Communications in Advanced Mathematical Sciences
Vol. 7, No. 1, 56-70, 2024
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1362711

Refinements and Reverses of Tensorial and Hadamard Product Inequalities for Selfadjoint Operators in Hilbert Spaces Related to Young's Result

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Abstract

Let H be a Hilbert space. In this paper we show among others that, if the selfadjoint operators A and B satisfy the condition $0 < m \leq A, B \leq M$, for some constants m, M, then

$$\begin{aligned}0 &\leq \frac{m}{M^2}\nu(1-\nu)\left(\frac{A^2\otimes 1+1\otimes B^2}{2}-A\otimes B\right)\\ &\leq (1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\\ &\leq \frac{M}{m^2}\nu(1-\nu)\left(\frac{A^2\otimes 1+1\otimes B^2}{2}-A\otimes B\right)\end{aligned}$$

for all $\nu \in [0,1]$. We also have the inequalities for Hadamard product

$$\begin{aligned}0 &\leq \frac{m}{M^2}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1-A\circ B\right)\\ &\leq [(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}\\ &\leq \frac{M}{m^2}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1-A\circ B\right)\end{aligned}$$

for all $\nu \in [0,1]$.

Keywords: Tensorial product, Hadamard product, Selfadjoint operators, Convex functions
2010 AMS: Primary 47A63, Secondary 47A99

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Received: 19-09-2023, **Accepted:** 19-02-2024, **Available online:** 04-03-2024
How to cite this article: S. S. Dragomir, *Refinements and Reverses of Tensorial and Hadamard Product Inequalities for Selfadjoint Operators in Hilbert Spaces Related to Young's Result*, Commun. Adv. Math. Sci., 7(1) (2024) 56-70.

1. Introduction

The famous *Young inequality* for scalars says that if a, $b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν*-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [1]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \leq S\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.3}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.3) is due to Tominaga [2] while the first one is due to Furuichi [3].

Kittaneh and Manasrah [4, 5] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \leq R\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{1.4}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

We also consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \leq K^R\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \tag{1.6}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [6] while the second by Liao et al. [7].

In [6] the authors also showed that $K^r(h) \geq S(h^r)$ for $h > 0$ and $r \in \left[0, \frac{1}{2}\right]$ implying that the lower bound in (1.6) is better than the lower bound from (1.3).

In the recent paper [8] we obtained the following reverses of Young's inequality as well:

$$0 \leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \leq \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{1.7}$$

and

$$1 \leq \frac{(1-\nu) a + \nu b}{a^{1-\nu} b^{\nu}} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right], \tag{1.8}$$

where $a, b > 0$, $\nu \in [0,1]$.

In [9], we obtained the following Young related inequalities:

Theorem 1.1. *For any $a, b > 0$ and $\nu \in [0,1]$ we have*

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \min\{a,b\} &\leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \\ &\leq \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a,b\} \end{aligned} \tag{1.9}$$

and

$$\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\right] \le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \le \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}\right]. \tag{1.10}$$

For an equivalent form and a different approach in proving the results (1.9) and (1.10) see [10].

The second inequalities in (1.9) and (1.10) are better than the corresponding results obtained by Furuichi and Minculete in [11] where instead of constant $\frac{1}{2}$ they had the constant 1. Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1,...,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of A_i is contained in I_i for $i = 1,...,k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1,...,k$; by following [12], we define

$$f(A_1,...,A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1,...,\lambda_k)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.11}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [12] extends the definition of Korányi [13] for functions of two variables and have the property that

$$f(A_1,...,A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1,...,t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \ge (\le) f(s) f(t) \text{ for all } s,t \in [0,\infty)$$

and if f is continuous on $[0,\infty)$, then [14, p. 173]

$$f(A \otimes B) \ge (\le) f(A) \otimes f(B) \text{ for all } A,\ B \ge 0. \tag{1.12}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.13}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0,1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of # and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B)\#(B \otimes A).$$

