Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Published online: 29.04.2025
DOI : 10.15672/hjms.1336986

Research Article

Fault tolerant metric resolvability of convex polytopes

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Abstract

Fault tolerant metric basis assigns a unique code in terms of distances to each node of the graph which works in the presence of faults. The distance based parameters are widely used in different fields like robot navigation, interconnection networks, sensor deployments, image processing and chemistry. The convex polytopes are stable and destroy resistant which make them attractive choice for interconnection networks and anti-tracking networks. The current article, computes the fault tolerant metric dimension (FTMD) of convex polytopes B_n, C_n, E_n and U_n. The applications of FTMD have also been considered in the manuscript.

Mathematics Subject Classification (2020). 05C12, 05C90

Keywords. convex polytopes, metric dimension, fault tolerant metric dimension

1. Introduction and preliminaries

Consider a connected graph G with the node set $V(G)$ and edge set $E(G)$. The distance between two nodes u and v is $d(u, v)$ which gives the length of a shortest path connecting these nodes. The distance of a node v from a set of nodes Π is $d(v, \Pi) = \min\{d(v, x)|x \in \Pi\}$. If $\Pi = \{x_1, x_2, \ldots, x_n\}$ is an ordered set of nodes, then $r(v|\Pi) = (d(v, x_1), d(v, x_2), \ldots, d(v, x_n))$ is called the code or representation of the node v with respect to Π. The set Π is known as a resolving set, if the codes $r(v|\Pi)$ are distinct for each $v \in V$. The least cardinality of a resolving set Π is called the MD of G symbolized as $dim(G)$ and Π itself is called metric basis (MB). In [6], Hernando et al. generalized the concept of MD to FTMD. If the codes $r(v|\Pi)$ are distinct in at least two places for each $v \in V(G)$, then Π is known as fault tolerant metric generator (FTMG) for G. A FTMG of least cardinality is known as fault tolerant metric basis (FTMB) and its cardinality the FTMD of G, symbolized as $dim_F(G)$. In [4] the notions of MD and FTMD were further generalized to $k-$metric dimension by Estrado et al. If the codes $r(v|\Pi)$ are distinct in at least k places for each $v \in V(G)$, then Π is known as $k-$metric generator for G. A connected graph G is $k-$metric dimensional if k is the largest integer such that there exists a $k-$metric basis for G. When $k = 1$ then it is MD and for $k = 2$ it is FTMD. The articles

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Received: 03.08.2023; Accepted: 23.03.2025

[7, 11, 19, 20] are referred for further discussion on FTMD.
The fault tolerance in a network enables it to sustain its working ability and functionality if any one of its parts stop working. This ensures the durability and lower repair cost of the networks. Tian et al. [18] studied convex polytopes in the context of anti-tracking networks. Convex polytopes have also been studied for metric dimension (MD) and partition dimension (PD). The MD of convex polytopes along with other families of graphs have been discussed in [1, 3, 8–10] whereas FTMD is discussed in [2, 15–17, 20]. The PD and fault tolerant partition dimension of rotationally symmetric graphs are computed in [12–14]. The computation of MD and FTMD is NP-hard [5] for general graphs which motivated the researchers to compute these parameters for special classes of graphs. Following the recent research work on MD and FTMD, the FTMD of convex polytopes for certain families have been computed in this article.

The following result on FTMD would be helpful in proving our claims.

Proposition 1.1 ([4]). *Let G be a $k-$metric dimensional graph with n number of nodes and k_1, k_2 be two integers such that $1 \leq k_1 < k_2 \leq k$, then $dim_{k_1}(G) < dim_{k_2}(G)$.*

The article is further divided into the following sections. The convex polytopes B_n, C_n, E_n and U_n are defined and exact values of FTMD for these families are computed in Section 2. The conclusion and open problems are given in Section 3.

2. Fault tolerant metric dimension of convex polytopes

The current section computes the exact values of FTMD of convex polytopes B_n, C_n, E_n and U_n. The first proposition in this section gives the MD of B_n, C_n, E_n [10] and U_n [9].

Proposition 2.1 ([10], [9]). *Consider the convex polytopes B_n, C_n, E_n and U_n for $n \geq 6$ then,*

(a) $dim(B_n) = 3$;
(b) $dim(C_n) = 3$;
(c) $dim(E_n) = 3$;
(c) $dim(U_n) = 3$.

2.1. FTMD of B_n

The convex polytope B_n is defined in [10] has $2n$ 4-sided faces, n 3-sided faces, n 5-sided faces, and a pair of n-sided faces with the following node and edge sets.

$V(B_n) = \{u_i, v_i, x_i, y_i, z : 1 \leq i \leq n\}$ and

$E(B_n) = \{u_iu_{i+1}; v_iv_{i+1}; y_iy_{i+1}; z_iz_{i+1}; u_iv_i; v_ix_i; v_{i+1}x_i; x_iy_i; y_iz_i : 1 \leq i \leq n\}$, where $u_{n+1} = u_1, v_{n+1} = v_1, x_{n+1} = x_1, y_{n+1} = y_1$ and $z_{n+1} = z_1$. The Figure 1 illustrates the graph of B_n. In the forthcoming result, we compute the FTMD of B_n.

Theorem 2.2. $dim_F(B_n) = 4$ *for $n \geq 6$.*

Proof. The proof comprises two parts.

Case 1: When n is odd with $n = 2\phi + 1$ and $\phi \geq 3$.

