}_A , where α is some function on \mathcal{U}_A .

(ii) [42, Proposition 2.1 (ii)] If (2.10) is satisfied on $\mathcal{U}_A \subset M$ then

$$A^2 - \frac{\text{tr}(A^2)}{n}g = \rho \left(A - \frac{\text{tr}(A)}{n}g \right)$$

and

$$\left(A - \frac{\text{tr}(A) - \rho}{n-2}g \right) \wedge \left(A - \frac{\text{tr}(A) - \rho}{n-2}g \right) = 0$$

on \mathcal{U}_A , where ρ is some function on \mathcal{U}_A .

Let (M, g) be a semi-Riemannian manifold of dimension $\dim M = n \geq 3$. We set

$$E = g \wedge S^2 + \frac{n-2}{2}S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}(S^2)}{2(n-1)}g \wedge g. \quad (2.11)$$

It is easy to check that the tensor E is a generalized curvature tensor. In the same way, we define the $(0, 4)$ -tensor $E(A)$ corresponding to a symmetric $(0, 2)$ -tensor A [42, eq. (1.7)]

$$E(A) = g \wedge A^2 + \frac{n-2}{2}A \wedge A - \text{tr}(A)g \wedge A + \frac{(\text{tr}(A))^2 - \text{tr}(A^2)}{2(n-1)}g \wedge g. \quad (2.12)$$

Let T be a generalized curvature tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$. We denote by $\text{Ric}(T)$, $\kappa(T)$ and $\text{Weyl}(T)$ the Ricci tensor, the scalar curvature and the Weyl tensor of the tensor T , respectively. We refer to [40, Section 2], [41, Section 3] or [48, Section 3] for definitions of the considered tensors. In particular, we have

$$\text{Weyl}(T) = T - \frac{1}{n-2}g \wedge \text{Ric}(T) + \frac{\kappa(T)}{2(n-2)(n-1)}g \wedge g.$$

Let A be a symmetric $(0, 2)$ -tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$. Let $E(A)$ be the tensor defined by (2.12). It is easy to check that $\text{Ric}(E(A))$ is a zero tensor. Therefore, we also have $\kappa(E(A)) = 0$. Any generalized curvature tensor T defined on a 3-dimensional semi-Riemannian manifold (M, g) is expressed by $T = g \wedge \text{Ric}(T) - (\kappa(T)/4)g \wedge g$ (see [42, Section 2, p. 383] and references therein). Thus we see that the tensor $T = E(A)$ on any 3-dimensional semi-Riemannian manifold (M, g) is a zero tensor. In particular, on any 3-dimensional semi-Riemannian manifold (M, g) we have $E = 0$.

Proposition 2.3. [42, Proposition 2.2] Let T be a generalized curvature tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$. If the following condition is satisfied at a point $x \in M$

$$T = \alpha_1 R + \frac{\alpha_2}{2} S \wedge S + \alpha_3 g \wedge S + \alpha_4 g \wedge S^2 + \frac{\alpha_5}{2} g \wedge g$$

then

$$\text{Weyl}(T) = \alpha_1 C + \frac{\alpha_2}{n-2} E$$

at this point, where the tensor E is defined by (2.11) and $\alpha_1, \dots, \alpha_5 \in \mathbb{R}$.

According to [36], a generalized curvature tensor T on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$, is called a Roter type tensor if

$$T = \frac{\phi}{2} \text{Ric}(T) \wedge \text{Ric}(T) + \mu g \wedge \text{Ric}(T) + \frac{\eta}{2} g \wedge g \tag{2.13}$$

on $\mathcal{U}_{\text{Ric}(T)} \cap \mathcal{U}_{\text{Weyl}(T)}$, where ϕ, μ and η are some functions on this set. Manifolds admitting Roter type tensors were investigated (e.g.) in [93]. We have

Proposition 2.4. Let T be a generalized curvature tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$, satisfying (2.13) on $\mathcal{U}_{\text{Ric}(T)} \cap \mathcal{U}_{\text{Weyl}(T)} \subset M$. Then the following relations hold on this set

(i) [48, Proposition 3.2]

$$(\text{Ric}(T))^2 = \alpha_1 \text{Ric}(T) + \alpha_2 g, \quad \alpha_1 = \kappa(T) + \phi^{-1}((n-2)\mu - 1), \quad \alpha_2 = \phi^{-1}(\mu\kappa(T) + (n-1)\eta),$$

- (a) $T \cdot T = L_T Q(g, T), \quad L_T = \phi^{-1}((n-2)(\mu^2 - \phi\eta) - \mu),$
- (b) $T \cdot \text{Weyl}(T) = L_T Q(g, \text{Weyl}(T)),$
- (c) $T \cdot T = Q(\text{Ric}(T), T) + (L_T + \phi^{-1}\mu)Q(g, \text{Weyl}(T)),$

$$\text{Weyl}(T) \cdot T = L_{\text{Weyl}(T)} Q(g, T), \quad L_{\text{Weyl}(T)} = L_T + \frac{1}{n-2} \left(\frac{\kappa(T)}{n-1} - \alpha_1 \right), \tag{2.14}$$

$$\text{Weyl}(T) \cdot \text{Weyl}(T) = L_{\text{Weyl}(T)} Q(g, \text{Weyl}(T)), \tag{2.15}$$

$$\text{Weyl}(T) \cdot T = Q(\text{Ric}(T), \text{Weyl}(T)) + \left(L_T - \frac{\kappa(T)}{n-1} \right) Q(g, \text{Weyl}(T)).$$

We also have

$$\begin{aligned} T \cdot \text{Weyl}(T) - \text{Weyl}(T) \cdot T &= \left(\frac{(n-1)\mu - 1}{(n-2)\phi} + \frac{\kappa(T)}{n-1} \right) Q(g, T) \\ &+ \frac{1}{n-2} Q(\text{Ric}(T), T) + \frac{\mu((n-1)\mu - 1) - (n-1)\phi\eta}{2(n-2)\phi} Q(\text{Ric}(T), g \wedge g), \end{aligned}$$

and, equivalently,

$$\begin{aligned} T \cdot \text{Weyl}(T) - \text{Weyl}(T) \cdot T &= \left(\phi^{-1} \left(\mu - \frac{1}{n-2} \right) + \frac{\kappa(T)}{n-1} \right) Q(g, T) \\ &+ \frac{1}{2} \left(\phi^{-1} \mu \left(\mu - \frac{1}{n-2} \right) - \eta \right) Q(\text{Ric}(T), g \wedge g). \end{aligned}$$

(ii) [93, Sections 1 and 4]

$$\begin{aligned} Q(\text{Ric}(T), \text{Weyl}(T)) &= \phi^{-1} \left(\frac{1}{n-2} - \mu \right) Q(g, T) \\ &+ \frac{1}{n-2} \left(L_T - \frac{\kappa(T)}{n-1} \right) Q(g, g \wedge \text{Ric}(T)). \end{aligned}$$

Moreover, if $L_T = \frac{\kappa(T)}{n-1}$, resp., $\kappa(T) = 0$, then we have

$$Q(\text{Ric}(T), \text{Weyl}(T)) = L_{\text{Weyl}(T)} Q(g, T),$$

and

$$T \cdot \text{Weyl}(T) - \text{Weyl}(T) \cdot T = -Q(\text{Ric}(T), \text{Weyl}(T)),$$

respectively. Moreover, if $L_{\text{Weyl}(T)} = 0$ at a point of $\mathcal{U}_{\text{Ric}(T)} \cap \mathcal{U}_{\text{Weyl}(T)}$ then at this point we have

$$\phi^{-1} \left(\frac{1}{n-2} - \mu \right) (T \cdot \text{Weyl}(T) - \text{Weyl}(T) \cdot T) = (n-2) \left(\phi^{-1} \mu \left(\mu - \frac{1}{n-2} \right) - \eta \right) Q(\text{Ric}(T), \text{Weyl}(T)).$$

For further results on generalizing curvature tensors satisfying some conditions we refer to [41, Section 3].

Let A be a symmetric $(0, 2)$ -tensor and T a $(0, k)$ -tensor, $k = 2, 3, \dots$. The tensor $Q(A, T)$ is called the *Tachibana tensor* of A and T , or the Tachibana tensor for short (see, e.g., [54]). Using the tensors g, R and S we can define the following $(0, 6)$ -Tachibana tensors: $Q(S, R), Q(g, R), Q(g, g \wedge S)$ and $Q(S, g \wedge S)$. We can check, by making use of (2.3)(a) and (2.4), that other $(0, 6)$ -Tachibana tensors constructed from g, R and S may be expressed by the four Tachibana tensors mentioned above or vanish identically on M .

According to [51, eq. (1.13), Theorem 3.4 (i)] the following identity is satisfied on any semi-Riemannian manifold (M, g) of dimension $n \geq 4$

$$C \cdot R + R \cdot C = R \cdot R + C \cdot C - \frac{1}{(n-2)^2} Q \left(g, g \wedge S^2 - \frac{\kappa}{n-1} g \wedge S \right). \quad (2.16)$$

From (2.16), by a suitable contraction, we get (cf. [53, Lemma 2.3], [61, p. 217])

$$C \cdot S = R \cdot S - \frac{1}{n-2} Q \left(g, S^2 - \frac{\kappa}{n-1} S \right).$$

Let (\bar{M}, \bar{g}) and (\tilde{N}, \tilde{g}) , $\dim \bar{M} = p, \dim \tilde{N} = n - p, 1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{U; x^a\}$ and $\{V; y^\alpha\}$, respectively. Let F be a positive smooth function on \bar{M} . It is well known that the *warped product* $\bar{M} \times_F \tilde{N}$ of (\bar{M}, \bar{g}) and (\tilde{N}, \tilde{g}) is the product manifold $\bar{M} \times \tilde{N}$ with the metric $g = \bar{g} \times_F \tilde{g}$ defined by (see, e.g., [106, 109])

$$\bar{g} \times_F \tilde{g} = \pi_1^* \bar{g} + (F \circ \pi_1) \pi_2^* \tilde{g},$$

where $\pi_1 : \bar{M} \times \tilde{N} \rightarrow \bar{M}$ and $\pi_2 : \bar{M} \times \tilde{N} \rightarrow \tilde{N}$ are the natural projections on \bar{M} and \tilde{N} , respectively. Let $\{U \times V; x^1, \dots, x^p, x^{p+1} = y^1, \dots, x^n = y^{n-p}\}$ be a product chart for $\bar{M} \times \tilde{N}$. The local components g_{ij} of the metric $g = \bar{g} \times_F \tilde{g}$ with respect to this chart are the following $g_{ij} = \bar{g}_{ab}$ if $i = a$ and $j = b$, $g_{ij} = F \tilde{g}_{\alpha\beta}$ if $i = \alpha$ and $j = \beta$, and $g_{ij} = 0$ otherwise, where $a, b, c, d, f \in \{1, \dots, p\}$, $\alpha, \beta, \gamma, \delta \in \{p+1, \dots, n\}$ and $h, i, j, k, r, s \in \{1, 2, \dots, n\}$. We will denote by bars (resp., by tildes) tensors formed from \bar{g} (resp., \tilde{g}). The local components

$$\Gamma_{ij}^h = \frac{1}{2} g^{hs} (\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij}), \quad \partial_j = \frac{\partial}{\partial x^j},$$

of the Levi-Civita connection ∇ of $\bar{M} \times_F \tilde{N}$ are the following (see, e.g., [106]):

$$\begin{aligned} \Gamma_{bc}^a &= \bar{\Gamma}_{bc}^a, \quad \Gamma_{\beta\gamma}^\alpha = \tilde{\Gamma}_{\beta\gamma}^\alpha, \quad \Gamma_{\alpha\beta}^a = -\frac{1}{2} \bar{g}^{ab} F_b \tilde{g}_{\alpha\beta}, \quad \Gamma_{\alpha\beta}^\alpha = \frac{1}{2F} F_a \delta_\beta^\alpha, \\ \Gamma_{\alpha b}^a &= \Gamma_{ab}^\alpha = 0, \quad F_a = \partial_a F = \frac{\partial F}{\partial x^a}, \quad \partial_a = \frac{\partial}{\partial x^a}. \end{aligned}$$

The local components

$$R_{hijk} = g_{hs} R_{ijk}^s = g_{hs} (\partial_k \Gamma_{ij}^s - \partial_j \Gamma_{ik}^s + \Gamma_{ij}^r \Gamma_{rk}^s - \Gamma_{ik}^r \Gamma_{rj}^s), \quad \partial_k = \frac{\partial}{\partial x^k},$$

of the Riemann-Christoffel curvature tensor R and the local components S_{ij} of the Ricci tensor S of the warped product $\bar{M} \times_F \tilde{N}$ which may not vanish identically are the following:

$$\begin{aligned} R_{abcd} &= \bar{R}_{abcd}, \quad R_{\alpha ab\beta} = -\frac{1}{2} T_{ab} \tilde{g}_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = F \tilde{R}_{\alpha\beta\gamma\delta} - \frac{1}{4} \Delta_1 F (\tilde{g}_{\alpha\delta} \tilde{g}_{\beta\gamma} - \tilde{g}_{\alpha\gamma} \tilde{g}_{\beta\delta}), \\ S_{ab} &= \bar{S}_{ab} - \frac{n-p}{2} \frac{1}{F} T_{ab}, \quad S_{\alpha\beta} = \tilde{S}_{\alpha\beta} - \frac{1}{2} (\text{tr}(T) + \frac{n-p-1}{2F} \Delta_1 F) \tilde{g}_{\alpha\beta}, \end{aligned}$$

where

$$T_{ab} = \bar{\nabla}_b F_a - \frac{1}{2F} F_a F_b, \quad \text{tr}(T) = \bar{g}^{ab} T_{ab} = \Delta F - \frac{1}{2F} \Delta_1 F,$$

$$\Delta F = \Delta_{\bar{g}} F = \bar{g}^{ab} \nabla_a F_b, \quad \Delta_1 F = \Delta_{1\bar{g}} F = \bar{g}^{ab} F_a F_b,$$

and T is the $(0, 2)$ -tensor with the local components T_{ab} . We can express the scalar curvature κ of $\bar{M} \times_F \tilde{N}$ by

$$\kappa = \bar{\kappa} + \frac{1}{F} \tilde{\kappa} - \frac{n-p}{F} (\text{tr}(T) + \frac{n-p-1}{4F} \Delta_1 F) = \bar{\kappa} + \frac{1}{F} \tilde{\kappa} - \frac{n-p}{F} (\Delta F + \frac{n-p-3}{4F} \Delta_1 F).$$

3. Pseudosymmetry type curvature conditions

It is well-known that if a semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, is locally symmetric then $\nabla R = 0$ on M (see, e.g., [107, Chapter 1.5]). This implies the following integrability condition $\mathcal{R}(X, Y) \cdot R = 0$, $X, Y \in \mathfrak{X}(M)$, in short

$$R \cdot R = 0. \tag{3.1}$$

Semi-Riemannian manifolds satisfying (3.1) are called *semisymmetric* [119] (see also [5, Chapter 8.5.3], [9, Chapter 20.7], [107, Chapter 1.6], [120, 121, 126]).

Semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds. A semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, is said to be *pseudosymmetric* if the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent at every point of M (see, e.g., [5, Chapter 8.5.3], [9, Chapter 20.7], [11, Section 15.1], [56, Chapter 6], [107, Chapter 12.4], [35, 40, 43, 56, 72, 86, 87, 116, 123, 125, 126, 127] and references therein). This is equivalent to

$$R \cdot R = L_R Q(g, R) \tag{3.2}$$

on $\mathcal{U}_R \subset M$, where L_R is some function on \mathcal{U}_R . Every semisymmetric manifold is pseudosymmetric. The converse statement is not true (see, e.g., [72]). A non-semisymmetric pseudosymmetric manifold is called *properly pseudosymmetric*. A pseudosymmetric manifold (M, g) , $\dim M = n \geq 3$, is called a *pseudosymmetric manifold of constant type* if the function L_R is constant on \mathcal{U}_R [101] (see also [88, 90, 102, 103, 104, 105]).

A semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, is called *Ricci-pseudosymmetric* if the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent at every point of M (see, e.g., [5, Chapter 8.5.3], [11, Section 15.1], [43]). This is equivalent on $\mathcal{U}_S \subset M$ to

$$R \cdot S = L_S Q(g, S), \tag{3.3}$$

where L_S is some function on \mathcal{U}_S . Every warped product manifold $\bar{M} \times_F \tilde{N}$ with a 1-dimensional (\bar{M}, \bar{g}) manifold and an $(n-1)$ -dimensional Einstein semi-Riemannian manifold (\tilde{N}, \tilde{g}) , $n \geq 3$, and a warping function F , is a Ricci-pseudosymmetric manifold, see, e.g., [16, Section 1] and [51, Example 4.1]. According to [84], a Ricci-pseudosymmetric manifold (M, g) , $\dim M = n \geq 3$, is called a *Ricci-pseudosymmetric manifold of constant type* if the function L_S is constant on \mathcal{U}_S .

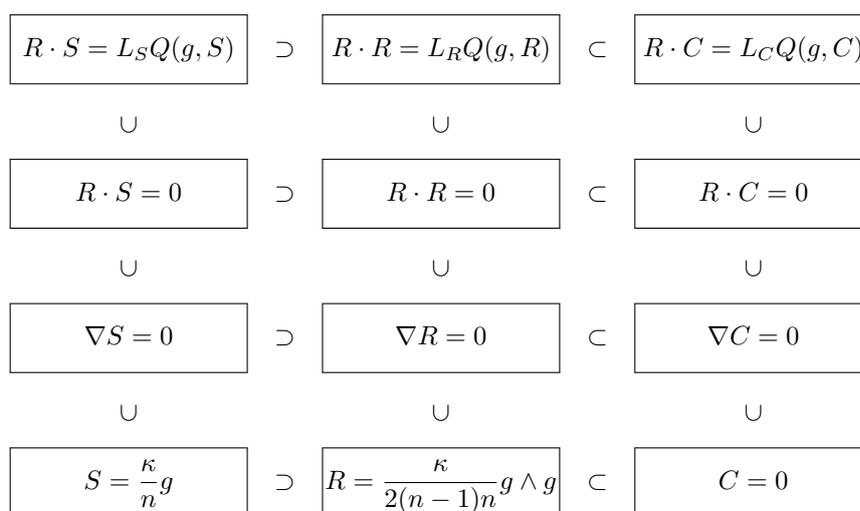
We mention that pseudosymmetric and Ricci-pseudosymmetric manifolds also are called *Deszcz symmetric* and *Deszcz Ricci-symmetric spaces*, respectively (see, e.g., [11]).

A semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$, is said to be *Weyl-pseudosymmetric* if the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent at every point of M [40, 43]. This is equivalent on $\mathcal{U}_C \subset M$ to

$$R \cdot C = L_1 Q(g, C), \tag{3.4}$$

where L_1 is some function on \mathcal{U}_C . We can easily check that on every Einstein manifold (M, g) , $\dim M \geq 4$, (3.4) turns into $R \cdot R = L_1 Q(g, R)$.

For a presentation of results on the problem of the equivalence of pseudosymmetry, Ricci-pseudosymmetry and Weyl-pseudosymmetry we refer to [43, Section 4]. Inclusions between mentioned above semi-Riemannian manifolds (M, g) , i.e., pseudosymmetric, Ricci-pseudosymmetric and Weyl-pseudosymmetric manifolds, can be presented in the following diagram [43, Section 4]



All inclusions in the above presentation are strict, provided that $\dim M = n \geq 4$.

A semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$, is said to be a *manifold with pseudosymmetric Weyl tensor* (to have a pseudosymmetric conformal Weyl tensor) if the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent at every point of M (see, e.g., [11, Section 15.1], [40, 43, 51]). This is equivalent on $\mathcal{U}_C \subset M$ to

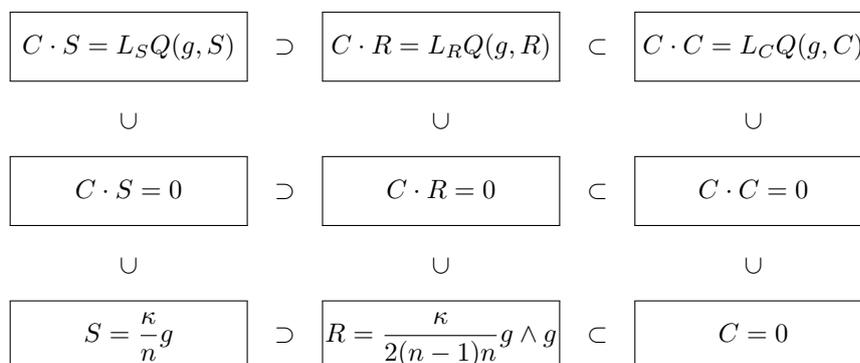
$$C \cdot C = L_C Q(g, C), \tag{3.5}$$

where L_C is some function on \mathcal{U}_C . Every warped product manifold $\bar{M} \times_F \tilde{N}$, with $\dim \bar{M} = \dim \tilde{N} = 2$, satisfies (3.5) (see, e.g., [40, 43, 51] and references therein). Thus in particular, the Schwarzschild spacetime, the Kottler spacetime and the Reissner-Nordström spacetime satisfy (3.5). Semi-Riemannian manifolds with pseudosymmetric Weyl tensor were investigated among other things in [40, 60, 73, 77].

We can show that $C \cdot R = LQ(g, R)$ implies $C \cdot C = LQ(g, C)$ (see [108, Proposition 2.1]). Let (M, g) , $\dim M = n \geq 4$, be a semi-Riemannian manifold satisfying $C \cdot R = LQ(g, R)$ on $\mathcal{U}_C \subset M$. From this we get easily $C \cdot S = LQ(g, S)$ on \mathcal{U}_C . Further, we have

$$\begin{aligned} C \cdot C &= C \cdot \left(R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{2(n-2)(n-1)} g \wedge g \right) \\ &= C \cdot R - \frac{1}{n-2} g \wedge (C \cdot S) + \frac{\kappa}{2(n-2)(n-1)} C \cdot (g \wedge g) = LQ(g, R) - \frac{L}{n-2} g \wedge Q(g, S) \\ &= LQ(g, R) - \frac{L}{n-2} Q(g, g \wedge S) = LQ \left(g, R - \frac{1}{n-2} g \wedge S \right) \\ &= LQ \left(g, R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{2(n-2)(n-1)} g \wedge g \right) = LQ(g, C). \end{aligned}$$

We also have the following diagram



All inclusions in the above presentation are strict, provided that $\dim M = n \geq 4$.

Warped product manifolds $\overline{M} \times_F \tilde{N}$, of dimension ≥ 4 , satisfying on $\mathcal{U}_C \subset \overline{M} \times_F \tilde{N}$, the condition

$$R \cdot R - Q(S, R) = LQ(g, C), \quad (3.6)$$

where L is some function on \mathcal{U}_C , were studied among others in [21]. In that paper necessary and sufficient conditions for $\overline{M} \times_F \tilde{N}$ to be a manifold satisfying (3.6) are given. Moreover, in that paper it was proved that any 4-dimensional warped product manifold $\overline{M} \times_F \tilde{N}$, with a 1-dimensional base $(\overline{M}, \overline{g})$, satisfies (3.6) [21, Theorem 4.1].

We refer to [16, 40, 43, 48, 51, 56, 60, 116] for details on semi-Riemannian manifolds satisfying (3.2) and (3.3)-(3.6), as well as other conditions of this kind, named pseudosymmetry type curvature conditions. We also refer to [60, Section 3] for a recent survey on manifolds satisfying such curvature conditions. It seems that the condition (3.2) is the most important condition of that family of curvature conditions (see, e.g., [51]). The Schwarzschild spacetime, the Kottler spacetime, the Reissner-Nordström spacetime, as well as the Friedmann-Lemaître-Robertson-Walker spacetimes are the "oldest" examples of pseudosymmetric warped product manifolds (see, e.g., [51, 56, 72, 116]).

4. Quasi-Einstein, 2-quasi-Einstein and partially Einstein manifolds

A semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, is said to be a *quasi-Einstein manifold* if

$$\text{rank}(S - \alpha g) = 1 \quad (4.1)$$

on $\mathcal{U}_S \subset M$, where α is some function on \mathcal{U}_S . It is known that every non-Einstein warped product manifold $\overline{M} \times_F \tilde{N}$ with a 1-dimensional $(\overline{M}, \overline{g})$ base manifold and a 2-dimensional manifold (\tilde{N}, \tilde{g}) or an $(n-1)$ -dimensional Einstein manifold (\tilde{N}, \tilde{g}) and a warping function F , $n \geq 4$, is a quasi-Einstein manifold (see, e.g., [16, 51]). A Riemannian manifold (M, g) , $\dim M = n \geq 3$, whose Ricci tensor has an eigenvalue of multiplicity $n-1$ is a quasi-Einstein manifold (cf. [37, Introduction]). Evidently, the converse statement is also true. We mention that quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and the investigation on quasi-umbilical hypersurfaces of conformally flat spaces (see, e.g., [43, 51] and references therein). Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied among others in [45, 58, 64, 84] (see also [43] and references therein). Quasi-Einstein manifolds satisfying some pseudosymmetry type curvature conditions were investigated recently in [2, 16, 40, 48, 60]. We mention that an example of a non-semisymmetric ($R \cdot R \neq 0$) Ricci-semisymmetric ($R \cdot S = 0$) quasi-Einstein hypersurface M in an Euclidean space \mathbb{E}^{n+1} , $n \geq 5$, was constructed in [1].

A semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, is called a *2-quasi-Einstein manifold* if

$$\text{rank}(S - \alpha g) \leq 2 \quad (4.2)$$

on $\mathcal{U}_S \subset M$ and $\text{rank}(S - \alpha g) = 2$ on some open non-empty subset of \mathcal{U}_S , where α is some function on \mathcal{U}_S (see, e.g., [53]).

Every non-Einstein and non-quasi-Einstein warped product manifold $\overline{M} \times_F \tilde{N}$ with a 2-dimensional base manifold $(\overline{M}, \overline{g})$ and a 2-dimensional manifold (\tilde{N}, \tilde{g}) or an $(n-2)$ -dimensional Einstein semi-Riemannian manifold (\tilde{N}, \tilde{g}) , when $n \geq 5$, and a warping function F satisfies (4.2) (see, e.g., [51, Theorem 6.1]). Thus some exact solutions of the Einstein field equations are non-conformally flat 2-quasi-Einstein manifolds. For instance, the Reissner-Nordström spacetime, as well as the Reissner-Nordström-de Sitter type spacetimes are such manifolds (see, e.g., [93]). It seems that the Reissner-Nordström spacetime is the "oldest" example of a non-conformally flat 2-quasi-Einstein warped product manifold. It is easy to see that every 2-quasi-umbilical hypersurface in a semi-Riemannian space of constant curvature is a 2-quasi-Einstein manifold (see, e.g., [53]).

We mention that Einstein warped product manifolds $\overline{M} \times_F \tilde{N}$, with $\dim \overline{M} = 1, 2$, were studied in [3, Chapter 9.J] (see also [3, Chapter 3.F]).

The semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, will be called a *partially Einstein manifold*, or a *partially Einstein space* (cf. [10, Foreword], [124, p. 20]), if at every point $x \in \mathcal{U}_S \subset M$ its Ricci operator S satisfies $S^2 = \lambda S + \mu Id_x$, or equivalently,

$$S^2 = \lambda S + \mu g, \quad (4.3)$$

where $\lambda, \mu \in \mathbb{R}$ and Id_x is the identity transformation of $T_x M$. Thus every quasi-Einstein manifold is a partially Einstein manifold. The converse statement is not true. Contracting (4.3) we get $\text{tr}(S^2) = \lambda\kappa + n\mu$. This together with (4.3) yields (cf. [41, Section 5])

$$S^2 - \frac{\text{tr}(S^2)}{n}g = \lambda \left(S - \frac{\kappa}{n}g \right). \quad (4.4)$$

In particular, a Riemannian manifold (M, g) , $\dim M = n \geq 3$, is a partially Einstein space if at every point $x \in U_S \subset M$ its Ricci operator S has exactly two distinct eigenvalues (principal Ricci curvatures) $\rho_1 = \rho_2 = \dots = \rho_p$ and $\rho_{p+1} = \rho_{p+2} = \dots = \rho_n$ with multiplicities p and $n - p$, respectively, where $1 \leq p \leq n - 1$. Now (4.3) and (4.4) yield

$$S^2 = (\rho_1 + \rho_{p+1})S - \rho_1\rho_{p+1}g \quad \text{and} \quad S^2 - \frac{\text{tr}(S^2)}{n}g = (\rho_1 + \rho_{p+1}) \left(S - \frac{\kappa}{n}g \right), \quad (4.5)$$

respectively. Evidently, if $p = 1$, or $p = n - 1$, then (M, g) is a quasi-Einstein manifold.

Remark 4.1. (i) Let (M, g) , $n = \dim M \geq 3$, be a semi-Riemannian manifold. In addition, let (M, g) be a conformally flat manifold when $n \geq 4$. Thus the set $U_C \subset M$ is empty. Now, by (1.3), the subsets U_S and U_R of M satisfy $U_S = U_R$. In view of [22, Lemma 1.2] (see also [66, Lemma 2.1]) we can state that on U_S any of the following three conditions is equivalent to each other: $R \cdot R = \rho Q(g, R)$, $R \cdot S = \rho Q(g, S)$ and

$$S^2 - \frac{\text{tr}(S^2)}{n}g = \left(\frac{\kappa}{n-1} + (n-2)\rho \right) \left(S - \frac{\kappa}{n}g \right). \quad (4.6)$$

where ρ is some function on U_S .

(ii) Let (M, g) , $n = \dim M \geq 3$, be a Riemannian manifold with vanishing Weyl conformal curvature tensor C such that its Ricci operator S has at every point of M exactly two Ricci principal curvatures ρ_1 and ρ_{p+1} with multiplicities p and $n - p$, respectively, where $1 \leq p < n$. Using (4.5) and (4.6) we can easily check that (M, g) is a pseudosymmetric manifold satisfying (3.2) with

$$L_R = \rho = \frac{1}{(n-2)(n-1)}((n-1)(\rho_1 + \rho_{p+1}) - \kappa). \quad (4.7)$$

Evidently, if $p = 1$ then (4.7) turns into

$$L_R = \rho = \frac{1}{n-1}\rho_1.$$

(iii) As an immediate consequence of the above presented results we obtain (cf. [126, Foreword, p. xviii]):

Riemannian spaces of dimensions ≥ 3 with vanishing Weyl conformal curvature tensor are Deszcz symmetric (i.e., pseudosymmetric) if and only if they are Einstein or partially Einstein.

(iv) (cf. [84, Proposition 2.1]) *Let (M, g) , $\dim M = n \geq 3$, be a semi-Riemannian manifold. We note that (4.1) holds at a point $x \in U_S \subset M$ if and only if $(S - \alpha g) \wedge (S - \alpha g) = 0$ at x , i.e.,*

$$\frac{1}{2}S \wedge S - \alpha g \wedge S + \frac{\alpha^2}{2}g \wedge g = 0. \quad (4.8)$$

From (4.8), by a suitable contraction, we get immediately

$$S^2 = (\kappa - (n-2)\alpha)S + \alpha((n-1)\alpha - \kappa)g. \quad (4.9)$$

Evidently, (4.9) is a special case of (4.3).