In 2007, Wada [15] obtained the following *Callebaut type inequalities* for tensorial product

$$(A\#B)\otimes(A\#B)\leq\frac{1}{2}\left[(A\#_{\alpha}B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_{\alpha}B)\right]\leq\frac{1}{2}(A\otimes B+B\otimes A) \tag{1.14}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\left\langle (A\circ B)e_j,e_j\right\rangle=\left\langle Ae_j,e_j\right\rangle\left\langle Be_j,e_j\right\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [16], we have the representation

$$A\circ B=\mathscr{U}^{*}(A\otimes B)\mathscr{U} \tag{1.15}$$

where $\mathscr{U}:H\to H\otimes H$ is the isometry defined by $\mathscr{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [14, p. 173]

$$f(A\circ B)\geq(\leq)f(A)\circ f(B)\ \text{ for all } A,\ B\geq 0. \tag{1.16}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1\ \text{for } A,\ B\geq 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\geq 1\ \text{for } A>0. \tag{1.17}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [17] showed that

$$A\circ B\leq\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2}\ \text{ for } A,\ B\geq 0$$

and Aujla and Vasudeva [18] gave an alternative upper bound

$$A\circ B\leq\left(A^2\circ B^2\right)^{1/2}\ \text{ for } A,\ B\geq 0.$$

It has been shown in [19] that $\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2}$ and $\left(A^2\circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by these results, in this paper we provide among others some upper and lower bounds for the Young differences

$$(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}$$

and

$$[(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}$$

for $\nu\in[0,1]$ and $A,\ B>0$.

2. Main Results

The first main result is as follows:

Theorem 2.1. *Assume that the selfadjoint operators A and B satisfy the condition $0<m\leq A,\ B\leq M$, then*

$$\begin{aligned}0&\leq\frac{1}{2}m\nu(1-\nu)\left[\left(\ln^2 A\right)\otimes 1+1\otimes\left(\ln^2 B\right)-2\ln A\otimes\ln B\right]\\&\leq(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\\&\leq\frac{1}{2}M\nu(1-\nu)\left[\left(\ln^2 A\right)\otimes 1+1\otimes\left(\ln^2 B\right)-2\ln A\otimes\ln B\right]\\&\leq\frac{1}{2}\nu(1-\nu)M(\ln M-\ln m)^2\end{aligned} \tag{2.1}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \le \frac{1}{8} m \left[\left(\ln^2 A \right) \otimes 1 + 1 \otimes \left(\ln^2 B \right) - 2 \ln A \otimes \ln B \right] \tag{2.2}$$
$$\le \frac{A \otimes 1 + 1 \otimes B}{2} - A^{1/2} \otimes B^{1/2}$$
$$\le \frac{1}{8} M \left[\left(\ln^2 A \right) \otimes 1 + 1 \otimes \left(\ln^2 B \right) - 2 \ln A \otimes \ln B \right]$$
$$\le \frac{1}{8} M \left(\ln M - \ln m \right)^2 .$$

Proof. If $t,\, s \in [m,M] \subset (0,\infty)$, then by (1.9) we get

$$0 \le \frac{1}{2} m \nu (1-\nu) (\ln t - \ln s)^2 \le (1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \tag{2.3}$$
$$\le \frac{1}{2} M \nu (1-\nu) (\ln t - \ln s)^2$$
$$\le \frac{1}{2} M \nu (1-\nu) (\ln M - \ln m)^2 .$$

If

$$A = \int_m^M t dE(t) \text{ and } B = \int_m^M s dF(s)$$

are the spectral resolutions of A and B, then by taking in (2.3) the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$, we get

$$0 \le \frac{1}{2} m \nu (1-\nu) \int_m^M \int_m^M (\ln t - \ln s)^2 dE(t) \otimes dF(s) \tag{2.4}$$
$$\le \int_m^M \int_m^M \left[(1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \right] dE(t) \otimes dF(s)$$
$$\le \frac{1}{2} M \nu (1-\nu) \int_m^M \int_m^M (\ln t - \ln s)^2 dE(t) \otimes dF(s)$$
$$\le \frac{1}{8} M (\ln M - \ln m)^2 \int_m^M \int_m^M dE(t) \otimes dF(s) .$$

Now, observe that, by (1.11)

$$\begin{aligned} \int_m^M \int_m^M (\ln t - \ln s)^2 dE(t) \otimes dF(s) &= \int_m^M \int_m^M \left(\ln^2 t - 2 \ln t \ln s + \ln^2 s \right) dE(t) \otimes dF(s) \\ &= \int_m^M \int_m^M \ln^2 t dE(t) \otimes dF(s) + \int_m^M \int_m^M \ln^2 s dE(t) \otimes dF(s) \\ &\quad - 2 \int_m^M \int_m^M \ln t \ln s dE(t) \otimes dF(s) \\ &= \left(\ln^2 A \right) \otimes 1 + 1 \otimes \left(\ln^2 B \right) - 2 \ln A \otimes \ln B, \end{aligned}$$