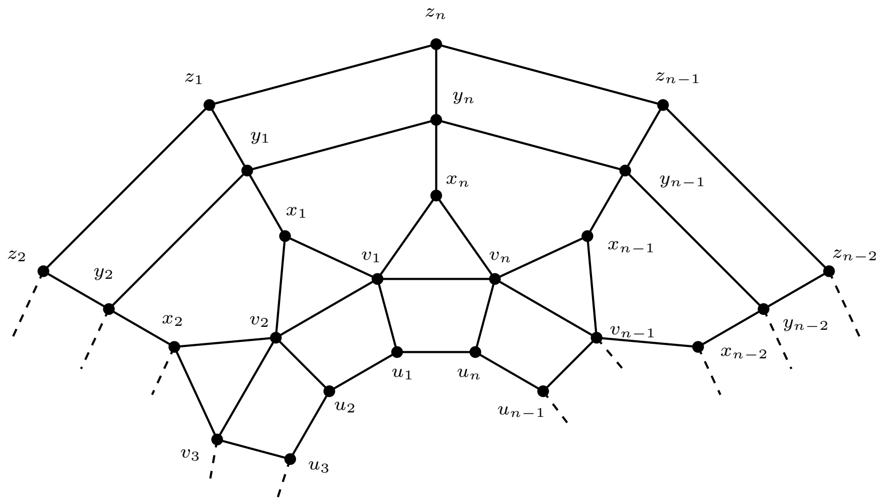

Figure 1. Convex Polytope B_n.

Consider the subset $\Pi = \{u_1, u_2, u_{\phi+1}, u_{\phi+3}\}$ of $V(B_n)$. The codes for all the nodes of B_n with respect to Π are given below.

$$r(u_\epsilon|\Pi) = \begin{cases} (0, 1, \phi, \phi-1) & \epsilon = 1; \\ (1, 0, \phi-1, \phi) & \epsilon = 2; \\ (\epsilon-1, \epsilon-2, \phi-\epsilon+1, \phi-\epsilon+3) & 3 \leq \epsilon \leq \phi; \\ (\phi, \phi-1, 0, 2) & \epsilon = \phi+1; \\ (\phi, \phi, 1, 1) & \epsilon = \phi+2; \\ (\phi-1, \phi, 2, 0) & \epsilon = \phi+3; \\ (n-\epsilon+1, n-\epsilon+2, \epsilon-\phi-1, \epsilon-\phi-3) & \phi+4 \leq \epsilon \leq n \end{cases}$$

$$r(v_\epsilon|\Pi) = \begin{cases} (1, 2, \phi+1, \phi) & \epsilon = 1; \\ (2, 1, \phi, \phi+1) & \epsilon = 2; \\ (\epsilon, \epsilon-1, \phi-\epsilon+2, \phi-\epsilon+4) & 3 \leq \epsilon \leq \phi+1; \\ (\phi+1, \phi+1, 2, 2) & \epsilon = \phi+2; \\ (n-\epsilon+2, n-\epsilon+3, \epsilon-\phi, \epsilon-\phi-2) & \phi+3 \leq \epsilon \leq n \end{cases}$$

$$r(x_\epsilon|\Pi) = \begin{cases} (2, 2, \phi+1, \phi+1) & \epsilon = 1; \\ (\epsilon+1, \epsilon, \phi-\epsilon+2, \phi-\epsilon+4) & 2 \leq \epsilon \leq \phi; \\ (n-\epsilon+2, \epsilon, \epsilon-\phi+1, \phi-\epsilon+4) & \phi+1 \leq \epsilon \leq \phi+2; \\ (n-\epsilon+2, n-\epsilon+3, \epsilon-\phi+1, \epsilon-\phi-1) & \phi+3 \leq \epsilon \leq n \end{cases}$$

$$r(y_\epsilon|\Pi) = \begin{cases} (3, 3, \phi+2, \phi+2) & \epsilon = 1; \\ (\epsilon+2, \epsilon+1, \phi-\epsilon+3, \phi-\epsilon+5) & 2 \leq \epsilon \leq \phi; \\ (n-\epsilon+3, \epsilon+1, \epsilon-\phi+2, \phi-\epsilon+5) & \phi+1 \leq \epsilon \leq \phi+2; \\ (n-\epsilon+3, n-\epsilon+4, \epsilon-\phi+2, \epsilon-\phi) & \phi+3 \leq \epsilon \leq n \end{cases}$$

$$r(z_\epsilon|\Pi) = \begin{cases} (4,4,\phi+3,\phi+3) & \epsilon = 1; \\ (\epsilon+3,\epsilon+2,\phi-\epsilon+4,\phi-\epsilon+6) & 2 \le \epsilon \le \phi; \\ (n-\epsilon+4,\epsilon+2,\epsilon-\phi+3,\phi-\epsilon+6) & \phi+1 \le \epsilon \le \phi+2; \\ (n-\epsilon+4,n-\epsilon+5,\epsilon-\phi+3,\epsilon-\phi+1) & \phi+3 \le \epsilon \le n \end{cases}$$

Case 2: When n is even with $n = 2\phi$ and $\phi \ge 3$.

Consider the subset $\Pi = \{u_1, u_2, u_{\phi+1}, u_{\phi+2}\}$ of $V(B_n)$. The codes for all the nodes of B_n with respect to Π are given below.