(v) Let ρ_1, ρ_2 and ρ_3 be principal Ricci curvatures of a 3-dimensional Riemannian manifold (M, g) . Moreover, let the condition

$$\rho_1 = \rho_2 \neq \rho_3 \quad (4.10)$$

be satisfied at every point of the set $U_S = U_R \subset M$. Thus

$$R \cdot R = \frac{\rho_3}{2}Q(g, R)$$

on U_R . Evidently, if $\rho_3 = 0$ on U_R then (M, g) is semisymmetric.

(vi) 3-dimensional Riemannian manifolds satisfying (4.10) were studied, among others things, in the following papers: [97, 99, 100, 101, 102, 103, 104]. We mention that an explicit classification of 3-dimensional semisymmetric Riemannian manifolds is given in [98].

Example 4.1. The warped product manifold $\overline{M} \times_F \widetilde{N}$ with a 1-dimensional manifold $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \pm 1$, and an $(n - 1)$ -dimensional semi-Riemannian Einstein manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 5$, assumed that it is not of constant curvature, and a warping function F , satisfies on $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$ [16, Theorem 4.1]

$$\begin{aligned} \text{rank}(S - \alpha g) &= 1, \quad R \cdot S = L_S Q(g, S), \\ (n - 2)(R \cdot C - C \cdot R) &= Q(S, R) - L_S Q(g, R), \quad \alpha = \frac{\kappa}{n - 1} - L_S, \quad L_S = -\frac{\text{tr}T}{2F}. \end{aligned}$$

Furthermore, using

$$\begin{aligned} Q(g, R) &= Q(g, C) - \frac{1}{2(n - 2)} Q(S, g \wedge g), \\ Q(S, R) &= Q(S, C) + \frac{1}{2(n - 2)} \left(\alpha - \frac{\kappa}{n - 1} \right) Q(S, g \wedge g), \end{aligned}$$

we obtain [51, Example 4.1]

$$(n - 2)(R \cdot C - C \cdot R) = Q(S, C) - L_S Q(g, C).$$

Example 4.2. Let M be an open connected non-empty subset of \mathbb{R}^5 endowed with the metric g of the form [94, 95, 96]

$$ds^2 = g_{ij} dx^i dx^j = dx^2 + dy^2 + du^2 + dv^2 + \rho^2 (xdu - ydv + dz)^2, \quad \rho = \text{const.} \neq 0.$$

The manifold (M, g) is a non-conformally flat manifold satisfying the following conditions [116]:

$$\begin{aligned} \nabla_X S(Y, Z) + \nabla_Y S(Z, X) + \nabla_Z S(X, Y) &= 0, \\ S &= \frac{\kappa}{2} g - \frac{3\kappa}{2} \eta \otimes \eta, \quad \eta = (0, 0, -\rho, -x\rho, y\rho), \quad \kappa = \rho^2, \quad S^2 = -\frac{\kappa}{2} S + \frac{\kappa^2}{2} g, \\ S \cdot R &= 2\kappa R - \frac{\kappa}{2} g \wedge S + \frac{\kappa^2}{4} g \wedge g, \quad C \cdot S = 0, \quad R \cdot S = -\frac{\kappa}{4} Q(g, S), \\ R \cdot R &= -\frac{\kappa}{4} Q(g, R), \quad R \cdot C = -\frac{\kappa}{4} Q(g, C), \\ C \cdot R &= -\frac{1}{3} Q(S, C) - \frac{\kappa}{3} Q(g, C), \quad C \cdot C = C \cdot R, \\ R \cdot C + C \cdot R &= -\frac{1}{3} Q(S, C) - \frac{7\kappa}{12} Q(g, C) \end{aligned}$$

and the condition of the form (*)

$$R \cdot C - C \cdot R = \frac{1}{3} Q(S, C) + \frac{\kappa}{12} Q(g, C).$$

The $(0, 4)$ -tensor $S \cdot R$ is defined by

$$(S \cdot R)(X, Y, W, Z) = R(SX, Y, W, Z) + R(X, SY, W, Z) + R(X, Y, SW, Z) + R(X, Y, W, SZ),$$

where $X, Y, W, Z \in \mathfrak{X}(M)$.

An important subclass of the class of partially Einstein manifolds, of dimension ≥ 4 , form some non-conformally flat and non-quasi-Einstein manifolds called Roter spaces. A semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$, satisfying on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ the following equation

$$R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \frac{\eta}{2} g \wedge g, \tag{4.11}$$

where ϕ, μ and η are some functions on this set, is called a *Roter type manifold*, or a *Roter manifold*, or a *Roter space* (see, e.g., [11, Section 15.5], [36, 51, 53, 56]). Equation (4.11) is called a *Roter equation* (see, e.g., [44, Section 1]). Roter spaces and in particular Roter hypersurfaces in semi-Riemannian spaces of constant curvature were studied in: [18, 36, 40, 48, 57, 58, 65, 69, 70, 80, 83, 92, 93]. We only mention that every Roter space (M, g) ,

$\dim M = n \geq 4$, satisfies on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ among other things the following conditions: (1.2), (3.2) and (3.3)-(3.6) (see, e.g., [43, 51] and references therein). We also refer to [42, Section 4] for a more detailed presentation of results on Roter spaces.

Let (M, g) , $n \geq 4$, be a non-partially-Einstein and non-conformally flat semi-Riemannian manifold. If its Riemann-Christoffel curvature R is at every point of $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ a linear combination of the Kulkarni-Nomizu products formed by the tensors $S^0 = g$ and $S^1 = S, S^2, \dots, S^{p-1}, S^p$, where p is some natural number ≥ 2 , then (M, g) is called a *generalized Roter type manifold*, or a *generalized Roter manifold*, or a *generalized Roter type space*, or a *generalized Roter space*. For instance, when $p = 2$, we have

$$R = \frac{\phi_2}{2} S^2 \wedge S^2 + \phi_1 S \wedge S^2 + \frac{\phi}{2} S \wedge S + \mu_1 g \wedge S^2 + \mu g \wedge S + \frac{\eta}{2} g \wedge g, \quad (4.12)$$

where $\phi, \phi_1, \phi_2, \mu_1, \mu$ and η are functions on $\mathcal{U}_S \cap \mathcal{U}_C$. Because (M, g) is a non-partially Einstein manifold, at least one of the functions μ_1, ϕ_1 and ϕ_2 is a non-zero function. Equation (4.12) is called a *Roter type equation* (see, e.g., [44, Section 1]). We refer to [44, 50, 51, 53, 60, 113, 115, 116, 117, 118] for results on manifolds (hypersurfaces) satisfying (4.12).

If (M, g) , $\dim M = n \geq 4$, is a semi-Riemannian manifold satisfying (3.5) on $\mathcal{U}_C \subset M$, i.e., $C \cdot C = L_C Q(g, C)$, then (2.16) yields

$$C \cdot R + R \cdot C = R \cdot R + LQ(g, C) - \frac{1}{(n-2)^2} Q(g, g \wedge S^2 - \kappa g \wedge S) \quad (4.13)$$

on \mathcal{U}_C . In addition, if (3.6) is satisfied on \mathcal{U}_C , i.e., $R \cdot R - Q(S, R) = LQ(g, C)$, then (4.13) turns into [51, Theorem 3.4 (ii)-(iii)]

$$C \cdot R + R \cdot C = Q(S, C) + (L + L_C)Q(g, C) - \frac{1}{(n-2)^2} Q\left(g, g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S\right). \quad (4.14)$$

Evidently, (4.14), by making use of (2.11) and the identity $Q(g, g \wedge g) = 0$ takes the form

$$C \cdot R + R \cdot C = Q(S, C) + (L + L_C)Q(g, C) - \frac{1}{(n-2)^2} Q(g, E).$$

We also have (cf. [51, Theorem 3.4 (iv)-(v)]): if (M, g) , $\dim M = n \geq 4$, is a quasi-Einstein semi-Riemannian manifold satisfying $\text{rank}(S - \alpha g) = 1$ (i.e., (4.1)), (3.5) and (3.6) then (4.14), by making use of (4.8) and (4.9), yields

$$C \cdot R + R \cdot C = Q(S, C) + (L + L_C)Q(g, C). \quad (4.15)$$

Example 4.3. The Gödel metric [85] is defined on $M = \mathbb{R}^4$ by

$$ds^2 = g_{ij} dx^i dx^j = a^2 \left(-(dx^1)^2 + \frac{e^{2x^1}}{2} (dx^2)^2 - (dx^3)^2 + (dx^4)^2 + 2e^{x^1} dx^2 dx^4 \right), \quad (4.16)$$

where a is a non-zero constant and $x^i \in \mathbb{R}$, $i, j \in \{1, 2, 3, 4\}$. For the metric (4.16) we have $\mathcal{U}_S = \mathcal{U}_C = M$ and

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0, \quad S = \kappa \omega \otimes \omega, \quad \kappa = \frac{1}{a^2}, \quad S^2 = \kappa S,$$

where ω is a 1-form and $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) = (0, a \exp(x^1), 0, a)$. Moreover, for the metric (4.16) we have [51, 60]:

$$R(SX, Y, Z, W) + R(SZ, Y, W, X) + R(SW, Y, X, Z) = 0, \quad R \cdot R = Q(S, R), \quad C \cdot C = \frac{\kappa}{6} Q(g, C).$$

Now (4.15) turns into

$$C \cdot R + R \cdot C = Q(S, C) + \frac{\kappa}{6} Q(g, C).$$

Theorem 4.1. (cf. [40, Proposition 3.2, Theorem 3.3, Theorem 4.4]) If (M, g) , $\dim M = n \geq 4$, is a semi-Riemannian manifold satisfying on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ conditions (3.5), (3.6) and

$$R \cdot S = Q(g, D), \quad (4.17)$$

where D is a symmetric $(0, 2)$ -tensor, then $R \cdot R = L_R Q(g, R)$ on this set, where L_R is some function on this set. Moreover, at every point of $\mathcal{U}_S \cap \mathcal{U}_C$ we have

- (i) $\text{rank}(S - \alpha_1 g) = 1$ and $\alpha_1 = \frac{1}{2} \left(\frac{\kappa}{n-1} - L + L_C \right)$, or
- (ii) $\text{rank}(S - \alpha_1 g) \geq 2$ and $\alpha_1 = \frac{1}{2} \left(\frac{\kappa}{n-1} - L + L_C \right)$ and (4.11).

Corollary 4.1. [51, Corollary 3.6] If (M, g) , $\dim M = n \geq 4$, is a semi-Riemannian manifold satisfying on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ conditions (3.5), (3.6) and (4.17) then

$$C \cdot R + R \cdot C = Q(S, C) + L_2 Q(g, C)$$

on $\mathcal{U}_S \cap \mathcal{U}_C$, where L_2 is some function on this set.

Theorem 4.2. [51, Theorem 3.1] If (M, g) , $\dim M = n \geq 4$, is a pseudosymmetric Einstein semi-Riemannian manifold satisfying $R \cdot R = L_R Q(g, R)$ (i.e. (3.2)) on $\mathcal{U}_R \subset M$, then on this set

$$\begin{aligned} R \cdot R - Q(S, R) &= \left(L_R - \frac{\kappa}{n} \right) Q(g, C), \\ C \cdot C &= \left(L_R - \frac{\kappa}{(n-1)n} \right) Q(g, C), \\ C \cdot R + R \cdot C &= Q(S, C) + \left(2L_R - \frac{\kappa}{n-1} \right) Q(g, C). \end{aligned}$$

Theorem 4.3. [51, Theorem 4.1] Let $\overline{M} \times_F \tilde{N}$ be the warped product manifold with a 1-dimensional manifold $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \pm 1$, and a 3-dimensional semi-Riemannian manifold (\tilde{N}, \tilde{g}) , and a warping function F . If (\tilde{N}, \tilde{g}) is not a space of constant curvature then (3.6) and (4.13) hold on $\mathcal{U}_C \subset \overline{M} \times_F \tilde{N}$. Moreover, if (\tilde{N}, \tilde{g}) is a quasi-Einstein manifold then (3.5) and (4.14) hold on $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \tilde{N}$.

The last theorem leads to the following results.

Theorem 4.4. [51, Theorem 4.3] Let $\overline{M} \times_F \tilde{N}$ be the warped product manifold with a 1-dimensional manifold $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \pm 1$, and an $(n-1)$ -dimensional quasi-Einstein semi-Riemannian manifold (\tilde{N}, \tilde{g}) , $n \geq 4$, and a warping function F , and let (\tilde{N}, \tilde{g}) be a conformally flat manifold, when $n \geq 5$. Then conditions (3.5), (3.6) and (4.14) are satisfied on $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \tilde{N}$.

Theorem 4.5. [51, Theorem 4.4] Let $\overline{M} \times \tilde{N}$ be the product manifold with a 1-dimensional manifold $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \pm 1$, and an $(n-1)$ -dimensional quasi-Einstein semi-Riemannian manifold (\tilde{N}, \tilde{g}) , $n \geq 4$, satisfying $\text{rank}(\tilde{S} - \rho \tilde{g}) = 1$ on $\mathcal{U}_{\tilde{S}} \subset \tilde{M}$, where ρ is some function on $\mathcal{U}_{\tilde{S}}$, and let (\tilde{N}, \tilde{g}) be a conformally flat manifold, when $n \geq 5$. Then on $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times \tilde{N}$ we have

$$\begin{aligned} (n-3)(n-2)\rho C &= g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S + (n-2)\rho \left(\frac{\kappa}{n-1} - \frac{\rho}{2} \right) g \wedge g, \\ C \cdot R + R \cdot C &= Q(S, C) + \left(\frac{\kappa}{(n-2)(n-1)} - \rho \right) Q(g, C). \end{aligned}$$

We present now the following results on Einstein and quasi-Einstein manifolds. Using (1.1) we can easily check that equation

$$g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}(S^2)}{2(n-1)} g \wedge g = 0 \tag{4.18}$$

is satisfied on any Einstein manifold (M, g) . Moreover, we also have

Theorem 4.6. [44, Lemma 2.1] If (M, g) , $n \geq 4$, is a quasi-Einstein manifold satisfying (4.1) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then (4.18) holds on this set.

We finish this section with the following results.

Theorem 4.7. [44, Lemma 2.2] If (M, g) , $n \geq 4$, is a Roter space satisfying (4.11) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then

$$C = \frac{\phi}{n-2} \left(g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}(S^2)}{2(n-1)} g \wedge g \right)$$

on this set.

Remark 4.2. (i) In view of [35, Lemma 3.2 (ii)], we can state that the following identity is satisfied on every semi-Riemannian manifold (M, g) , $n = \dim M \geq 3$, with vanishing Weyl conformal curvature tensor C

$$R \cdot R - Q(S, R) = \frac{1}{(n-2)^2} Q(g, g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S). \tag{4.19}$$

(ii) On every 3-dimensional semi-Riemannian manifold (M, g) the identity $R \cdot R = Q(S, R)$ is satisfied [35, Theorem 3.1]. Thus (4.19) reduces to

$$Q(g, g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S) = 0.$$

From this, by suitable contractions we easily get (4.18).

(iii) From (i) we easily deduce that on every semi-Riemannian conformally flat manifold (M, g) , $n = \dim M \geq 4$, conditions: $R \cdot R = Q(S, R)$ and (4.18) are equivalent.