$$\begin{aligned} \int_m^M \int_m^M \left[(1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \right] dE(t) \otimes dF(s) &= (1-\nu) \int_m^M \int_m^M t dE(t) \otimes dF(s) + \nu \int_m^M \int_m^M s dE(t) \otimes dF(s) \\ &\quad - \int_m^M \int_m^M t^{1-\nu} s^{\nu} dE(t) \otimes dF(s) \\ &= (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \end{aligned}$$

and

$$\int_m^M \int_m^M dE(t) \otimes dF(s) = 1 \otimes 1 = 1.$$

By employing (2.4), we then get the desired result (2.1). □

Corollary 2.2. *With the assumptions of Theorem 2.1,*

$$\begin{aligned}0 &\leq \frac{1}{2}m\nu(1-\nu)\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right] \\ &\leq [(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu} \\ &\leq \frac{1}{2}M\nu(1-\nu)\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right] \\ &\leq \frac{1}{2}\nu(1-\nu)M(\ln M-\ln m)^2\end{aligned} \tag{2.5}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}0 &\leq \frac{1}{8}m\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right] \\ &\leq \frac{A+B}{2}\circ 1-A^{1/2}\circ B^{1/2} \\ &\leq \frac{1}{8}M\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right] \\ &\leq \frac{1}{8}M(\ln M-\ln m)^2.\end{aligned} \tag{2.6}$$

Proof. The proof follows from Theorem 2.1 by taking to the left $\mathscr{U}^*$, to the right $\mathscr{U}$, where $\mathscr{U}: H\to H\otimes H$ is the isometry defined by $\mathscr{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$ and utilizing the representation (1.15). □

Remark 2.3. *If we take $B=A$ in Corollary 2.2, then we get*

$$\begin{aligned}0 &\leq m\nu(1-\nu)\left[\left(\ln^2 A\right)\circ 1-\ln A\circ\ln A\right]\leq A\circ 1-A^{1-\nu}\circ A^{\nu} \\ &\leq M\nu(1-\nu)\left[\left(\ln^2 A\right)\circ 1-\ln A\circ\ln A\right] \\ &\leq \frac{1}{2}\nu(1-\nu)M(\ln M-\ln m)^2\end{aligned} \tag{2.7}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}0 &\leq \frac{1}{4}m\left[\left(\ln^2 A\right)\circ 1-\ln A\circ\ln A\right]\leq A\circ 1-A^{1/2}\circ A^{1/2} \\ &\leq \frac{1}{4}M\left[\left(\ln^2 A\right)\circ 1-\ln A\circ\ln A\right]\leq \frac{1}{8}M(\ln M-\ln m)^2.\end{aligned} \tag{2.8}$$

Theorem 2.4. *With the assumptions of Theorem 2.1, we have*

$$\begin{aligned}0 &\leq \frac{m}{2M^2}\nu(1-\nu)\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right) \\ &\leq (1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu} \\ &\leq \frac{M}{2m^2}\nu(1-\nu)\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right)\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2\end{aligned} \tag{2.9}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}0 &\leq \frac{m}{8M^2}\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right) \\ &\leq \frac{A\otimes 1+1\otimes B}{2}-A^{1/2}\otimes B^{1/2} \\ &\leq \frac{M}{8m^2}\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right)\leq \frac{M}{8m^2}(M-m)^2.\end{aligned} \tag{2.10}$$

Proof. We observe that

$$0<\frac{1}{\max\{a,b\}}\leq\frac{\ln a-\ln b}{a-b}\leq\frac{1}{\min\{a,b\}},$$

which implies that

$$0<\frac{1}{\max^2\{a,b\}}\leq\left(\frac{\ln a-\ln b}{a-b}\right)^2\leq\frac{1}{\min^2\{a,b\}}$$

for all $a,b>0$.

By making use of (1.9), we derive

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)(b-a)^2\frac{\min\{a,b\}}{\max^2\{a,b\}}\\&\leq\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\min\{a,b\}\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\\&\leq\frac{1}{2}\nu(1-\nu)(b-a)^2\frac{\max\{a,b\}}{\min^2\{a,b\}}.\end{aligned}\tag{2.11}$$

If $t,\, s\in[m,M]\subset(0,\infty)$, then by (2.11) we get

$$\begin{aligned}0&\leq\frac{m}{2M^2}\nu(1-\nu)(t-s)^2\leq(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}\\&\leq\frac{M}{2m^2}\nu(1-\nu)(t-s)^2.\end{aligned}\tag{2.12}$$