$$r(u_\epsilon|\Pi) = \begin{cases} (0,1,\phi,\phi-1) & \epsilon = 1; \\ (1,0,\phi-1,\phi) & \epsilon = 2; \\ (\epsilon-1,\epsilon-2,\phi-\epsilon+1,\phi-\epsilon+2) & 3 \le \epsilon \le \phi; \\ (\phi,\phi-1,0,1) & \epsilon = \phi+1; \\ (\phi-1,\phi,1,0) & \epsilon = \phi+2; \\ (n-\epsilon+1,n-\epsilon+2,\epsilon-\phi-1,\epsilon-\phi-2) & \phi+3 \le \epsilon \le n \end{cases}$$

$$r(v_\epsilon|\Pi) = \begin{cases} (1,2,\phi+1,\phi) & \epsilon = 1; \\ (\epsilon,\epsilon-1,\phi-\epsilon+2,\phi-\epsilon+3) & 2 \le \epsilon \le \phi+1; \\ (n-\epsilon+2,n-\epsilon+3,\epsilon-\phi,\epsilon-\phi-1) & \phi+2 \le \epsilon \le n \end{cases}$$

$$r(x_\epsilon|\Pi) = \begin{cases} (2,2,\phi+1,\phi+1) & \epsilon = 1; \\ (\epsilon+1,\epsilon-1,\phi-\epsilon+2,\phi-\epsilon+3) & 2 \le \epsilon \le \phi; \\ (\phi+1,\phi+1,2,2) & \epsilon = \phi+1; \\ (n-\epsilon+2,n-\epsilon+3,\epsilon-\phi+1,\epsilon-\phi) & \phi+2 \le \epsilon \le n \end{cases}$$

$$r(y_\epsilon|\Pi) = \begin{cases} (3,3,\phi+2,\phi+2) & \epsilon = 1; \\ (\epsilon+2,\epsilon+1,\phi-\epsilon+3,\phi-\epsilon+4) & 2 \le \epsilon \le \phi; \\ (\phi+2,\phi+2,3,3) & \epsilon = \phi+1; \\ (n-\epsilon+3,n-\epsilon+4,\epsilon-\phi+2,\epsilon-\phi+1) & \phi+2 \le \epsilon \le n \end{cases}$$

$$r(z_\epsilon|\Pi) = \begin{cases} (4,4,\phi+3,\phi+3) & \epsilon = 1; \\ (\epsilon+3,\epsilon+2,\phi-\epsilon+4,\phi-\epsilon+5) & 2 \le \epsilon \le \phi; \\ (\phi+3,\phi+3,4,4) & \epsilon = \phi+1; \\ (n-\epsilon+4,n-\epsilon+5,\epsilon-\phi+3,\epsilon-\phi+2) & \phi+2 \le \epsilon \le n \end{cases}$$

The codes in both cases clearly specify that Π is FTMG; therefore, $dim_F(B_n) \le 4$. Hence, from Propositions 1.1 and 2.1, we conclude that $dim_F(B_n) = 4$ for $n \ge 6$. □

2.2. FTMD of C_n

The convex polytope C_n is defined in [10] consists of $3n$ 3-sided faces, n 4-sided faces, n 5-sided faces, and a pair of n-sided faces with the following node and edge sets.

$V(C_n) = \{u_i, v_i, x_i, y_i, z : 1 \le i \le n\}$ and

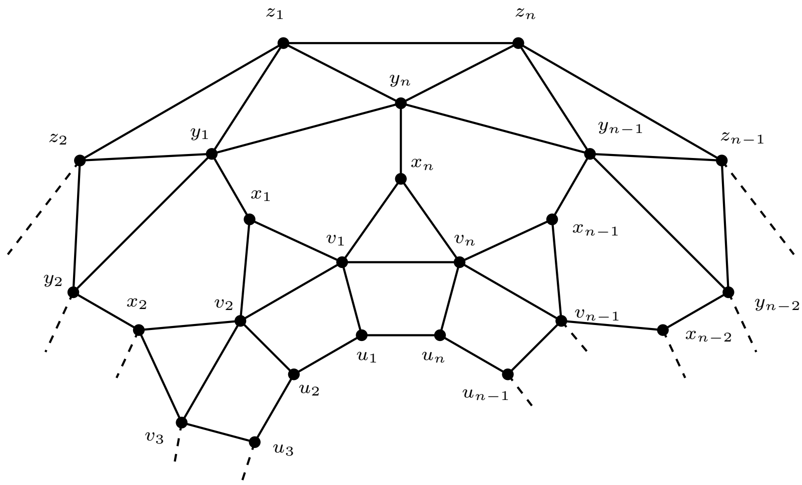

Figure 2. Convex Polytope C_n.

$E(C_n) = \{u_iu_{i+1}; v_iv_{i+1}; y_iy_{i+1}; z_iz_{i+1}; u_iv_i; v_ix_i; v_{i+1}x_i; x_iy_i; y_iz_i; y_iz_{i+1} : 1 \leq i \leq n\}$, where $u_{n+1} = u_1, v_{n+1} = v_1, x_{n+1} = x_1, y_{n+1} = y_1$ and $z_{n+1} = z_1$. The Figure 2 illustrates the graph of C_n.

In the forthcoming result, we compute the FTMD of C_n.

Theorem 2.3. *$dim_F(C_n) = 4$ for $n \geq 6$.*

Proof. The proof comprises two parts.

Case 1: When n is odd with $n = 2\phi + 1$ and $\phi \geq 3$.

Consider the subset $\Pi = \{u_1, u_2, u_{\phi+1}, u_{\phi+3}\}$ of $V(C_n)$. The codes for all the nodes of C_n with respect to Π are given below.