Remark 4.3. Warped product manifolds $\bar{M} \times_F \tilde{N}$, with a 1-dimensional base manifold (\bar{M}, \bar{g}) , $\bar{g}_{11} = \pm 1$, and an $(n-1)$ -dimensional Einsteinian or non-Einsteinian fiber (\tilde{N}, \tilde{g}) , $n \geq 4$, satisfying the condition

$$R \cdot C - C \cdot R = LQ(S, R),$$

were studied in [2].

5. Warped product manifolds with 2-dimensional base manifold

Let $\bar{M} \times_F \tilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold (\bar{M}, \bar{g}) and an $(n-2)$ -dimensional semi-Riemannian manifold (\tilde{N}, \tilde{g}) , $n \geq 4$, and a warping function F , and let (\tilde{N}, \tilde{g}) be a space of constant curvature when $n \geq 5$.

Let S_{hk} and C_{hijk} be the local components of the Ricci tensor S and the Weyl conformal curvature tensor C of $\bar{M} \times_F \tilde{N}$, respectively. We have

$$S_{ad} = \frac{\bar{\kappa}}{2} g_{ab} - \frac{n-2}{2F} T_{ab}, \quad S_{\alpha\beta} = \tau_1 g_{\alpha\beta}, \quad S_{a\alpha} = 0, \tag{5.1}$$

$$\tau_1 = \frac{\tilde{\kappa}}{(n-2)F} - \frac{\text{tr}(T)}{2F} - (n-3) \frac{\Delta_1 F}{4F^2}, \quad \Delta_1 F = \Delta_{1\bar{g}} F = \bar{g}^{ab} F_a F_b, \tag{5.2}$$

$$T_{ab} = \bar{\nabla}_a F_b - \frac{1}{2F} F_a F_b, \quad \text{tr}(T) = \bar{g}^{ab} T_{ab},$$

where T is the $(0, 2)$ -tensor with the local components T_{ab} . We also have [51, eqs. (5.10)-(5.14)]

$$C_{abcd} = \frac{(n-3)\rho_1}{n-1} (g_{ad}g_{bc} - g_{ac}g_{bd}), \tag{5.3}$$

$$C_{\alpha bc\beta} = -\frac{(n-3)\rho_1}{(n-2)(n-1)} g_{bc}g_{\alpha\beta}, \tag{5.4}$$

$$C_{\alpha\beta\gamma\delta} = \frac{2\rho_1}{(n-2)(n-1)} (g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}), \tag{5.5}$$

$$C_{abc\delta} = C_{ab\alpha\beta} = C_{a\alpha\beta\gamma} = 0, \tag{5.6}$$

where

$$\rho_1 = \frac{\bar{\kappa}}{2} + \frac{\tilde{\kappa}}{(n-3)(n-2)F} + \frac{1}{2F} \left(\Delta F - \frac{\Delta_1 F}{F} \right), \quad \Delta F = \bar{g}^{ab} \bar{\nabla}_a F_b.$$

If we set [51, eqs. (5.13)]

$$\rho = \frac{2(n-3)}{n-1} \rho_1 = \frac{2(n-3)}{n-1} \left(\frac{\bar{\kappa}}{2} + \frac{\tilde{\kappa}}{(n-3)(n-2)F} + \frac{1}{2F} \left(\Delta F - \frac{\Delta_1 F}{F} \right) \right) \tag{5.7}$$

then (5.3)-(5.6) turn into [51, eqs. (5.14)]

$$C_{abcd} = \frac{\rho}{2}(g_{ad}g_{bc} - g_{ac}g_{bd}), \quad C_{\alpha bc\beta} = -\frac{\rho}{2(n-2)}g_{bc}g_{\alpha\beta},$$

$$C_{\alpha\beta\gamma\delta} = \frac{\rho}{(n-3)(n-2)}(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}), \quad C_{abcd} = C_{ab\alpha\beta} = C_{\alpha\beta\gamma\delta} = 0.$$

Further, by making use of the formulas for the local components $(C \cdot C)_{hijklm}$ and $Q(g, C)_{hijklm}$ of the tensors $C \cdot C$ and $Q(g, C)$, i.e.

$$(C \cdot C)_{hijklm} = g^{rs}(C_{rijk}C_{shlm} + C_{hrjk}C_{sil m} + C_{hir k}C_{sjlm} + C_{hijr}C_{sklm}),$$

$$Q(g, C)_{hijklm} = g_{hl}C_{mijk} + g_{il}C_{hmjk} + g_{jl}C_{himk} + g_{kl}C_{hijm} - g_{hm}C_{lijk} - g_{im}C_{hljk} - g_{jm}C_{hil k} - g_{km}C_{hijl},$$

we obtain [51, eqs. (7.7)-(7.8)]

$$(C \cdot C)_{\alpha abcd\beta} = -\frac{(n-1)\rho^2}{4(n-2)^2}g_{\alpha\beta}(g_{ab}g_{cd} - g_{ac}g_{bd}),$$

$$(C \cdot C)_{\alpha\alpha\beta\gamma d\delta} = \frac{(n-1)\rho^2}{4(n-2)^2(n-3)}g_{ad}(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}),$$

$$Q(g, C)_{\alpha abcd\beta} = \frac{(n-1)\rho}{2(n-2)}g_{\alpha\beta}(g_{ab}g_{cd} - g_{ac}g_{bd}),$$

$$Q(g, C)_{\alpha\alpha\beta\gamma d\delta} = -\frac{(n-1)\rho}{2(n-2)(n-3)}g_{ad}(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}).$$

Theorem 5.1. [51, Theorem 7.1 (i)] Let $\bar{M} \times_F \tilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold (\bar{M}, \bar{g}) and an $(n-2)$ -dimensional semi-Riemannian manifold (\tilde{N}, \tilde{g}) , $n \geq 4$, and a warping function F , and let (\tilde{N}, \tilde{g}) be a space of constant curvature, when $n \geq 5$.

(i) The following equation is satisfied on $\mathcal{U}_C \subset \bar{M} \times_F \tilde{N}$:

$$C \cdot C = L_C Q(g, C),$$

$$L_C = -\frac{\rho}{2(n-2)} = -\frac{n-3}{(n-2)(n-1)} \left(\frac{\bar{\kappa}}{2} + \frac{\tilde{\kappa}}{(n-3)(n-2)F} + \frac{1}{2F} \left(\Delta F - \frac{\Delta_1 F}{F} \right) \right), \tag{5.8}$$

where the function ρ is defined by (5.7).

(ii) Equation (3.6) is satisfied on $\mathcal{U}_C \subset \bar{M} \times_F \tilde{N}$, where the functions τ_1 and L are defined by (5.2) and

$$L = -\frac{n-2}{(n-1)\rho} \left(\bar{\kappa} \left(\tau_1 + \frac{\text{tr}(T)}{2F} \right) + \frac{n-3}{4F^2} (\text{tr}(T^2) - (\text{tr}(T))^2) \right), \tag{5.9}$$

T and T^2 are the $(0, 2)$ -tensors with the local components

$$T_{ab} = \bar{\nabla}_a F_b - \frac{1}{2F} F_a F_b, \quad T_{ad}^2 = T_{ac} \bar{g}^{cd} T_{db}, \quad \text{tr}(T) = \bar{g}^{ab} T_{ab}, \quad \text{tr}(T^2) = \bar{g}^{ab} T_{ab}^2.$$

(iii) The following equation is satisfied on $\mathcal{U}_C \subset \bar{M} \times_F \tilde{N}$

$$C \cdot R + R \cdot C = Q(S, C) + (L_C + L)Q(g, C) - \frac{1}{(n-2)^2} Q \left(g, g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S \right),$$

where L_C and L are functions defined by (5.8) and (5.9).

We have (see, eq. (5.1))

$$S_{ad} = \frac{\bar{\kappa}}{2} g_{ab} - \frac{n-2}{2F} T_{ab}, \quad S_{\alpha\beta} = \tau_1 g_{\alpha\beta}, \quad S_{a\alpha} = 0,$$

where τ_1 is defined by (5.2).

We define now on $\mathcal{U}_S \subset \bar{M} \times_F \tilde{N}$ the $(0, 2)$ -tensor A by $A = S - \tau_1 g$. We can check that $\text{rank}(A) = 2$ at a point of \mathcal{U}_S if and only if $\text{tr}(A^2) - (\text{tr}(A))^2 \neq 0$ at this point [51, Section 6]. At all points of \mathcal{U}_S , at which $\text{rank}(A) = 2$, we set

$$\tau_2 = (\text{tr}(A^2) - (\text{tr}(A))^2)^{-1}. \tag{5.10}$$

Let V be the set of all points of $\mathcal{U}_S \cap \mathcal{U}_C$ at which: $\text{rank}(A) = 2$ and S_{ad} is not proportional to g_{ad} .

Theorem 5.2. [51, Theorem 7.1 (ii)] Let $\overline{M} \times_F \tilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ and an $(n - 2)$ -dimensional semi-Riemannian manifold (\tilde{N}, \tilde{g}) , $n \geq 4$, and a warping function F , and let (\tilde{N}, \tilde{g}) be a space of constant curvature when $n \geq 5$. Then on the set $V \subset \mathcal{U}_S \cap \mathcal{U}_C$ (defined above) we have:

$$C = -\frac{(n-1)\rho\tau_2}{(n-3)(n-2)} \left(g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}(S^2)}{2(n-1)} g \wedge g \right),$$

$$R \cdot C + C \cdot R = Q(S, C) + \left(L - \frac{\rho}{2(n-2)} + \frac{n-3}{(n-2)(n-1)\rho\tau_2} \right) Q(g, C),$$

where ρ , τ_1 and τ_2 are defined by (5.7), (5.2) and (5.10), respectively.

Theorem 5.3. [51, Theorem 6.2] Let $\overline{M} \times_F \tilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ and an $(n - 2)$ -dimensional semi-Riemannian manifold (\tilde{N}, \tilde{g}) , $n \geq 4$, and a warping function F , and let (\tilde{N}, \tilde{g}) be an Einstein space, when $n \geq 5$. On the set $V \subset \mathcal{U}_S \cap \mathcal{U}_C$ we have

$$R \cdot S = (\phi_1 - 2\tau_1\phi_2 + \tau_1^2\phi_3)Q(g, S) + (\phi_2 - \tau_1\phi_3)Q(g, S^2) + \phi_3Q(S, S^2),$$

$$\phi_1 = \frac{2\tau_1 - \bar{\kappa}}{2(n-2)}, \quad \phi_2 = \frac{1}{n-2}, \quad \phi_3 = \frac{\tau_2(2\kappa - \bar{\kappa} - 2(n-1)\tau_1)}{n-2},$$

where τ_1 is defined by (5.2).

Theorems 4.5 and 4.6 imply

Theorem 5.4. [44, Proposition 2.3] If (M, g) , $n \geq 4$, is a semi-Riemannian manifold satisfying (4.1) or (4.11) at every point of $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then the following equation is satisfied on this set

$$\tau C = g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}(S^2)}{2(n-1)} g \wedge g, \tag{5.11}$$

where τ is some function on $\mathcal{U}_S \cap \mathcal{U}_C$.

Theorem 5.4, [51, Theorem 7.1 (ii)] and [65, Theorem 4.1] imply

Theorem 5.5. [44, Theorem 2.4] Let $\overline{M} \times_F \tilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$, an $(n - 2)$ -dimensional semi-Riemannian manifold (\tilde{N}, \tilde{g}) , $n \geq 4$, a warping function F , and let (\tilde{N}, \tilde{g}) be a space of constant curvature when $n \geq 5$. Then (5.11) holds on $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \tilde{N}$.

Remark 5.1. Let $\overline{M} \times_F \tilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ and an $(n - 2)$ -dimensional semi-Riemannian manifold (\tilde{N}, \tilde{g}) , $n \geq 4$, and a warping function F , and let (\tilde{N}, \tilde{g}) be a space of constant curvature when $n \geq 5$.

(i) From [65, Theorem 4.1] it follows that at all points of the set $\mathcal{U}_S \cap \mathcal{U}_C$, at which S_{ad} is proportional to g_{ad} and $\text{rank}(A) = 2$, the Riemann-Christoffel curvature tensor R is a linear combination of the Kulkarni-Nomizu products $S \wedge S$, $g \wedge S$ and $g \wedge g$, i.e., (4.11) is satisfied. Thus, in view of [43, Theorem 6.7] (see also [51, Theorem 3.2]), we have $R \cdot R = L_R Q(g, R)$ and in a consequence $R \cdot S = L_R Q(g, S)$, for some function L_R .

(ii) On the set $V \subset \mathcal{U}_S \cap \mathcal{U}_C$ we also have [51, Section 7]

$$R \cdot C = Q(S, C) + \left(L + \frac{n-3}{(n-2)(n-1)\rho\tau_2} \right) Q(g, C) + \frac{(n-1)\rho\tau_2}{(n-2)^2} g \wedge Q(S, S^2)$$

$$+ \frac{1}{2(n-2)^2} Q \left(\left(\frac{\rho}{2} + (n-1)\rho\tau_1^2\tau_2 \right) S - (n-1)\rho\tau_1\tau_2 S^2, g \wedge g \right),$$

$$C \cdot R = -\frac{\rho}{2(n-2)} Q(g, C) - \frac{(n-1)\rho\tau_2}{(n-2)^2} g \wedge Q(S, S^2)$$

$$- \frac{1}{2(n-2)^2} Q \left(\left(\frac{\rho}{2} + (n-1)\rho\tau_1^2\tau_2 \right) S - (n-1)\rho\tau_1\tau_2 S^2, g \wedge g \right),$$

and in a consequence

$$R \cdot C - C \cdot R = Q(S, C) + \left(L + \frac{n-3}{(n-2)(n-1)\rho\tau_2} + \frac{\rho}{2(n-2)} \right) Q(g, C) + \frac{2(n-1)\rho\tau_2}{(n-2)^2} g \wedge Q(S, S^2)$$

$$+ \frac{1}{(n-2)^2} Q \left(\left(\frac{\rho}{2} + (n-1)\rho\tau_1^2\tau_2 \right) S - (n-1)\rho\tau_1\tau_2 S^2, g \wedge g \right),$$

where ρ , τ_1 and τ_2 are defined by (5.7), (5.2) and (5.10), respectively.

(iii) A short presentation on quasi-Einstein and 2-quasi-Einstein warped product manifolds satisfying conditions of the form (*) is given in [46] (see also [39]).

6. Semi-Riemannian manifolds satisfying (1.4)

We can check that on any Einstein manifold (M, g) , $\dim M = n \geq 4$, the tensors $Q(g, R)$, $Q(S, R)$, $Q(g, C)$ and $Q(S, C)$ satisfy

$$\frac{\kappa}{n}Q(g, R) = Q(S, R) = Q(S, C) = \frac{\kappa}{n}Q(g, C). \tag{6.1}$$

Further, in [63, Theorem 3.1] it was stated that on every Einstein manifold (M, g) , $\dim M = n \geq 4$, the following identity is satisfied

$$R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n}Q(g, R). \tag{6.2}$$

The remarks above lead to the problem of investigation of curvature properties of non-Einstein and non-conformally flat semi-Riemannian manifolds (M, g) , $\dim M = n \geq 4$, satisfying at every point of M the curvature condition of the following form: the derivation-commutator $R \cdot C - C \cdot R$ is proportional to $Q(g, R)$, $Q(S, R)$, $Q(g, C)$ and $Q(S, C)$. Such conditions are strongly related to some pseudosymmetry type curvature conditions, see, e.g., [43] and references therein.