If

$$A=\int_m^M t\,dE(t)\text{ and }B=\int_m^M s\,dF(s)$$

are the spectral resolutions of A and B, then by taking in (2.12) the double integral $\int_m^M\int_m^M$ over $dE(t)\otimes dF(s)$, we get

$$\begin{aligned}0&\leq\frac{m}{2M^2}\nu(1-\nu)\int_m^M\int_m^M(t-s)^2E(t)\otimes dF(s)\\&\leq\int_m^M\int_m^M\left[(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}\right]E(t)\otimes dF(s)\\&\leq\frac{M}{2m^2}\nu(1-\nu)\int_m^M\int_m^M(t-s)^2E(t)\otimes dF(s).\end{aligned}\tag{2.13}$$

Since, by (1.11)

$$\begin{aligned}\int_m^M\int_m^M(t-s)^2E(t)\otimes dF(s)&=\int_m^M\int_m^M\left(t^2-2ts+s^2\right)E(t)\otimes dF(s)\\&=\int_m^M\int_m^M t^2E(t)\otimes dF(s)+\int_m^M\int_m^M s^2E(t)\otimes dF(s)-\int_m^M\int_m^M 2tsE(t)\otimes dF(s)\\&=A^2\otimes 1+1\otimes B^2-2A\otimes B,\end{aligned}$$

then by (2.13) we derive the first part of (2.9).

The last part follows by the fact that

$$(t-s)^2\leq(M-m)^2$$

for all $t,\, s\in[m,M]$. □

Corollary 2.5. *With the assumptions of Theorem 2.1, we have the following inequalities for the Hadamard product*

$$\begin{aligned} 0 &\le \frac{m}{M^2}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \\ &\le [(1-\nu)A+\nu B]\circ 1 - A^{1-\nu}\circ B^{\nu} \\ &\le \frac{M}{m^2}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \le \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 \end{aligned} \tag{2.14}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned} 0 &\le \frac{m}{4M^2}\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \le \frac{A+B}{2}\circ 1 - A^{1/2}\circ B^{1/2} \\ &\le \frac{M}{4m^2}\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \le \frac{M}{8m^2}(M-m)^2. \end{aligned} \tag{2.15}$$

The proof of this corollary is similar to the one of Corollary 2.2 by utilizing Theorem 2.4 and we omit the details.

Remark 2.6. *If we take $B=A$ in Corollary 2.5, then we get*

$$\begin{aligned} 0 &\le \frac{m}{M^2}\nu(1-\nu)\left(A^2\circ 1 - A\circ A\right) \le A - A^{1-\nu}\circ A^{\nu} \\ &\le \frac{M}{m^2}\nu(1-\nu)\left(A^2\circ 1 - A\circ A\right) \le \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 \end{aligned} \tag{2.16}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned} 0 &\le \frac{m}{4M^2}\left(A^2\circ 1 - A\circ A\right) \le A\circ 1 - A^{1/2}\circ A^{1/2} \\ &\le \frac{M}{4m^2}\left(A^2\circ 1 - A\circ A\right) \le \frac{M}{8m^2}(M-m)^2. \end{aligned} \tag{2.17}$$

Further, we also have:

Theorem 2.7. *Assume that the selfadjoint operators A and B satisfy the condition $0<A,\, B\le M$, then*

$$\begin{aligned} 0 &\le (1-\nu)A\otimes 1 + \nu 1\otimes B - A^{1-\nu}\otimes B^{\nu} \\ &\le M\nu(1-\nu)\left(\frac{A^{-1}\otimes B + A\otimes B^{-1}}{2} - 1\right) \end{aligned} \tag{2.18}$$

for all $\nu\in[0,1]$.

In particular,

$$0 \le \frac{A\otimes 1 + 1\otimes B}{2} - A^{1/2}\otimes B^{1/2} \le \frac{1}{4}M\left(\frac{A^{-1}\otimes B + A\otimes B^{-1}}{2} - 1\right). \tag{2.19}$$

Proof. Recall that if $a,\, b>0$ and

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } a\ne b, \\ b \text{ if } a=b \end{cases}$$

is the *logarithmic mean* and $G(a,b):=\sqrt{ab}$ is the *geometric mean*, then $L(a,b)\ge G(a,b)$ for all $a,\, b>0$.