$$r(u_\epsilon|\Pi) = \begin{cases} (0, 1, \phi, \phi - 1) & \epsilon = 1; \\ (1, 0, \phi - 1, \phi) & \epsilon = 2; \\ (\epsilon - 1, \epsilon - 2, \phi - \epsilon + 1, \phi - \epsilon + 3) & 3 \leq \epsilon \leq \phi; \\ (\phi, \phi - 1, 0, 2) & \epsilon = \phi + 1; \\ (\phi, \phi, 1, 1) & \epsilon = \phi + 2; \\ (\phi - 1, \phi, 2, 0) & \epsilon = \phi + 3; \\ (n - \epsilon + 1, n - \epsilon + 2, \epsilon - \phi - 1, \epsilon - \phi - 3) & \phi + 4 \leq \epsilon \leq n \end{cases}$$

$$r(v_\epsilon|\Pi) = \begin{cases} (1, 2, \phi + 1, \phi) & \epsilon = 1; \\ (2, 1, \phi, \phi + 1) & \epsilon = 2; \\ (\epsilon, \epsilon - 1, \phi - \epsilon + 2, \phi - \epsilon + 4) & 3 \leq \epsilon \leq \phi + 1; \\ (\phi + 1, \phi + 1, 2, 2) & \epsilon = \phi + 2; \\ (n - \epsilon + 2, n - \epsilon + 3, \epsilon - \phi, \epsilon - \phi - 2) & \phi + 3 \leq \epsilon \leq n \end{cases}$$

$$r(x_\epsilon|\Pi) = \begin{cases} (2, 2, \phi + 1, \phi + 1) & \epsilon = 1; \\ (\epsilon + 1, \epsilon, \phi - \epsilon + 2, \phi - \epsilon + 4) & 2 \leq \epsilon \leq \phi; \\ (n - \epsilon + 2, \epsilon, \epsilon - \phi + 1, \phi - \epsilon + 4) & \phi + 1 \leq \epsilon \leq \phi + 2; \\ (n - \epsilon + 2, n - \epsilon + 3, \epsilon - \phi + 1, \epsilon - \phi - 1) & \phi + 3 \leq \epsilon \leq n \end{cases}$$

$$r(y_\epsilon|\Pi)=\begin{cases}(3,3,\phi+2,\phi+2) & \epsilon=1;\\ (\epsilon+2,\epsilon+1,\phi-\epsilon+3,\phi-\epsilon+5) & 2\le\epsilon\le\phi;\\ (n-\epsilon+3,\epsilon+1,\epsilon-\phi+2,\phi-\epsilon+5) & \phi+1\le\epsilon\le\phi+2;\\ (n-\epsilon+3,n-\epsilon+4,\epsilon-\phi+2,\epsilon-\phi) & \phi+3\le\epsilon\le n\end{cases}$$

$$r(z_\epsilon|\Pi)=\begin{cases}(4,4,\phi+3,\phi+3) & \epsilon=1;\\ (\epsilon+2,4,\phi-\epsilon+4,\phi+3) & 2\le\epsilon\le3;\\ (\epsilon+2,\epsilon+1,\phi-\epsilon+4,\phi-\epsilon+6) & 4\le\epsilon\le\phi,\phi\ge4;\\ (\phi+3,\epsilon+1,4,\phi-\epsilon+6) & \phi+1\le\epsilon\le\phi+2;\\ (n-\epsilon+4,n-\epsilon+5,\epsilon-\phi+2,4) & \phi+3\le\epsilon\le\phi+4;\\ (n-\epsilon+4,n-\epsilon+5,\epsilon-\phi+2,\epsilon-\phi) & \phi+5\le\epsilon\le n,\phi\ge4\end{cases}$$

Case 2: When n is even with $n=2\phi$ and $\phi\ge3$.

Consider the subset $\Pi=\{u_1,u_2,u_{\phi+1},u_{\phi+2}\}$ of $V(C_n)$. The codes for all the nodes of C_n with respect to Π are given below.

$$r(u_\epsilon|\Pi)=\begin{cases}(0,1,\phi,\phi-1) & \epsilon=1;\\ (1,0,\phi-1,\phi) & \epsilon=2;\\ (\epsilon-1,\epsilon-2,\phi-\epsilon+1,\phi-\epsilon+2) & 3\le\epsilon\le\phi;\\ (\phi,\phi-1,0,1) & \epsilon=\phi+1;\\ (\phi-1,\phi,1,0) & \epsilon=\phi+2;\\ (n-\epsilon+1,n-\epsilon+2,\epsilon-\phi-1,\epsilon-\phi-2) & \phi+3\le\epsilon\le n\end{cases}$$

$$r(v_\epsilon|\Pi)=\begin{cases}(1,2,\phi+1,\phi) & \epsilon=1;\\ (\epsilon,\epsilon-1,\phi-\epsilon+2,\phi-\epsilon+3) & 2\le\epsilon\le\phi+1;\\ (n-\epsilon+2,n-\epsilon+3,\epsilon-\phi,\epsilon-\phi-1) & \phi+2\le\epsilon\le n\end{cases}$$

$$r(x_\epsilon|\Pi)=\begin{cases}(2,2,\phi+1,\phi+1) & \epsilon=1;\\ (\epsilon+1,\epsilon,\phi-\epsilon+2,\phi-\epsilon+3) & 2\le\epsilon\le\phi;\\ (\phi+1,\phi+1,2,2) & \epsilon=\phi+1;\\ (n-\epsilon+2,n-\epsilon+3,\epsilon-\phi+1,\epsilon-\phi) & \phi+2\le\epsilon\le n\end{cases}$$

$$r(y_\epsilon|\Pi)=\begin{cases}(3,3,\phi+2,\phi+2) & \epsilon=1;\\ (\epsilon+2,\epsilon+1,\phi-\epsilon+3,\phi-\epsilon+4) & 2\le\epsilon\le\phi;\\ (\phi+2,\phi+2,3,3) & \epsilon=\phi+1;\\ (n-\epsilon+3,n-\epsilon+4,\epsilon-\phi+2,\epsilon-\phi+1) & \phi+2\le\epsilon\le n\end{cases}$$

$$r(z_\epsilon|\Pi)=\begin{cases}(4,4,\phi+3,\phi+2) & \epsilon=1;\\ (\epsilon+2,4,\phi-\epsilon+4,\phi-\epsilon+5) & 2\le\epsilon\le3;\\ (\epsilon+2,\epsilon+1,\phi-\epsilon+4,\phi-\epsilon+5) & 4\le\epsilon\le\phi,\phi\ge4;\\ (n-\epsilon+4,\epsilon+1,4,4) & \phi+1\le\epsilon\le\phi+2;\\ (n-\epsilon+4,n-\epsilon+5,\epsilon-\phi+2,\epsilon-\phi+1) & \phi+3\le\epsilon\le n\end{cases}$$