Semi-Riemannian manifolds (M, g) , $\dim M = n \geq 4$, satisfying at every point of M the following condition

$$\text{the tensors } R \cdot C - C \cdot R \text{ and } Q(S, C) \text{ are linearly dependent,} \tag{6.3}$$

were investigated in [48]. It is obvious that (6.3) is satisfied at every point of M at which C vanishes. It is also clear that (6.1) and (6.2) imply that

$$R \cdot C - C \cdot R = \frac{1}{n-1}Q(S, C)$$

holds on any Einstein manifold (M, g) , $\dim M = n \geq 4$. Therefore we will restrict our considerations to manifolds (M, g) , $\dim M = n \geq 4$, satisfying (6.3) on the set $\mathcal{U}_S \cap \mathcal{U}_C \subset M$. Thus on $\mathcal{U}_S \cap \mathcal{U}_C$ we have (1.4), i.e., $R \cdot C - C \cdot R = LQ(S, C)$, where L is some function on this set. We mention that if the tensor $R \cdot C - C \cdot R$ vanishes on $\mathcal{U}_S \cap \mathcal{U}_C$ then on this set [63, Theorem 4.1]:

$$\begin{aligned} R \cdot C &= C \cdot R = 0, \\ Q(S, C) &= 0. \end{aligned} \tag{6.4}$$

On the other hand, if $Q(S, C)$ vanishes on $\mathcal{U}_S \cap \mathcal{U}_C$ then at every point $x \in \mathcal{U}_S \cap \mathcal{U}_C$ we have:

- (i) if $\text{rank } S = 1$ at x then $\kappa = 0$ and (6.4) hold at x [48, Section 3], or
- (ii) if $\text{rank } S > 1$ at x then $C \cdot R = 0$ and $R \cdot C = \frac{\kappa}{n-1}Q(g, C)$ hold at x [48, Section 4].

Thus we see that in the case (ii), if a manifold satisfies (1.4) then its scalar curvature must vanish on $\mathcal{U}_S \cap \mathcal{U}_C$.

The main result of [48, Section 3] states that pseudosymmetric manifolds satisfying some additional curvature conditions are quasi-Einstein manifolds satisfying the conditions: $C \cdot C = 0$, $C \cdot R = 0$, and (1.4) with $L = \frac{1}{n-1}$. Precisely we have

Theorem 6.1. [48, Theorem 3.4] *Let (M, g) , $\dim M = n \geq 4$, be a semi-Riemannian manifold. If the following conditions:*

$$\begin{aligned} R \cdot R &= \frac{\kappa}{(n-1)n}Q(g, R), \\ R \cdot R - Q(S, R) &= -\frac{(n-2)\kappa}{(n-1)n}Q(g, C), \\ R \cdot C &= \frac{1}{n-1}Q(S, C) \end{aligned}$$

are satisfied on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$, then on this set we have:

$$\begin{aligned} C \cdot C &= 0, \\ \text{rank}(S - \frac{\kappa}{n}g) &= 1, \\ C \cdot R &= 0, \\ (n-1)(R \cdot C - C \cdot R) &= Q(S, C). \end{aligned}$$

Theorem 6.2. [48, Proposition 3.9] Let T be a generalized curvature tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$.

(i) If the conditions

$$\begin{aligned} Q(\text{Ric}(T), T) &= 0, \\ \text{rankRic}(T) &= 1 \end{aligned} \tag{6.5}$$

are satisfied on $\mathcal{U}_{\text{Ric}(T)} \cap \mathcal{U}_{\text{Weyl}(T)} \subset M$, then on this set we have

$$\begin{aligned} \kappa(T) &= 0, \\ T \cdot \text{Weyl}(T) = \text{Weyl}(T) \cdot T &= Q(\text{Ric}(T), \text{Weyl}(T)) = 0. \end{aligned} \tag{6.6}$$

(ii) If the conditions (6.5) and

$$Q(\text{Ric}(T), \text{Weyl}(T)) = 0$$

are satisfied on $\mathcal{U}_{\text{Ric}(T)} \cap \mathcal{U}_{\text{Weyl}(T)} \subset M$, then on this set we have (6.6) and

$$T \cdot \text{Weyl}(T) = \text{Weyl}(T) \cdot T = 0.$$

In [48, Section 3] an example of warped product manifolds satisfying assumptions of [48, Theorem 3.4] is also given.

Let (M, g) , $\dim M = n \geq 4$, be a semi-Riemannian manifold with parallel Weyl conformal curvature tensor, i.e. $\nabla C = 0$ on M . It is obvious that the last condition implies $R \cdot C = 0$. Moreover, let the manifold (M, g) be neither conformally flat nor locally symmetric. Such manifolds are called *essentially conformally symmetric manifolds*, e.c.s. manifolds/metrics, or ECS manifolds/metrics, in short (see, e.g., [23, 24, 26, 30, 31]). E.c.s. manifolds are semisymmetric manifolds ($R \cdot R = 0$ [23, Theorem 9]) satisfying $\kappa = 0$ and $Q(S, C) = 0$ ([23, Theorems 7 and 8]). In addition,

$$FC = \frac{1}{2}S \wedge S \tag{6.7}$$

holds on M , where F is some function on M , called the *fundamental function* [24]. At every point of M we also have $\text{rank}S \leq 2$ [24, Theorem 5]. We mention that the local structure of e.c.s. manifolds is already determined. We refer to [25, 28] for results related to this subject. We also mention that certain e.c.s. metrics are realized on compact manifolds [27, 29, 30, 31, 32, 33, 34].

Equation (6.7), by suitable contraction, leads immediately to $S^2 = \kappa S$, which by $\kappa = 0$, reduces to $S^2 = 0$. Evidently, $\text{tr}(S^2) = 0$. Now using (6.7) we get

$$FC = \frac{n-2}{2(n-2)}S \wedge S = \frac{1}{n-2} \left(g \wedge S^2 + \frac{n-2}{2}S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}(S^2)}{2(n-1)}g \wedge g \right).$$

Thus we have

Theorem 6.3. [42, Theorem 6.1] Condition (5.11), with $\tau = (n-2)F$, is satisfied on every essentially conformally symmetric manifold (M, g) .

We assume that $F = 0$ at $x \in M$. Now (6.7) implies $\text{rank}S \leq 1$ at x . It is clear that if S vanishes then

$$R \cdot C = C \cdot R = Q(S, C) = 0 \tag{6.8}$$

holds at x . If $\text{rank}S = 1$ then in view of [48, Proposition 3.9(ii)] we also have (6.8) at x . Next, we assume that F is non-zero at $x \in M$. Thus $\text{rank}S = 2$ at x . Now (6.7) turns into (2.13) with $T = R$, $\text{Ric}(T) = S$, $\phi = F^{-1}$, $\mu = \frac{1}{n-2}$ and $\eta = 0$. Therefore (2.14) and (2.15) reduce to $C \cdot R = 0$ and $C \cdot C = 0$, respectively. Consequently, (6.8) holds at x . Thus we have

Theorem 6.4. [48, Theorem 4.1] Condition (6.8) is satisfied on every essentially conformally symmetric manifold (M, g) .

Thus we see that e.c.s. manifolds satisfy (1.4). We also mention that the tensor $C \cdot C$ of every e.c.s. manifold is the zero tensor ([48, Remark 4.2(ii)]).

E.c.s. warped product manifolds were investigated in [91]. In that paper examples of such manifolds are given [48, Remark 4.2(i)].

In [48, Section 5] Roter type manifolds satisfying (1.4) are investigated. In [48, Theorem 5.2] it was stated that if (M, g) , $\dim M = n \geq 4$, is a Roter type manifold with vanishing scalar curvature κ on $\mathcal{U} \subset M$ then (1.4), with $L = -1$, holds on this set.

Theorem 6.5. [48, Theorem 5.2] Let (M, g) , $\dim M = n \geq 4$, be a semi-Riemannian manifold satisfying (4.11) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$. If $\kappa = 0$ on $\mathcal{U}_S \cap \mathcal{U}_C$ then (1.4), with $L = -1$, holds on this set.

This result is also an immediate consequence of the fact that every Roter space (M, g) satisfies (1.2) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ (see Section 4). However in the proof of the last theorem formula (1.2) was not applied.

We also have

Theorem 6.6. [48, Theorem 5.3] Let (M, g) , $\dim M = n \geq 4$, be a semi-Riemannian manifold satisfying (4.11) and (1.4) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$, and let $\mathcal{U}_1 \subset \mathcal{U}_S \cap \mathcal{U}_C$ be the set of all points at which the functions L and L_C , defined by (1.4), (2.14) and (2.15) (for $T = R$), respectively, are nowhere zero on this set. Then we have on \mathcal{U}_1 : $L = -1$ and $\kappa = 0$.

In [48, Example 5.4] it was shown that under some conditions the Cartesian product of two semi-Riemannian spaces of constant curvature satisfies assumptions of Theorem 6.6 (i.e., [48, Theorem 5.3]).

7. 2-quasi-umbilical hypersurfaces

Let M , $\dim M \geq 4$, be a hypersurface in an $(n + 1)$ -dimensional Riemannian manifold N . According to [13, 124] (see also [53]), the hypersurface M is said to be *quasi-umbilical*, resp., *2-quasi-umbilical*, at a point $x \in M$ if it has a principal curvature with multiplicity $n - 1$, resp., $n - 2$, i.e., when the principal curvatures of M at x are the following $\lambda_1, \lambda_2 = \lambda_3 = \dots = \lambda_n$, resp., $\lambda_1, \lambda_2, \lambda_3 = \lambda_4 = \dots = \lambda_n$.

Let M , $\dim M \geq 4$, be a hypersurface in an $(n + 1)$ -dimensional semi-Riemannian manifold N . The hypersurface M is said to be *quasi-umbilical* (see, e.g., [71, 84]) resp., *2-quasi-umbilical* (see, e.g., [74, 84]) at a point $x \in M$ if $\text{rank}(H - \alpha g) = 1$, resp., $\text{rank}(H - \alpha g) = 2$ holds at x , for some $\alpha \in \mathbb{R}$.

We recall that a hypersurface M , $\dim M = n \geq 4$, in a semi-Riemannian conformally flat manifold N is quasi-umbilical at a point $x \in M$ if and only if its Weyl conformal curvature tensor C vanishes at this point [71, Theorem 4.1]. In particular, M is non-quasi-umbilical at a point $x \in M$ if and only if its Weyl conformal curvature tensor C is non-zero at x , i.e., $x \in \mathcal{U}_C \subset M$.

Let $N_s^{n+1}(c)$, $n \geq 3$, be a semi-Riemannian space of constant curvature $c = \frac{\tilde{\kappa}}{n(n+1)}$ with signature $(s, n + 1 - s)$, where $\tilde{\kappa}$ is its scalar curvature. Let M be a connected hypersurface isometrically immersed in $N_s^{n+1}(c)$. We have (see, e.g., [47])

$$R_{hijk} = \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}) + \frac{\tilde{\kappa}}{n(n+1)}(g_{hk}g_{ij} - g_{hj}g_{ik}), \quad \varepsilon = \pm 1, \tag{7.1}$$

where g_{hk} , R_{hijk} and H_{hk} , respectively, are the local components of the metric tensor g , the Riemann-Christoffel curvature tensor R and the second fundamental tensor H of M , respectively. We mention that ε (given in the last formula) is the definiteness sign of the normal lines to M . Contracting (7.1) with g^{ij} and g^{kh} , respectively, we obtain

$$S_{hk} = \varepsilon(\text{tr}(H)H_{hk} - H_{hk}^2) + \frac{(n-1)\tilde{\kappa}}{n(n+1)}g_{hk},$$

$$\frac{\kappa}{n-1} = \frac{\varepsilon}{n-1}((\text{tr}(H))^2 - \text{tr}(H^2)) + \frac{\tilde{\kappa}}{n+1},$$

respectively, where $\text{tr}(H) = g^{hk}H_{hk}$, $\text{tr}(H^2) = g^{hk}H_{hk}^2$ and S_{hk} are the local components of the Ricci tensor S of M and κ is the scalar curvature of M . It is known that (3.6) holds on M . Precisely,

$$R \cdot R - Q(S, R) = -\frac{(n-2)\tilde{\kappa}}{n(n+1)}Q(g, C) \tag{7.2}$$

on M [71, Lemma 2.1] (see also [19, Lemma 2.1 (ii)]). Evidently, if $n = 3$ then (7.2) turns into

$$R \cdot R = Q(S, R). \tag{7.3}$$

If the ambient space is a semi-Euclidean space \mathbb{E}_s^{n+1} , $n \geq 3$, then (7.2) also reduces to (7.3). As it was proved in [20, Lemma 4.1], we also have the following identity on M in $N_s^{n+1}(c)$, $n \geq 3$,

$$R \cdot R - \frac{\tilde{\kappa}}{n(n+1)}Q(g, R) = -\frac{1}{2}Q(H^2, H \wedge H). \tag{7.4}$$

Further we have

Theorem 7.1. [51, Theorem 3.7] *Let M be a hypersurface isometrically immersed in $N_s^{n+1}(c)$, $n \geq 4$. Then*

$$\begin{aligned} C \cdot R + R \cdot C &= Q(S, C) - \frac{(n-2)\tilde{\kappa}}{n(n+1)}Q(g, C) + C \cdot C \\ &\quad - \frac{1}{(n-2)^2}Q\left(g, g \wedge S^2 + \frac{n-2}{2}S \wedge S - \kappa g \wedge S\right) \end{aligned}$$

holds on M . Moreover, if the condition $C \cdot C = L_C Q(g, C)$ (i.e., (3.5)) is satisfied on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then on this set we have

$$\begin{aligned} C \cdot R + R \cdot C &= Q(S, C) + \left(L_C - \frac{(n-2)\tilde{\kappa}}{n(n+1)}\right)Q(g, C) \\ &\quad - \frac{1}{(n-2)^2}Q\left(g, g \wedge S^2 + \frac{n-2}{2}S \wedge S - \kappa g \wedge S\right), \end{aligned} \tag{7.5}$$

and in addition, if M is a quasi-Einstein hypersurface satisfying the condition $\text{rank}(S - \alpha g) = 1$ (i.e., (4.1)), on $\mathcal{U}_S \cap \mathcal{U}_C$ then on this set we have

$$C \cdot R + R \cdot C = Q(S, C) + \left(L_C - \frac{(n-2)\tilde{\kappa}}{n(n+1)}\right)Q(g, C).$$

It is known that every 2-quasi-umbilical hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfies (3.5) (see, e.g., [51, Theorem 3.8]). We have

Theorem 7.2. [51, Theorem 3.8] *If M is a 2-quasi-umbilical hypersurface isometrically immersed in $N_s^{n+1}(c)$, $n \geq 4$, then (7.5) holds on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$.*

Theorem 7.3. [53, Proposition 4.2] *Let M be a hypersurface in an Euclidean space \mathbb{E}^{n+1} , $n \geq 4$, having exactly three distinct principal curvatures λ_1, λ_2 and λ_3 satisfying at every point of M : $\lambda_1 = 0, \lambda_2 = -(n-2)\lambda$ and $\lambda_3 = \lambda_4 = \dots = \lambda_n = \lambda \neq 0$. Then M is a minimal, 2-quasi-umbilical and 2-quasi-Einstein hypersurface satisfying: (1.7), (7.3) and*

$$\text{rank}\left(S - \frac{\kappa}{(n-2)(n-1)}g\right) = 2,$$

$$\begin{aligned} S &= -H^2, \\ \kappa &= -\text{tr}(H^2) = -(n-2)(n-1)\lambda^2, \\ S^2 &= -(\phi^2 + \psi)S + \phi\psi H, \\ S^3 &= -(\phi^2 + 2\psi)S^2 - \psi^2 S, \end{aligned}$$

$$\begin{aligned} R &= \frac{1}{2(\phi\psi)^2} (S^2 + (\phi^2 + \psi)S) \wedge (S^2 + (\phi^2 + \psi)S), \\ R \cdot S &= \phi Q(H, S) = \frac{n-1}{\kappa} Q(S, S^2), \\ C \cdot S &= \phi Q(H, S) + \frac{\phi^2}{n-2} Q(g, S) - \frac{\phi\psi}{n-2} Q(g, H) \\ &= \frac{\kappa}{(n-2)(n-1)} Q(g, S) - \frac{1}{n-2} Q(g, S^2) + \frac{n-1}{\kappa} Q(S, S^2), \end{aligned} \tag{7.6}$$

$$\begin{aligned} (n-2)R \cdot C &= (n-2)Q(S, R) - \phi g \wedge Q(H, S), \\ (n-2)C \cdot R &= (n-3)Q(S, R) - \phi H \wedge Q(g, S), \\ C \cdot C &= 0, \end{aligned}$$

where $\phi = -(n-3)\lambda$ and $\psi = (n-2)\lambda^2$. Moreover, we have

$$(n-2)(R \cdot C - C \cdot R) = Q(S, R) + \phi(H \wedge Q(g, S) - g \wedge Q(H, S)). \tag{7.7}$$

We note that (7.6) is a particular form of the Roter type equation (4.12).