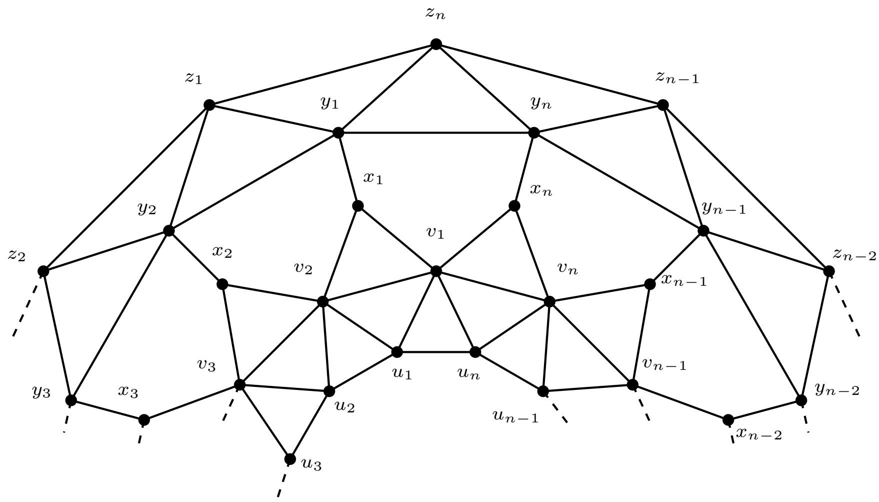

Figure 3. Convex Polytope E_n.

The codes in both cases clearly specify that Π is FTMG; therefore, $dim_F(C_n) \leq 4$. Hence, from Propositions 1.1 and 2.1, we conclude that $dim_F(C_n) = 4$ for $n \geq 6$.

□

2.3. FTMD of E_n

The convex polytope E_n can be obtained from C_n by adding the new edges $u_i v_{i+1}$ in it and consisting of $3n$ 3-sided faces, n 4-sided faces, n 5-sided faces, and a pair of n-sided faces with the following node and edge sets.

$V(E_n) = \{u_i, v_i, x_i, y_i, z : 1 \leq i \leq n\}$ and

$E(E_n) = \{u_i u_{i+1}; v_i v_{i+1}; y_i y_{i+1}; z_i z_{i+1}; u_i v_i; u_i v_{i+1}; v_i x_i; v_{i+1} x_i; x_i y_i; y_i z_i; y_{i+1} z_i : 1 \leq i \leq n\}$, where $u_{n+1} = u_1, v_{n+1} = v_1, x_{n+1} = x_1, y_{n+1} = y_1$ and $z_{n+1} = z_1$. The Figure 3 illustrates the graph of E_n.

In the forthcoming result, we compute the FTMD of E_n.

Theorem 2.4. *$dim_F(E_n) = 4$ for $n \geq 6$.*

Proof. The proof comprises two parts.

Case 1: When n is odd with $n = 2\phi + 1$ and $\phi \geq 3$.

Consider the subset $\Pi = \{u_1, u_2, u_{\phi+1}, u_{\phi+3}\}$ of $V(E_n)$. The codes for all the nodes of E_n with respect to Π are given below.

$$r(u_\epsilon|\Pi) = \begin{cases} (0, 1, \phi, \phi - 1) & \epsilon = 1; \\ (1, 0, \phi - 1, \phi) & \epsilon = 2; \\ (\epsilon - 1, \epsilon - 2, \phi - \epsilon + 1, \phi - \epsilon + 3) & 3 \leq \epsilon \leq \phi; \\ (\phi, \phi - 1, 0, 2) & \epsilon = \phi + 1; \\ (\phi, \phi, 1, 1) & \epsilon = \phi + 2; \\ (\phi - 1, \phi, 2, 0) & \epsilon = \phi + 3; \\ (n - \epsilon + 1, n - \epsilon + 2, \epsilon - \phi - 1, \epsilon - \phi - 3) & \phi + 4 \leq \epsilon \leq n \end{cases}$$

$$r(v_\epsilon|\Pi)=\begin{cases}(1,1,\phi,\phi) & \epsilon=1;\\(2,1,\phi-1,\phi+1) & \epsilon=2;\\(\epsilon,\epsilon-1,\phi-\epsilon+1,\phi-\epsilon+3) & 3\le\epsilon\le\phi;\\(n-\epsilon+1,\epsilon-1,\epsilon-\phi,\phi-\epsilon+3) & \phi+1\le\epsilon\le\phi+2;\\(n-\epsilon+1,n-\epsilon+2,\epsilon-\phi,\epsilon-\phi-2) & \phi+3\le\epsilon\le n\end{cases}$$

$$r(x_\epsilon|\Pi)=\begin{cases}(2,2,\phi,\phi+1) & \epsilon=1;\\(3,2,\phi-1,\phi+1) & \epsilon=2;\\(\epsilon+1,\epsilon,\phi-\epsilon+1,\phi-\epsilon+3) & 3\le\epsilon\le\phi-1,\phi\ge4;\\(\phi+1,\epsilon,2,\phi-\epsilon+3) & \phi\le\epsilon\le\phi+1;\\(n-\epsilon+1,n-\epsilon+2,\epsilon-\phi+1,2) & \phi+2\le\epsilon\le\phi+3;\\(n-\epsilon+1,n-\epsilon+2,\epsilon-\phi+1,\epsilon-\phi-1) & \phi+4\le\epsilon\le n-1,\phi\ge4;\\(2,2,\phi+1,\phi) & \epsilon=n\end{cases}$$

$$r(y_\epsilon|\Pi)=\begin{cases}(3,3,\phi+1,\phi+2) & \epsilon=1;\\(\epsilon+2,\epsilon+1,\phi-\epsilon+2,\phi-\epsilon+4) & 2\le\epsilon\le\phi-1;\\(\phi+2,\epsilon+1,3,\phi-\epsilon+4) & \phi\le\epsilon\le\phi+1;\\(n-\epsilon+2,n-\epsilon+3,\epsilon-\phi+2,3) & \phi+2\le\epsilon\le\phi+3;\\(n-\epsilon+2,n-\epsilon+3,\epsilon-\phi+2,\epsilon-\phi) & \phi+4\le\epsilon\le n-1,\phi\ge4;\\(3,3,\phi+2,\phi+1) & \epsilon=n\end{cases}$$

$$r(z_\epsilon|\Pi)=\begin{cases}(4,4,\phi+3,\phi+3) & \epsilon=1;\\(\epsilon+3,\epsilon+2,\phi-\epsilon+2,\phi-\epsilon+4) & 2\le\epsilon\le\phi-2,\phi\ge4;\\(\epsilon+3,\epsilon+2,4,\phi-\epsilon+4) & \phi-1\le\epsilon\le\phi;\\(n-\epsilon+2,n-\epsilon+3,\epsilon-\phi+3,4) & \phi+1\le\epsilon\le\phi+2,\phi=3;\\(n-\epsilon+2,n-\epsilon+3,\epsilon-\phi+3,4) & \phi+1\le\epsilon\le\phi+3,\phi\ge4;\\(n-\epsilon+2,n-\epsilon+3,\epsilon-\phi+3,\epsilon-\phi+1) & \phi+4\le\epsilon\le n-2,\phi\ge5;\\(4,4,n-\epsilon+\phi+2,\epsilon-\phi+1) & n-1\le\epsilon\le n\end{cases}$$

Case 2: When n is even with $n=2\phi$ and $\phi\ge3$.

Case 2.1: When $n=6$ It is easy to check that $\Pi=\{u_1,u_4,z_1,z_4\}$ is a FTMG.

Case 2.2: When $n>6$ and $\phi\ge4$. Consider the subset $\Pi=\{u_1,u_3,u_{\phi+1},u_{\phi+3}\}$ of $V(E_n)$. The codes for all the nodes of E_n with respect to Π are given below.

$$r(u_\epsilon|\Pi) = \begin{cases} (0,2,\phi,\phi-2) & \epsilon = 1; \\ (1,1,\phi-1,\phi-1) & \epsilon = 2; \\ (2,0,\phi-2,\phi) & \epsilon = 3; \\ (\epsilon-1,\epsilon-3,\phi-\epsilon+1,\phi-\epsilon+3) & 4 \leq \epsilon \leq \phi; \\ (\phi,\phi-2,0,2) & \epsilon = \phi+1; \\ (\phi-1,\phi-1,1,1) & \epsilon = \phi+2; \\ (\phi-2,\phi,2,0) & \epsilon = \phi+3; \\ (n-\epsilon+1,n-\epsilon+3,\epsilon-\phi-1,\epsilon-\phi-3) & \phi+4 \leq \epsilon \leq n \end{cases}$$

$$r(v_\epsilon|\Pi) = \begin{cases} (1,2,\phi,\phi-1) & \epsilon = 1; \\ (2,1,\phi-1,\phi) & \epsilon = 2; \\ (\epsilon,\epsilon-2,\phi-\epsilon+1,\phi-\epsilon+3) & 3 \leq \epsilon \leq \phi; \\ (n-\epsilon+1,\epsilon-1,\epsilon-\phi,\phi-\epsilon+3) & \phi+1 \leq \epsilon \leq \phi+2; \\ (n-\epsilon+1,n-\epsilon+3,\epsilon-\phi,\epsilon-\phi-2) & \phi+3 \leq \epsilon \leq n \end{cases}$$

$$r(x_\epsilon|\Pi) = \begin{cases} (2,2,\phi,\phi) & \epsilon = 1; \\ (\epsilon+1,2,\phi-\epsilon+1,\phi-\epsilon+3) & 2 \leq \epsilon \leq 3; \\ (5,3,\phi-2,\phi-1) & \epsilon = 4, \phi = 4; \\ (5,3,\phi-3,\phi-1) & \epsilon = 4, \phi > 4; \\ (\epsilon+1,\epsilon-1,2,\phi-\epsilon+3) & 5 \leq \epsilon \leq \phi, \phi > 4; \\ (n-\epsilon+1,\epsilon-1,\epsilon-\phi+1,2) & \phi+1 \leq \epsilon \leq \phi+2; \\ (n-\epsilon+1,n-\epsilon+3,\epsilon-\phi+1,\epsilon-\phi-1) & \phi+3 \leq \epsilon \leq n-1; \\ (2,3,\phi+1,\phi-1) & \epsilon = n \end{cases}$$

$$r(y_\epsilon|\Pi) = \begin{cases} (3,3,\phi+1,\phi+1) & \epsilon = 1; \\ (\epsilon+2,3,\phi-\epsilon+2,\phi-\epsilon+4) & 2 \leq \epsilon \leq 3; \\ (6,4,\phi-1,\phi) & \epsilon = 4, \phi = 4; \\ (6,4,\phi-2,\phi) & \epsilon = 4, \phi > 4; \\ (\epsilon+1,\epsilon,3,\phi-\epsilon+4) & 5 \leq \epsilon \leq \phi, \phi > 4; \\ (n-\epsilon+2,\epsilon,\epsilon-\phi+2,3) & \phi+1 \leq \epsilon \leq \phi+2; \\ (n-\epsilon+2,n-\epsilon+4,\epsilon-\phi+2,\epsilon-\phi) & \phi+3 \leq \epsilon \leq n-1; \\ (3,4,\phi+2,\phi) & \epsilon = n \end{cases}$$