Biharmonic hypersurfaces with three distinct principal curvatures in an Euclidean 5-space \mathbb{E}^5 were investigated in [79]. The main result of [79, Theorem 3.2] states that every biharmonic hypersurface M with three distinct principal curvatures in \mathbb{E}^5 is minimal. The principal curvatures of M are the following: $\lambda_1 = 0$, $\lambda_2 = -2\lambda$ and $\lambda_3 = \lambda_4 = \lambda \neq 0$, where λ is some function on M . Curvature properties of such hypersurfaces are expressed in the last theorem, provided that $n = 4$ (see also [53, Theorem 4.3]).

We refer to [110] for a survey of results on biharmonic hypersurfaces.

Let now M be a type number two hypersurface in $N_s^{n+1}(c)$, $n \geq 3$, i.e., let $\text{rank}(H) = 2$ at every point of M (see, e.g., [14]). Using (2.7), (2.8), (2.9) and (7.4) we obtain on M (cf. [20, Theorem 4.2])

$$H^3 = \text{tr}(H)H^2 + \frac{\text{tr}(H^2) - (\text{tr}(H))^2}{2}H, \tag{7.8}$$

$$Q(H, H \wedge H^2) = 0, \tag{7.9}$$

$$R \cdot R = \frac{\tilde{\kappa}}{n(n+1)}Q(g, R). \tag{7.10}$$

Thus we have

Theorem 7.4. (cf. [20, Theorem 4.2]) *Every type number two hypersurface M in $N_s^{n+1}(c)$, $n \geq 3$, is a pseudosymmetric manifold of constant type satisfying (7.8), (7.9) and (7.10).*

We mention that type number two hypersurfaces in 4-dimensional Riemannian space of constant curvature were investigated in [88]. We also note that in [75, Section 5] it was stated that the Cartan hypersurface in a 4-dimensional sphere is a pseudosymmetric manifold of constant type. Evidently, (7.8) is a special form of the equation

$$H^3 = \text{tr}(H)H^2 + \psi H, \tag{7.11}$$

where ψ is a function on M . We refer to Remark 8.7 (ii) of this paper for results on hypersurfaces in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (7.11).

As an immediate consequence of Proposition 2.1 and [42, Remark 7.1 (iii)] we obtain the following

Theorem 7.5. *If M , $\dim M = n \geq 4$, is a type number two hypersurface isometrically immersed in a semi-Riemannian conformally flat manifold N , $\dim N = n + 1$, then $C \cdot C = L_C Q(g, C)$ (i.e., (3.5)) holds on $\mathcal{U}_C \subset M$, where L_C is some function on this set.*

Let M be a type number two hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. Thus, in view of the last theorem, (3.5) holds on $\mathcal{U}_C \subset M$. Moreover, in view of Theorem 7.1, (7.5) is satisfied on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$. In addition, we note that (7.10) implies

$$R \cdot C = \frac{\tilde{\kappa}}{n(n+1)}Q(g, C). \tag{7.12}$$

Now using Theorem 7.1, Theorem 7.5 and (7.12) we get easily the following result.

Theorem 7.6. *Let M be a type number two hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. Then the following condition of the form (*) is satisfied on $\mathcal{U}_S \cap \mathcal{U}_C$*

$$C \cdot R - R \cdot C = Q(S, C) + L_1 Q(g, C) - \frac{1}{(n-2)^2}Q\left(g, g \wedge S^2 + \frac{n-2}{2}S \wedge S - \kappa g \wedge S\right),$$

where L_1 is some function on this set. Moreover, if M is a quasi-Einstein hypersurface satisfying the condition $\text{rank}(S - \alpha g) = 1$ (i.e., (4.1)) on $\mathcal{U}_S \cap \mathcal{U}_C$ then on this set we have

$$C \cdot R - R \cdot C = Q(S, C) + L_1 Q(g, C).$$

We refer to [58, Section 5] for further results on type number two hypersurfaces in $N_s^{n+1}(c)$, $n \geq 4$, satisfying curvatures conditions of pseudosymmetry type.

8. Hypersurfaces satisfying (1.6)

Let M be a hypersurface isometrically immersed in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$ a curvature condition of the kind: the tensor $R \cdot C$, $C \cdot R$ or $R \cdot C - R \cdot C$ is a linear combination of the tensor $R \cdot R$ and of a finite sum of the Tachibana tensors of the form $Q(A, T)$, where A is a symmetric $(0, 2)$ -tensor and T a generalized curvature tensor. As it was mentioned in Introduction, if such condition is satisfied on \mathcal{U}_H then (1.6) holds on this set. We present now the following results on hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.6) on $\mathcal{U}_H \subset M$.

Theorem 8.1. [112, Proposition 5.1, eq. (29)] *If M is a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.6) on $\mathcal{U}_H \subset M$, for some functions ψ and ρ on \mathcal{U}_H , then on \mathcal{U}_H we have*

$$\begin{aligned} R \cdot C &= Q(S, R) - \frac{(n-2)\tilde{\kappa}}{n(n+1)}Q(g, R) + \frac{\alpha_2}{2}Q(S, g \wedge g) + \frac{\rho}{2(n-2)}Q(H, g \wedge g), \\ C \cdot R &= \frac{n-3}{n-2}Q(S, R) + \alpha_1 Q(g, R) + \frac{\alpha_2}{2}Q(S, g \wedge g), \end{aligned}$$

$$(n-2)(R \cdot C - C \cdot R) = Q(S, R) + \frac{\rho}{2}Q(H, g \wedge g) + \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \frac{\kappa}{n-1} - \varepsilon\psi \right) Q(g, R), \tag{8.1}$$

$$(n-2)C \cdot C = (n-3)Q(S, R) + (n-2)\alpha_1 Q(g, R) + \frac{\alpha_1 - \alpha_2}{2}Q(S, g \wedge g) + \frac{n-3}{2(n-2)}\rho Q(H, g \wedge g),$$

$$\begin{aligned} R \cdot S &= \frac{\tilde{\kappa}}{n(n+1)}Q(g, S) + \rho Q(g, H), \\ \alpha_1 &= \frac{1}{n-2} \left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2 - 3n + 3)\tilde{\kappa}}{n(n+1)} \right), \end{aligned} \tag{8.2}$$

$$\alpha_2 = -\frac{(n-3)\tilde{\kappa}}{(n-2)n(n+1)}. \tag{8.3}$$

Theorem 8.2. [55, Proposition 4.2] *Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.6) on $\mathcal{U}_H \subset M$, for some functions ψ and ρ on \mathcal{U}_H .*

(i) *The following conditions are satisfied on \mathcal{U}_H :*

$$Q(\rho H - \alpha_3 S - S^2, g \wedge g) = 0,$$

$$\rho H = S^2 + \alpha_3 S + \frac{\lambda}{n}g, \quad \lambda = \rho \operatorname{tr}(H) - \kappa\alpha_3 - \operatorname{tr}(S^2), \tag{8.4}$$

$$\alpha_3 = (n-2)^2 \left(\frac{1}{n-2}(\alpha_1 - \alpha_2) - 2\alpha_2 - \frac{\tilde{\kappa}}{n(n+1)} \right) - \frac{\kappa}{n-1} = \varepsilon\psi - \frac{2(n-1)\tilde{\kappa}}{n(n+1)}, \tag{8.5}$$

where α_1 and α_2 are defined by (8.2) and (8.3), respectively. Moreover,

$$\begin{aligned} R \cdot S &= Q(g, S^2) + \left(\varepsilon\psi - \frac{(2n-3)\tilde{\kappa}}{n(n+1)} \right) Q(g, S), \\ R \cdot S^2 &= Q(S, S^2) + \rho_1 Q(g, S^2) + \rho_2 Q(g, S), \\ S^3 &= \left(-2\varepsilon\psi + \frac{3(n-1)\tilde{\kappa}}{n(n+1)} \right) S^2 + \rho_2 S + \rho_3 g, \end{aligned}$$

hold on \mathcal{U}_H , where the functions ρ_1 , ρ_2 and ρ_3 are defined by

$$\begin{aligned} \rho_1 &= -\frac{(n-2)\tilde{\kappa}}{n(n+1)} - \alpha_3, \\ \rho_2 &= -\frac{\lambda}{n} - \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} + \alpha_3 \right) \alpha_3, \\ \rho_3 &= \frac{1}{n} \left(\operatorname{tr}(S^3) + \left(2\varepsilon\psi - \frac{3(n-1)\tilde{\kappa}}{n(n+1)} \right) \operatorname{tr}(S^2) - \kappa\rho_2 \right). \end{aligned}$$

(ii) If at a point $x \in \mathcal{U}_H$ we have $S^2 = \beta_1 S + \beta_2 g$, for some $\beta_1, \beta_2 \in \mathbb{R}$, then $\rho = 0$, $\beta_1 = \alpha_3$ and $\beta_2 = -(\lambda/n)$ at this point.

Theorems 8.1 and 8.2 (precisely, (8.1), (8.4) and (8.5)) lead to the following result.

Theorem 8.3. *If M is a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.6) on $\mathcal{U}_H \subset M$, for some functions ψ and ρ on \mathcal{U}_H then*

$$\begin{aligned} (n-2)(R \cdot C - C \cdot R) &= Q(S, R) + \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \frac{\kappa}{n-1} - \varepsilon\psi \right) Q(g, R) \\ &\quad + \frac{1}{2}Q(S^2, g \wedge g) + \left(\frac{\varepsilon\psi}{2} - \frac{(n-1)\tilde{\kappa}}{n(n+1)} \right) Q(S, g \wedge g) \end{aligned}$$

on \mathcal{U}_H .

Theorem 8.4. [55, Theorem 4.5] *Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.6) on $\mathcal{U}_H \subset M$, for some functions ψ and ρ on \mathcal{U}_H . If the tensor $C \cdot C$ and a generalized curvature tensor T satisfy $C \cdot C = Q(g, T)$ on \mathcal{U}_H , then on this set we have*

$$T = \left(\frac{\kappa}{n-1} + \frac{2\varepsilon\psi}{n-1} - \frac{\tilde{\kappa}}{n+1} \right) C + \frac{\lambda}{2}g \wedge g - \frac{n-3}{(n-2)^2(n-1)} \left(g \wedge S^2 + \frac{n-2}{2}S \wedge S - \kappa g \wedge S \right),$$

where λ is some function on \mathcal{U}_H .

Further, we also have the following results obtained in [55, Section 4].

Theorem 8.5. *Let M be a hypersurface isometrically immersed in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.6) on $\mathcal{U}_H \subset M$.*

(i) [55, Theorem 4.6] *If on \mathcal{U}_H the tensor $Q(S, R)$ is equal to the Tachibana tensor $Q(g, T_1)$, where T_1 is a generalized curvature tensor, then any of the tensors: $R \cdot R$, $R \cdot C$, $C \cdot R$, $R \cdot C - C \cdot R$ and $C \cdot C$ is equal to some Tachibana tensor $Q(g, T_2)$, where T_2 is a linear combination of the tensors R , $g \wedge g$, $g \wedge S$, $g \wedge S^2$ and $S \wedge S$.*

(ii) [55, Proposition 4.7] *The following conditions are satisfied on \mathcal{U}_H :*

$$\begin{aligned} C \cdot C &= \frac{n-3}{n-2}R \cdot C + \frac{1}{n-2} \left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(2n-3)\tilde{\kappa}}{n(n+1)} \right) Q(g, C), \\ (n-2)C \cdot R + R \cdot C &= (n-2)Q(S, C) + \left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n-1)^2\tilde{\kappa}}{n(n+1)} \right) Q(g, C) \\ &\quad - \frac{1}{(n-2)}Q \left(g, g \wedge S^2 + \frac{n-2}{2}S \wedge S - \kappa g \wedge S \right). \end{aligned}$$

Theorem 8.6. [55, Theorem 4.8] *Let M be a hypersurface isometrically immersed in $N_s^{n+1}(c)$, $n \geq 4$, such that (4.1) and (7.11) are satisfied on $\mathcal{U}_H \subset M$, where ψ is a function on this set. Then (1.2) holds on \mathcal{U}_H if and only if the following two conditions hold on this set*

$$(a) \frac{\kappa}{n-1} = \frac{\tilde{\kappa}}{n+1} \quad \text{and} \quad (b) \quad Q \left(S - \frac{\kappa}{n}g, C \right) = 0. \tag{8.6}$$

For further results related to Theorem 8.5 (i) we refer to [42, Section 8].

Remark 8.1. (i) Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. As it was proved in [20, Proposition 3.1 (ii), Proposition 3.2] the conditions (7.11) and

$$R \cdot S = \frac{\tilde{\kappa}}{n(n+1)} Q(g, S) \tag{8.7}$$

are equivalent on $\mathcal{U}_H \subset M$. If (7.11) holds on $\mathcal{U}_H \subset M$ then on this set we have [38, Theorem 3.1)]

$$(n-2)(R \cdot C - C \cdot R) = Q(S, R) + \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \frac{\kappa}{n-1} - \varepsilon\psi \right) Q(g, R). \tag{8.8}$$

(ii) Let now M be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, such that at every point of M there are principal curvatures $0, \dots, 0, \lambda, \dots, \lambda, -\lambda, \dots, -\lambda$, with the same multiplicity of λ and $-\lambda$,

and λ is a positive function on M . Thus we have on M : $\text{tr}(H) = 0$ and $H^3 = \lambda^2 H$, and, in a consequence, we also have (8.7) and (8.8). We mention that the Cartan hypersurfaces, as well as generalized Cartan hypersurfaces [7, Section 6] have at every point principal curvatures $0, \dots, 0, \lambda, \dots, \lambda, -\lambda, \dots, -\lambda$, with the same multiplicity of λ and $-\lambda$, and λ is a positive function on M . Curvature properties of pseudosymmetry type of Cartan hypersurfaces (see, e.g., [5, Chapter 3], [110, Chapter 7.5]) are given in [38, 75, 76]. Both classes of the considered hypersurfaces are austere hypersurfaces [4].

9. Hypersurfaces satisfying (1.4)

In [43] a survey on some family of generalized Einstein metric conditions was given. Those curvature conditions are strongly related to pseudosymmetry. In particular, [43, Section 6] contains results of non-Einstein and non-conformally flat semi-Riemannian manifolds (M, g) , of dimension $n \geq 4$, satisfying conditions of the form: the tensor $R \cdot C - C \cdot R$ is proportional to the Tachibana tensor: $Q(g, R)$, $Q(S, R)$, $Q(g, C)$ or $Q(S, C)$. More precisely, those conditions are considered on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$. Among other results in that section it was shown that some hypersurfaces M isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfy (1.4) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$. We recall that an example of a hypersurface having mentioned properties was constructed in [58, Section 5]. We also mention that semi-Riemannian manifolds satisfying (1.4) were investigated in [48].