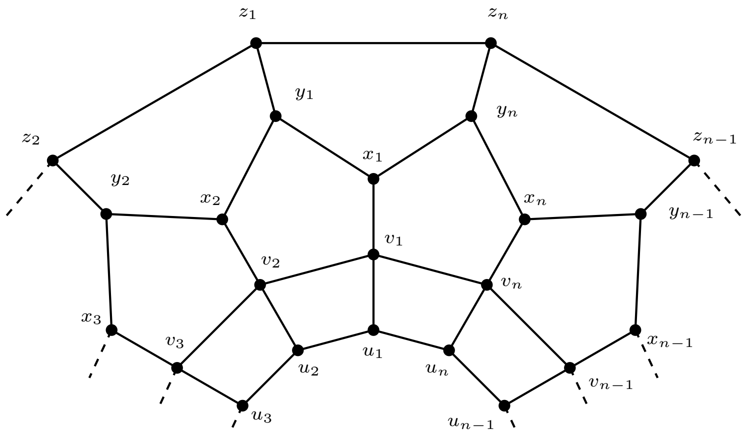

Figure 4. Convex Polytope U_n.

$$r(z_\epsilon|\Pi) = \begin{cases} (4, 4, \phi+1, \phi+2) & \epsilon = 1; \\ (\epsilon+3, 4, \phi-\epsilon+2, \phi-\epsilon+4) & \epsilon = 2; \\ (\epsilon+3, \epsilon+2, \phi-\epsilon+2, \phi-\epsilon+4) & 3 \leq \epsilon \leq \phi-2, \phi > 4; \\ (\phi+2, \phi, 4, 5) & \epsilon = \phi-1; \\ (n-\epsilon+2, \epsilon+1, 4, 4) & \phi \leq \epsilon \leq \phi+1; \\ (4, n-\epsilon+4, \epsilon-\phi+3, 4) & \phi+2 \leq \epsilon \leq \phi+3, \phi = 4; \\ (n-\epsilon+2, n-\epsilon+4, \epsilon-\phi+3, 4) & \phi+2 \leq \epsilon \leq \phi+3, \phi > 4; \\ (n-\epsilon+2, n-\epsilon+4, \epsilon-\phi+3, \epsilon-\phi+1) & \phi+4 \leq \epsilon \leq n-2, \phi > 5; \\ (4, n-\epsilon+4, \phi+2, \epsilon-\phi+1) & n-1 \leq \epsilon \leq n \end{cases}$$

The codes in both cases clearly specify that Π is FTMG; therefore, $dim_F(E_n) \leq 4$. Hence, from Propositions 1.1 and 2.1, we conclude that $dim_F(E_n) = 4$ for $n \geq 6$.

□

2.4. FTMD of U_n

The convex polytope U_n are defined in [9] has n 4-sided faces, $2n$ 5-sided faces, and a pair of n-sided faces with the following node and edge sets.

$V(U_n) = \{u_i, v_i, x_i, y_i, z : 1 \leq i \leq n\}$ and

$E(U_n) = \{u_iu_{i+1}; v_iv_{i+1}; z_iz_{i+1}; u_iv_i; v_ix_i; x_iy_i; x_{i+1}y_i; y_iz_i : 1 \leq i \leq n\}$, where $u_{n+1} = u_1, v_{n+1} = v_1, x_{n+1} = x_1, y_{n+1} = y_1$ and $z_{n+1} = z_1$. The Figure 4 illustrates the graph of U_n.

In the forthcoming result, we compute the FTMD of U_n.

Theorem 2.5. *$dim_F(U_n) = 4$ for $n \geq 6$.*

Proof. The proof comprises two parts.

Case 1: When n is odd with $n = 2\phi + 1$ and $\phi \geq 3$.

Consider the subset $\Pi = \{u_1, u_2, u_{\phi+1}, u_{\phi+3}\}$ of $V(U_n)$. The codes for all the nodes of U_n with respect to Π are given below.

$$r(u_\epsilon|\Pi) = \begin{cases} (0, 1, \phi, \phi-1) & \epsilon = 1; \\ (1, 0, \phi-1, \phi) & \epsilon = 2; \\ (\epsilon-1, \epsilon-2, \phi-\epsilon+1, \phi-\epsilon+3) & 3 \le \epsilon \le \phi; \\ (\phi, \phi-1, 0, 2) & \epsilon = \phi+1; \\ (\phi, \phi, 1, 1) & \epsilon = \phi+2; \\ (\phi-1, \phi, 2, 0) & \epsilon = \phi+3; \\ (n-\epsilon+1, n-\epsilon+2, \epsilon-\phi-1, \epsilon-\phi-3) & \phi+4 \le \epsilon \le n \end{cases}$$

$$r(v_\epsilon|\Pi) = \begin{cases} (1, 2, \phi+1, \phi) & \epsilon = 1; \\ (2, 1, \phi, \phi+1) & \epsilon = 2; \\ (\epsilon, \epsilon-1, \phi-\epsilon+2, \phi-\epsilon+4) & 3 \le \epsilon \le \phi+1; \\ (\phi+1, \phi+1, 2, 2) & \epsilon = \phi+2; \\ (n-\epsilon+2, n-\epsilon+3, \epsilon-\phi, \epsilon-\phi-2) & \phi+3 \le \epsilon \le n \end{cases}$$