Let M be a hypersurface isometrically immersed in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.4) on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H$. We recall that (1.5) holds on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H$, where α and β are some functions defined on this set.

According to [82, Proposition 3.3], the Riemann-Christoffel curvature tensor R of M is expressed on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H$ by a linear combination of the Kulkarni-Nomizu products $S \wedge S$, $g \wedge S$ and $g \wedge g$ formed by the Ricci tensor S and the metric tensor g of M . Precisely, we have (4.11), i.e.,

$$R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \frac{\eta}{2} g \wedge g,$$

where ϕ , μ and η are some functions on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H$ (see [49, eqs. (5.2)]). We also can express the tensors $C \cdot C$, $Q(g, C)$ and $Q(S, C)$ by some linear combinations of the Tachibana tensors formed by the tensors g and H . In [49, Theorem 5.1] it was stated that if the scalar curvature κ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, vanishes on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H \subset M$ then

$$R \cdot C - C \cdot R = -Q(S, C) \tag{9.1}$$

on this set. From that theorem it follows immediately [49, Corollary 5.3] that if M is a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having at every point exactly two distinct principal curvatures, and if its scalar curvature κ vanishes on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H \subset M$ then (9.1) holds on this set. In [49, Examples 5.4, 5.5 and 5.7] examples of non-conformally flat and non-Einstein hypersurfaces, with $\kappa = 0$, having at every point exactly two distinct principal curvatures are presented.

As it was mentioned in Introduction, if at every point of \mathcal{U}_H of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, one of the tensors $R \cdot C$, $C \cdot R$ or $R \cdot C - C \cdot R$ is a linear combination of the tensor $R \cdot R$ and a finite sum of the Tachibana tensors of the form $Q(A, T)$, where A is a symmetric $(0, 2)$ -tensor and T a generalized curvature tensor, then (1.6) holds on \mathcal{U}_H [47, Corollary 4.1]. Thus in particular, if (1.4) is satisfied on \mathcal{U}_H then (1.6) holds on this set. Hypersurfaces in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.6), or in particular (1.6) with $\rho = 0$, i.e., (7.11), were studied in several papers: [20, 38, 45, 47, 54, 58, 62, 64, 65, 70, 75, 81, 84, 111, 112, 113, 114, 115]. Section 6 of [49] contains some results on hypersurfaces satisfying (1.6). In Section 7 of [49] hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.4) on $\mathcal{U}_H \subset M$ were investigated. The main result of this section (see, [49, Theorem 7.3]) states that if M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$ the equalities: (7.11) and $\text{rank}(S - \alpha g) = 1$, for some function α on \mathcal{U}_H , then on this set

$$\text{rank} \left(S - \left(\frac{\kappa}{n-1} - \frac{\tilde{\kappa}}{n(n+1)} \right) g \right) = 1, \tag{9.2}$$

$$(n-2)(R \cdot C - C \cdot R) = Q(S, C) - \frac{\tilde{\kappa}}{n(n+1)} Q(g, C). \tag{9.3}$$

In particular, if the ambient space is a semi-Euclidean space \mathbb{E}_s^{n+1} , $n \geq 4$, then (9.2) and (9.3) turn into

$$\text{rank} \left(S - \frac{\kappa}{n-1} g \right) = 1, \tag{9.4}$$

$$(n-2)(R \cdot C - C \cdot R) = Q(S, C) \tag{9.5}$$

respectively.

Let M be a hypersurface in an Euclidean space \mathbb{E}^{n+1} , $n = 2p + 1$, $p \geq 2$, having at every point three principal curvatures $\lambda_1 = \lambda \neq 0$, $\lambda_2 = -\lambda$ and $\lambda_3 = 0$, provided that the multiplicity of λ_1 , as well as of λ_2 is p . Clearly, M is an austere hypersurface [89, p. 102]. Evidently, $M = \mathcal{U}_H$ and (7.11), (9.4) and (9.5) hold on M (see, [49, Example 7.5(i)] for details). We recall that in [1] it was stated that M is a non-semisymmetric ($R \cdot R \neq 0$) Ricci-symmetric ($R \cdot S = 0$) hypersurface. Further results on hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.4) on $\mathcal{U}_H \subset M$, with $L \neq -1$, are given in [49, Proposition 7.1] and [49, Example 7.5(ii)-(iv)]. We also have

Theorem 9.1. [49, Theorem 7.4] *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.4) with $L = -1$ on \mathcal{U}_H then on this set we have: $\kappa = 0$ and*

$$(n - 1)(R \cdot C - C \cdot R) = Q \left(g, \left(\frac{(n - 1)\tilde{\kappa}}{n(n + 1)} - \varepsilon\psi \right) R + \left(\frac{2(n - 1)\tilde{\kappa}}{n(n + 1)} - \varepsilon\psi \right) g \wedge S - \frac{1}{2(n - 2)} S \wedge S - g \wedge S^2 \right),$$

where $\tilde{\kappa}$ is the scalar curvature of $N_s^{n+1}(c)$, κ the scalar curvature of M and the function ψ is defined by (1.6).

In [49, Example 7.6] it was stated that some tubular hypersurfaces with vanishing scalar curvature satisfy (1.4) with $L = -1$.

10. Hypersurfaces satisfying (1.8)

In Section 5 of [55] we consider hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.8) on $\mathcal{U}_H \subset M$.

We recall that a semi-Riemannian manifold (M, g) , $n \geq 3$, is said to be a quasi-Einstein manifold (see Section 4) if (4.1) holds on $\mathcal{U}_S \subset M$, where α is some function on this set. Let M be a quasi-Einstein hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.8). We have

Theorem 10.1. [55, Theorem 5.1] *If M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.8) and (4.1) on $\mathcal{U}_H \subset M$, for some functions α , L_1 and L_2 , then on this set we have (7.11) and*

$$\begin{aligned} (n - 2)(R \cdot C - C \cdot R) &= Q(S, C) - \frac{\tilde{\kappa}}{n(n + 1)} Q(g, C), \\ \alpha &= \frac{\kappa}{n - 1} - \frac{\tilde{\kappa}}{n(n + 1)}, \\ R \cdot C - C \cdot R &= \frac{\kappa}{n - 1} Q(g, C) - Q(S, C) + B, \end{aligned}$$

where ψ is some function on \mathcal{U}_H , and the function α and the $(0, 6)$ -tensor B are defined by (4.1) and

$$B = Q \left(S - \frac{1}{n - 1} \left(\frac{(n - 2)\kappa}{n - 1} + \frac{\tilde{\kappa}}{n(n + 1)} \right) g, C \right),$$

respectively. Moreover, (1.2) is satisfied on \mathcal{U}_H if and only if (8.6) holds on this set.

We consider non-quasi-Einstein hypersurfaces satisfying (1.8). Precisely, we consider (1.8) at all points of \mathcal{U}_H at which the following condition is satisfied

$$\text{rank}(S - \alpha g) > 1, \text{ for any } \alpha \in \mathbb{R}. \tag{10.1}$$

We have

Theorem 10.2. [55, Theorem 5.2] *Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.8) on $\mathcal{U}_H \subset M$. If at a point $x \in \mathcal{U}_H$ (10.1) is satisfied then at this point we have (1.2) and*

$$(n - 1)Q(S, R) = Q \left(g, \left(\varepsilon\psi + \kappa - \frac{(n - 1)\tilde{\kappa}}{n(n + 1)} \right) R + \left(\varepsilon\psi - \frac{2(n - 1)\tilde{\kappa}}{n(n + 1)} \right) g \wedge S + g \wedge S^2 - \frac{1}{2} S \wedge S \right).$$

Theorem 10.3. [55, Theorem 5.4] *If at every point of a non-quasi-Einstein hypersurface M in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, the difference tensor $R \cdot C - C \cdot R$ is a linear combination of the Tachibana tensors $Q(g, C)$ and $Q(S, C)$, then (1.2) holds on M .*

11. Chen ideal submanifolds

Let M be a submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 2, m \geq 1$. Let g be the Riemannian metric induced on M from the standard metric on \mathbb{E}^{n+m} , ∇ the corresponding Levi-Civita connection on M , and R, S, τ and C , the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of g , respectively. For the scalar curvature τ of (M, g) we use the calibration

$$\tau(p) = \sum_{i < j} K(p, e_i(p) \wedge e_j(p))$$

where $K(p, \pi)$ denotes the Riemannian sectional curvature of (M, g) at the point p for a plane section π in the tangent space $T_p M$. For each point p in M , considering the number

$$(\inf K)(p) := \inf \{K(p, \pi) | \pi \text{ is a plane section in } T_p M\},$$

B.-Y. Chen (see [6, 9]) introduced the $\delta(2)$ -curvature by

$$(\delta(2))(p) = \delta(p) := \tau(p) - (\inf K)(p).$$

This $\delta(2)$ -, for short, δ -curvature of Chen thus is a well defined real function on M which clearly is a Riemannian invariant of (M, g) .

From [9] (see also [6, 8, 12]), we have the following basic result which, in particular, answered a question raised by S.S. Chern [15] long before, concerning intrinsic obstructions on Riemannian manifolds in view of minimal immersibility in Euclidean spaces.

Theorem 11.1. [6] (see also [68, Theorem 1]) For any submanifold M of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 2, m \geq 1$,

$$\delta \leq \frac{n^2(n-2)}{2(n-1)} H^2, \tag{11.1}$$

and in (11.1) equality holds at a point $p \in M$ if and only if, with respect to some suitable adapted orthonormal frame $\{e_i, \xi_\alpha\}$ around p on M in \mathbb{E}^{n+m} , the shape operators are given by

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & z & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & z \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \beta > 1,$$

where $z = a + b$ and $\inf K = ab - \sum_{\beta > 1} (c_\beta^2 + d_\beta^2) : M \rightarrow \mathbb{R}$.

Evidently, if $m = 1$ then $\inf K = ab$.

With respect to the above theorem, one has the following definition [6, 8, 52, 67, 122], also see [68, Definition 1]. Let M be a submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 2, m \geq 1$. It is called a Chen ideal submanifold if, at each of its points, the Chen's basic inequality (11.1) in the Theorem 11.1 is actually an equality.

Let M be a Chen ideal submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4, m \geq 1$. We use the notations as in Theorem 11.1. The Riemann-Christoffel curvature tensor R satisfies:

$$\begin{cases} R(e_1, e_2, e_2, e_1) = \inf K = ab - \sum_{\beta=2}^m (c_\beta^2 + d_\beta^2); \\ R(e_1, e_i, e_i, e_1) = az \quad \text{for } i \geq 3; \\ R(e_2, e_i, e_i, e_2) = bz \quad \text{for } i \geq 3; \\ R(e_i, e_j, e_j, e_i) = z^2 \quad \text{for } 3 \leq i < j \leq n. \end{cases} \tag{11.2}$$

The other values of $R(e_u, e_v, e_w, e_t)$ are null. The Ricci tensor S satisfies:

$$\begin{cases} S(e_1, e_1) = \inf K + (n-2)az; \\ S(e_2, e_2) = \inf K + (n-2)bz; \\ S(e_i, e_i) = (n-2)z^2 \quad \text{for } 3 \leq i \leq n; \\ S(e_u, e_v) = 0 \quad \text{for } 1 \leq u < v \leq n. \end{cases} \tag{11.3}$$

The scalar curvature τ is given by

$$\tau = \sum_{i=1}^n S(e_i, e_i) = 2 \inf K + (n - 1)(n - 2)z^2. \tag{11.4}$$

The Weyl conformal curvature tensor C of M is determined by the following relations:

$$\begin{cases} C(e_1, e_2, e_2, e_1) = \frac{(n - 3) \inf K}{n - 1}; \\ C(e_1, e_i, e_i, e_1) = -\frac{(n - 3) \inf K}{(n - 1)(n - 2)} \text{ for } i \geq 3 \\ C(e_2, e_i, e_i, e_2) = -\frac{(n - 3) \inf K}{(n - 1)(n - 2)} \text{ for } i \geq 3 \\ C(e_i, e_j, e_j, e_i) = \frac{2 \inf K}{(n - 1)(n - 2)} \text{ for } 3 \leq i < j \leq n. \end{cases} \tag{11.5}$$

From (11.5) it follows (cf. [52], [67, Theorem F]) that every Chen ideal submanifold M of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$, has a pseudosymmetric Weyl conformal curvature C , i.e., it satisfies the identity

$$C \cdot C = L_C Q(g, C), \quad L_C = -\frac{(n - 3) \inf K}{(n - 1)(n - 2)}. \tag{11.6}$$

As it was mentioned in Section 8, at every point of a hypersurface M in a space form $\tilde{N}^{n+1}(c)$, $n \geq 4$, the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent. Precisely, (7.2) holds on M . Thus, in particular, $R \cdot R - Q(S, R) = 0$ on every Chen ideal hypersurface M in \mathbb{E}^{n+1} , $n \geq 4$.

Now let us compute the difference $R \cdot R - Q(S, R)$ on the Chen ideal submanifold M of codimension m in \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. With respect to the notations in Theorem 11.1 and from the equalities (11.2), (11.3) and (11.5), we can prove the following

Theorem 11.2. [68, Theorem 2] *The identity*

$$R \cdot R - Q(S, R) = \frac{ab - \inf K}{\inf K} (n - 2)z^2 Q(g, C)$$

holds on the subset \mathcal{U}_C of every Chen ideal submanifold M of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. In addition, at each point $p \in M$ where C vanishes ($\inf K = 0$), the following equalities hold:

$$\begin{cases} (R \cdot R - Q(S, R))(e_1, e_2, Z, W; e_1, e_i) = (n - 3)abz^2 \langle (e_2 \wedge_g e_i)(Z), W \rangle, \\ (R \cdot R - Q(S, R))(e_1, e_j, Z, W; e_1, e_i) = -abz^2 \langle (e_j \wedge_g e_i)(Z), W \rangle, \\ (R \cdot R - Q(S, R))(e_1, e_2, Z, W; e_2, e_i) = -(n - 3)abz^2 \langle (e_1 \wedge_g e_i)(Z), W \rangle, \\ (R \cdot R - Q(S, R))(e_2, e_j, Z, W; e_2, e_i) = -abz^2 \langle (e_j \wedge_g e_i)(Z), W \rangle, \end{cases}$$

and the other values of $(R \cdot R - Q(S, R))(e_u, e_v, Z, W; e_w, e_t)$ being null.

As it was proved in [51], every warped product manifold $\overline{M} \times_F \tilde{N}$ of a 2-dimensional base manifold $(\overline{M}, \overline{g})$ and an $(n - 2)$ -dimensional fibre, which is a space of constant curvature (\tilde{N}, \tilde{g}) , $n \geq 4$, with the warping function F , satisfies

$$C \cdot C = -\frac{(n - 3)\rho}{(n - 2)(n - 1)} Q(g, C), \quad \rho = \frac{\bar{\tau}}{2} + \frac{\tilde{\tau}}{(n - 3)(n - 2)F} + \frac{\Delta F}{2F} - \frac{\Delta_1 F}{2F^2}, \tag{11.7}$$

where $\Delta F = g^{ab} \nabla_b F_a$, $\Delta_1 F = g^{ab} F_a F_b$, and $\bar{\tau}, \tilde{\tau}$ are the scalar curvatures of the base and the fibre, respectively, see to [51] for details.