$$r(x_\epsilon|\Pi) = \begin{cases} (2, 3, \phi+2, \phi+1) & \epsilon = 1; \\ (\epsilon+1, \epsilon, \phi-\epsilon+3, \phi+2) & 2 \le \epsilon \le 3; \\ (\epsilon+1, \epsilon, \phi-\epsilon+3, \phi-\epsilon+5) & 4 \le \epsilon \le \phi+1; \\ (\phi+2, \phi+2, 3, 3) & \epsilon = \phi+2; \\ (n-\epsilon+3, n-\epsilon+4, \epsilon-\phi+1, \epsilon-\phi-1) & \phi+3 \le \epsilon \le n \end{cases}$$

$$r(y_\epsilon|\Pi) = \begin{cases} (3, 3, \phi+2, \phi+2) & \epsilon = 1; \\ (\epsilon+2, \epsilon+1, \phi-\epsilon+3, \phi-\epsilon+5) & 2 \le \epsilon \le \phi; \\ (n-\epsilon+3, \epsilon+1, \epsilon-\phi+2, \phi-\epsilon+5) & \phi+1 \le \epsilon \le \phi+2; \\ (n-\epsilon+3, n-\epsilon+4, \epsilon-\phi+2, \epsilon-\phi) & \phi+3 \le \epsilon \le n \end{cases}$$

$$r(z_\epsilon|\Pi) = \begin{cases} (4, 4, \phi+3, \phi+3) & \epsilon = 1; \\ (\epsilon+3, \epsilon+2, \phi-\epsilon+4, \phi-\epsilon+6) & 2 \le \epsilon \le \phi; \\ (n-\epsilon+4, \epsilon+2, \epsilon-\phi+3, \phi-\epsilon+6) & \phi+1 \le \epsilon \le \phi+2; \\ (n-\epsilon+4, n-\epsilon+5, \epsilon-\phi+3, \epsilon-\phi+1) & \phi+3 \le \epsilon \le n \end{cases}$$

Case 2: When n is even with $n = 2\phi$ and $\phi \ge 3$.

Consider the subset $\Pi = \{u_1, u_2, u_{\phi+1}, u_{\phi+2}\}$ of $V(U_n)$. The codes for all the nodes of U_n with respect to Π are given below.

$$r(u_\epsilon|\Pi) = \begin{cases} (0, 1, \phi, \phi-1) & \epsilon = 1; \\ (1, 0, \phi-1, \phi) & \epsilon = 2; \\ (\epsilon-1, \epsilon-2, \phi-\epsilon+1, \phi-\epsilon+2) & 3 \le \epsilon \le \phi; \\ (\phi, \phi-1, 0, 1) & \epsilon = \phi+1; \\ (\phi-1, \phi, 1, 0) & \epsilon = \phi+2; \\ (n-\epsilon+1, n-\epsilon+2, \epsilon-\phi-1, \epsilon-\phi-2) & \phi+3 \le \epsilon \le n \end{cases}$$

$$r(v_\epsilon|\Pi) = \begin{cases} (1, 2, \phi + 1, \phi) & \epsilon = 1; \\ (\epsilon, \epsilon - 1, \phi - \epsilon + 2, \phi - \epsilon + 3) & 2 \leq \epsilon \leq \phi + 1; \\ (n - \epsilon + 2, n - \epsilon + 3, \epsilon - \phi, \epsilon - \phi - 1) & \phi + 2 \leq \epsilon \leq n \end{cases}$$

$$r(x_\epsilon|\Pi) = \begin{cases} (2, 3, \phi + 2, \phi + 1) & \epsilon = 1; \\ (\epsilon + 1, \epsilon, \phi - \epsilon + 3, \phi - \epsilon + 4) & 2 \leq \epsilon \leq \phi; \\ (\phi + 2, \phi + 1, 2, 3) & \epsilon = \phi + 1; \\ (n - \epsilon + 3, n - \epsilon + 4, \epsilon - \phi + 1, \epsilon - \phi) & \phi + 2 \leq \epsilon \leq n \end{cases}$$

$$r(y_\epsilon|\Pi) = \begin{cases} (3, 3, \phi + 2, \phi + 2) & \epsilon = 1; \\ (\epsilon + 2, \epsilon + 1, \phi - \epsilon + 3, \phi - \epsilon + 4) & 2 \leq \epsilon \leq \phi; \\ (\phi + 2, \phi + 2, 3, 3) & \epsilon = \phi + 1; \\ (n - \epsilon + 3, n - \epsilon + 4, \epsilon - \phi + 2, \epsilon - \phi + 1) & \phi + 2 \leq \epsilon \leq n \end{cases}$$

$$r(z_\epsilon|\Pi) = \begin{cases} (4, 4, \phi + 3, \phi + 3) & \epsilon = 1; \\ (\epsilon + 3, \epsilon + 2, \phi - \epsilon + 4, \phi - \epsilon + 5) & 2 \leq \epsilon \leq \phi; \\ (\phi + 3, \phi + 3, 4, 4) & \epsilon = \phi + 1; \\ (n - \epsilon + 4, n - \epsilon + 5, \epsilon - \phi + 3, \epsilon - \phi + 2) & \phi + 2 \leq \epsilon \leq n \end{cases}$$

The codes in both cases clearly specify that Π is FTMG; therefore, $dim_F(U_n) \leq 4$. Hence, from Propositions 1.1 and 2.1, we conclude that $dim_F(U_n) = 4$ for $n \geq 6$.

□

3. Conclusion

In this manuscript, we infer that the FTMD of the convex polytopes B_n, C_n, E_n and U_n is 4 and does not depend on the order of these families. FTMB provides codes of least length for the graphs which are effective even in the presence of faults. The applications of FTMB can be realized in the deployment of sensors, robot navigation, transportation and interconnection networks.

Open Problem: Compute the FTMD of generalized Petersen graphs.

Open Problem: Compute the FTMD of mesh related networks.

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