As we noted in Section 1, some warped product manifolds $\overline{M} \times_F \tilde{N}$ with a 2-dimensional Riemannian manifold $(\overline{M}, \overline{g})$ and an $(n - 2)$ -dimensional unit sphere \mathbb{S}^{n-2} , are related to Chen ideal submanifolds. Namely, according to [17], every non-trivial and non-minimal Chen ideal submanifold M of dimension n in the

Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$ is isometric to an open subset of a warped product $\overline{M} \times_F \mathbb{S}^{n-2}$ of a 2-dimensional base manifold $(\overline{M}, \overline{g})$ and an $(n-2)$ -dimensional unit sphere \mathbb{S}^{n-2} , where the warping function F is a solution of some second order quasilinear elliptic partial differential equation in the plane. Thus we see that (11.7) holds on M . Furthermore, from (11.6) and (11.7) it follows that $\inf K$ is expressed on M by

$$\inf K = \frac{\overline{\tau}}{2} + \frac{1}{F} + \frac{\Delta F}{2F} - \frac{\Delta_1 F}{2F^2}.$$

Since the scalar curvature τ of M is given by (11.4) and satisfies (see, e.g., [51])

$$\tau = \overline{\tau} + \frac{(n-3)(n-2)}{F} - \frac{(n-2)\Delta F}{F} - \frac{(n-2)(n-5)\Delta_1 F}{4F^2},$$

we get

$$(n-2)z^2 = \frac{n-4}{F} - \frac{\Delta F}{F} - \frac{(n-6)\Delta_1 F}{4F^2}.$$

In [51], among other things, it was shown that the Hessian of the function $f = \sqrt{F}$ is proportional to the metric \overline{g} on \overline{M} .

We mention that Chen ideal submanifolds which are semisymmetric were classified in [78]. We have

Theorem 11.3. [78] (see also [68, Theorem 3]) *A Chen ideal submanifold M of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 3$, $m \geq 1$, is semisymmetric if and only if M is minimal (in which case M is $(n-2)$ -ruled) or M is a round hypercone in some totally geodesic subspace \mathbb{E}^{n+1} of \mathbb{E}^{n+m} .*

Chen ideal pseudosymmetric submanifolds were classified in [52, 67]. We have

Theorem 11.4. [52, 67] (see also [68, Theorem 4]) *A Chen ideal submanifold M of codimension m in \mathbb{E}^{n+m} ($n \geq 3$, $m \geq 1$) is pseudosymmetric if and only if*

(i) *either M is semisymmetric (see Theorem 11.3),*

(ii) *or at every point p of M where $R \cdot R \neq 0$, the 2D normal section $\Sigma_{\tilde{\pi}}^2 \subset \mathbb{E}^{2+m}$ of M^n at p in the direction of the tangent plane $\tilde{\pi} \subset T_p M^n$ for which the sectional curvature function $K(p, \pi)$ at p attains its minimal value ($\inf K$) (p) is pseudo-umbilical at p , or equivalently, if p is a spherical point of the projection $\overline{\Sigma}_{\tilde{\pi}}^2 \subset \mathbb{E}^3$ of this 2D normal section $\overline{\Sigma}_{\tilde{\pi}}^2$ on the space \mathbb{E}^3 spanned by $\tilde{\pi}$ and the mean curvature vector $\vec{H}(p)$ of M^n in \mathbb{E}^{n+m} at p (and in this case $L_R = \frac{n^2}{2(n-1)^2} H^2$, where H is the mean curvature of M^n in \mathbb{E}^{n+m}).*

The above presented part of this section based on [68, Section 3]) and the remains part of this section on [68, Section 5]. We present now results on Chen ideal submanifolds in Euclidean spaces whose difference tensor $R \cdot C - C \cdot R$ can be expressed in terms of some of the Tachibana tensors $Q(g, R)$, $Q(S, R)$, $Q(g, C)$, $Q(S, C)$, $Q(g, g \wedge S)$, $Q(S, g \wedge S)$.

Theorem 11.5. [68, Theorem 6] *Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. Then there exist two real valued functions L_1 and L_2 on M such that (1.8), i.e.,*

$$R \cdot C - C \cdot R = L_1 Q(S, C) + L_2 Q(g, C),$$

if and only if there exists an orthonormal tangent frame field $\{e_1, \dots, e_n\}$ and an orthonormal normal frame field $\{\xi_1, \dots, \xi_m\}$ such that the shape operators $A_\alpha := A_{\xi_\alpha}$, $1 \leq \alpha \leq m$, are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & \epsilon a & 0 & \cdots & 0 \\ 0 & 0 & (1+\epsilon)a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (1+\epsilon)a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \beta \geq 2,$$

where $\epsilon = \pm 1$, a, c_β, d_β (for $2 \leq \beta \leq m$) are real functions on M such that

$$\sum_{\beta=2}^m (c_\beta^2 + d_\beta^2) = \epsilon a^2 - \inf K, \quad L_1 = -1, \quad L_2 = -\frac{2 \inf K + 2(n-1)(n-2)(1+\epsilon)a^2}{n-1}.$$

In this case, M is a Roter space. In addition one has one of the following two cases:

(i) either $\epsilon = -1$ and M is a semisymmetric and minimal submanifold (see Theorem 11.3) such that

$$R \cdot C - C \cdot R = \frac{2 \inf K}{n-1} Q(g, C) - Q(S, C),$$

(ii) or $\epsilon = +1$ and M is a properly pseudosymmetric and non-minimal submanifold (see Theorem 11.4) such that

$$R \cdot C - C \cdot R = -\frac{2 \inf K + 4(n-1)(n-2)a^2}{n-1} Q(g, C) - Q(S, C).$$

Corollary 11.1. [68, Corollary 3] Let M be a Chen ideal submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. Then the difference tensor $R \cdot C - C \cdot R$ and the Tachibana tensor $Q(g, C)$ are linearly dependent if and only if M is conformally flat.

Corollary 11.2. [68, Corollary 4] Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. Then there exists a real valued function L on M such that

$$R \cdot C - C \cdot R = LQ(S, C),$$

if and only if M is not minimal, and there exists an orthonormal tangent frame field $\{e_1, \dots, e_n\}$ and an orthonormal normal frame field $\{\xi_1, \dots, \xi_m\}$ such that the shape operators $A_\alpha := A_{\xi_\alpha}$, $1 \leq \alpha \leq m$, are given by

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & 2a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \beta \geq 2,$$

where a, c_β, d_β (for $2 \leq \beta \leq m$) are real functions on M such that

$$\sum_{\beta=2}^m (c_\beta^2 + d_\beta^2) = (2n^2 - 6n + 5) a^2 > 0$$

and $L = -1$. In this case, M^n is properly pseudosymmetric (see Theorem 11.4).

Theorem 11.6. [68, Theorem 5] Let M be a non-conformally flat Chen ideal submanifold of codimension m in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. Then there exist two real valued functions L_3 and L_4 on M such that (1.9) is satisfied on M if and only if there exists an orthonormal tangent frame field $\{e_1, \dots, e_n\}$ and an orthonormal normal frame field $\{\xi_1, \dots, \xi_m\}$ such that the shape operators $A_\alpha := A_{\xi_\alpha}$, $1 \leq \alpha \leq m$ are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & \epsilon a & 0 & \cdots & 0 \\ 0 & 0 & (1 + \epsilon)a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (1 + \epsilon)a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \beta \geq 2,$$

where $\epsilon = \pm 1$, a, c_β, d_β (for $2 \leq \beta \leq m$) are real functions on M such that

$$\sum_{\beta=2}^m (c_\beta^2 + d_\beta^2) = \epsilon a^2 - \inf K, \quad 2 \inf K - (1 + \epsilon)a^2 \neq 0, \quad \inf K \neq (n-2)(1 + \epsilon)a^2$$

and

$$L_3 = \frac{(n-3) \inf K - (n-2)(1 + \epsilon)a^2}{(n-1)(n-2)} \frac{2 \inf K}{2 \inf K - (1 + \epsilon)a^2},$$

$$(1 + \epsilon)a^2 \left(L_4 - \frac{1}{(n-2)} \frac{\inf K}{2 \inf K - (1 + \epsilon)a^2} \right) = 0.$$

In this case, M is a Roter space. In addition one has one of the following two cases:

(i) either $\epsilon = -1$ and M is a semisymmetric and minimal submanifold (see Theorem 11.3) such that

$$R \cdot C - C \cdot R = \frac{(n-3) \inf K}{(n-1)(n-2)} Q(g, R),$$

(ii) or $\epsilon = +1$ and M is a properly pseudosymmetric and non minimal submanifold (see Theorem 11.4) such that

$$R \cdot C - C \cdot R = \frac{(n-3) \inf K - 2(n-2)a^2}{(n-1)(n-2)} \frac{\inf K}{\inf K - a^2} Q(g, R) + \frac{1}{2(n-2)} \frac{\inf K}{\inf K - a^2} Q(S, R).$$

Corollary 11.3. [68, Corollary 1] Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. Then there exists a real valued function L on M such that

$$R \cdot C - C \cdot R = LQ(g, R)$$

if and only if M is minimal and there exists an orthonormal tangent frame field $\{e_1, \dots, e_n\}$ and an orthonormal normal frame field $\{\xi_1, \dots, \xi_m\}$ such that the shape operators $A_\alpha := A_{\xi_\alpha}$, $1 \leq \alpha \leq m$, are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & -a & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \beta \geq 2,$$

where a, c_β, d_β (for $2 \leq \beta \leq m$) are real functions on M such that

$$\sum_{\beta=2}^m (c_\beta^2 + d_\beta^2) = -a^2 - \inf K, \quad L = \frac{(n-3) \inf K}{(n-1)(n-2)}.$$

In this case, M is semisymmetric (see Theorem 11.3).

Corollary 11.4. [68, Corollary 2] Let M be a Chen ideal submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. Then the difference tensor $R \cdot C - C \cdot R$ and the Tachibana tensor $Q(S, R)$ are linearly dependent if and only if M is conformally flat ($\inf K = 0$).

Theorem 11.7. [68, Theorem 7] Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. Then there exist two real valued functions L_5 and L_6 on M such that (1.10) is satisfied on M if and only if there exists an orthonormal tangent frame field $\{e_1, \dots, e_n\}$ and an orthonormal normal frame field $\{\xi_1, \dots, \xi_m\}$ such that the shape operators $A_\alpha := A_{\xi_\alpha}$, $1 \leq \alpha \leq m$, are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & \epsilon a & 0 & \cdots & 0 \\ 0 & 0 & (1+\epsilon)a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (1+\epsilon)a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \beta \geq 2,$$

where $\epsilon = \pm 1$, a, c_β, d_β (for $2 \leq \beta \leq m$) are real functions on M such that

$$\sum_{\beta=2}^m (c_\beta^2 + d_\beta^2) = \epsilon a^2 - \inf K, \quad \inf K \neq (n-2)(1+\epsilon)a^2$$

and moreover

$$L_5 = -\frac{2(n-2)(1+\epsilon)a^2 \inf K (\inf K + (n-2)(1+\epsilon)a^2)}{n-1 (\inf K - (n-2)(1+\epsilon)a^2)^2},$$

$$L_6 = -\frac{1 \inf K [(n-3) \inf K + (n-1)(n-2)(1+\epsilon)a^2]}{(n-1)(n-2) (\inf K - (n-2)(1+\epsilon)a^2)^2}.$$

In this case, M is a Roter space. In addition one has one of the following two cases:

(i) either $\epsilon = -1$ and M is a semisymmetric and minimal submanifold (see Theorem 11.3) such that

$$R \cdot C - C \cdot R = -\frac{n-3}{(n-1)(n-2)}Q(S, g \wedge S),$$

(ii) or $\epsilon = +1$ and M is a properly pseudosymmetric and non-minimal submanifold (see Theorem 11.4) such that

$$R \cdot C - C \cdot R = -\frac{4(n-2)a^2 \inf K (\inf K + 2(n-2)a^2)}{n-1 (\inf K - 2(n-2)a^2)^2}Q(g, g \wedge S) - \frac{1}{(n-1)(n-2)} \frac{\inf K ((n-3) \inf K + 2(n-1)(n-2)a^2)}{(\inf K - 2(n-2)a^2)^2}Q(S, g \wedge S).$$

Corollary 11.5. [68, Corollary 5] Let M be a non-conformally Chen ideal submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. Then there exists a real valued function L on M such that

$$R \cdot C - C \cdot R = LQ(g, g \wedge S),$$

if and only if M is not minimal, and there exists an orthonormal tangent frame field $\{e_1, \dots, e_n\}$ and an orthonormal normal frame field $\{\xi_1, \dots, \xi_m\}$ such that the shape operators $A_\alpha := A_{\xi_\alpha}$, $1 \leq \alpha \leq m$, are given by

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & 2a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \beta \geq 2,$$

where a, c_β, d_β (for $2 \leq \beta \leq m$) are real functions on M such that

$$\sum_{\beta=2}^m (c_\beta^2 + d_\beta^2) = \frac{2n^2 - 5n + 1}{n-1}a^2 > 0, \quad L = -\frac{2a^2}{n-2}.$$

In this case, M is a properly pseudosymmetric manifold (see Theorem 11.4).

Corollary 11.6. [68, Corollary 6] Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. Then there exists a real valued function L on M such that

$$R \cdot C - C \cdot R = LQ(S, g \wedge S),$$

if and only if one has one of the two cases which follow:

(i) either M is minimal, and there exists an orthonormal tangent frame field $\{e_1, \dots, e_n\}$ and an orthonormal normal frame field $\{\xi_1, \dots, \xi_m\}$ such that the shape operators $A_\alpha := A_{\xi_\alpha}$, $1 \leq \alpha \leq m$, are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & -a & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \beta \geq 2,$$

where a, c_β, d_β (for $2 \leq \beta \leq m$) are real functions on M such that

$$\sum_{\beta=2}^m (c_\beta^2 + d_\beta^2) = -a^2 - \inf K, \quad L = -\frac{n-3}{(n-1)(n-2)},$$

(ii) or M is not minimal, and there exists an orthonormal tangent frame field $\{e_1, \dots, e_n\}$ and an orthonormal normal frame field $\{\xi_1, \dots, \xi_m\}$ such that the shape operators $A_\alpha := A_{\xi_\alpha}$, $1 \leq \alpha \leq m$, are given by

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & 2a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \beta \geq 2,$$

where a, c_β, d_β (for $2 \leq \beta \leq m$) are real functions on M such that

$$\sum_{\beta=2}^m (c_\beta^2 + d_\beta^2) = (2n - 3)a^2, \quad L = \frac{1}{2(n-1)(n-2)}.$$

In the first case, M is a semisymmetric manifold (see Theorem 11.3). In the second case, M is a properly pseudosymmetric manifold (see Theorem 11.4).

Corollary 11.7. [68, Corollary 7] Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. If M is minimal, then

$$R \cdot C - C \cdot R = -\frac{n-3}{(n-1)(n-2)}Q(S, g \wedge S).$$

Corollary 11.8. [68, Corollary 8] A Chen ideal submanifold M of dimension $n \geq 4$ in the Euclidean space \mathbb{E}^{n+m} satisfies the curvature condition $R \cdot C - C \cdot R = 0$ if and only if M is conformally flat.

According to [68, eq. (28)] (see also [42, Theorem 4.1]), as an immediate consequence of Theorem 11.7, for Chen ideal and Roter submanifolds, we express the difference tensor $R \cdot C - C \cdot R$ as a linear combination of the Tachibana tensors $Q(g, g \wedge S)$, $Q(S, g \wedge S)$, in terms of $\inf K$ and the scalar curvature τ .

Corollary 11.9. [68, Corollary 9] Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space \mathbb{E}^{n+m} , $n \geq 4$, $m \geq 1$. If M is a Roter space, then

$$R \cdot C - C \cdot R = -\frac{2(\tau - 2 \inf K)(\tau + 2(n-2) \inf K)}{(n-1)(\tau - 2n \inf K)^2}Q(g, g \wedge S) - \frac{2(n-1) \inf K(\tau + 2(n-4) \inf K)}{(n-2)(\tau - 2n \inf K)^2}Q(S, g \wedge S).$$

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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