Then from (1.9) we have for $a\ne b$ that

$$\begin{aligned} (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} &\le \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a,b\} \\ &= \frac{1}{2}\nu(1-\nu)(b-a)^2\left(\frac{\ln a - \ln b}{b-a}\right)^2 \max\{a,b\} \\ &\le \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{ab}\max\{a,b\} \\ &= \frac{1}{2}\nu(1-\nu)\left(\frac{b}{a}+\frac{a}{b}-2\right)\max\{a,b\}, \end{aligned}$$

which implies that

$$(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq\frac{1}{2}\nu(1-\nu)\left(\frac{b}{a}+\frac{a}{b}-2\right)\max\{a,b\} \tag{2.20}$$

for all $a,\,b>0$.

If $t,\,s\in[m,M]\subset(0,\infty)$, then by (2.20) we get

$$\begin{aligned}(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}&\leq\frac{1}{2}\nu(1-\nu)\left(\frac{s}{t}+\frac{t}{s}-2\right)\max\{t,s\}\\&\leq\frac{1}{2}M\nu(1-\nu)\left(\frac{s}{t}+\frac{t}{s}-2\right).\end{aligned} \tag{2.21}$$

By taking in (2.21) the double integral $\int_m^M\int_m^M$ over $dE(t)\otimes dF(s)$, we get

$$\int_m^M\int_m^M\left[(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}\right]dE(t)\otimes dF(s)\leq\frac{1}{2}M\nu(1-\nu)\int_m^M\int_m^M\left(\frac{s}{t}+\frac{t}{s}-2\right)dE(t)\otimes dF(s). \tag{2.22}$$

Since

$$\begin{aligned}\int_m^M\int_m^M\left(\frac{s}{t}+\frac{t}{s}-2\right)dE(t)\otimes dF(s)=&\int_m^M\int_m^M t^{-1}sE(t)\otimes dF(s)+\int_m^M\int_m^M ts^{-1}dE(t)\otimes dF(s)\\&-\int_m^M\int_m^M dE(t)\otimes dF(s)\\=&A^{-1}\otimes B+A\otimes B^{-1}-2,\end{aligned}$$

hence by (2.22) we derive (2.18). □

Corollary 2.8. *With the assumptions of Theorem 2.7, we have the inequalities for the Hadamard product*

$$\begin{aligned}0&\leq[(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}\\&\leq M\nu(1-\nu)\left(\frac{A^{-1}\circ B+A\circ B^{-1}}{2}-1\right)\end{aligned} \tag{2.23}$$

for all $\nu\in[0,1]$.

In particular,

$$0\leq\frac{A+B}{2}\circ 1-A^{1/2}\circ B^{1/2}\leq\frac{1}{4}M\left(\frac{A^{-1}\circ B+A\circ B^{-1}}{2}-1\right). \tag{2.24}$$

The proof of this corollary is similar to the one of Corollary 2.2 by utilizing Theorem 2.7.

We observe that, if we take $B=A$ in Corollary 2.8, then we get

$$0\leq A\circ 1-A^{1-\nu}\circ A^{\nu}\leq M\nu(1-\nu)\left(A^{-1}\circ A-1\right) \tag{2.25}$$

for all $\nu\in[0,1]$.

In particular,

$$0\leq A\circ 1-A^{1/2}\circ A^{1/2}\leq\frac{1}{8}M\left(A^{-1}\circ A-1\right). \tag{2.26}$$

We also have the following multiplicative results:

Theorem 2.9. *Assume that the selfadjoint operators A and B satisfy the condition* $0<m\leq A,\,B\leq M$, *then*

$$\begin{aligned}A^{1-\nu}\otimes B^{\nu}&\leq\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right]A^{1-\nu}\otimes B^{\nu}\\&\leq(1-\nu)A\otimes 1+\nu 1\otimes B\\&\leq\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right]A^{1-\nu}\otimes B^{\nu}\end{aligned} \tag{2.27}$$

for all $\nu \in [0,1]$.

In particular,

$$A^{1-\nu} \otimes B^{\nu} \le \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] A^{1/2} \otimes B^{1/2} \le \frac{A \otimes 1 + 1 \otimes B}{2} \le \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] A^{1/2} \otimes B^{1/2}. \tag{2.28}$$

Proof. Since

$$\frac{(b-a)^2}{\max^2\{a,b\}} = \left(\frac{\max\{a,b\} - \min\{a,b\}}{\max\{a,b\}}\right)^2 = \left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2$$

and

$$\frac{(b-a)^2}{\min^2\{a,b\}} = \left(\frac{\max\{a,b\} - \min\{a,b\}}{\min\{a,b\}}\right)^2 = \left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2,$$

hence by (1.10) we derive

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \le \frac{(1-\nu)a + \nu b}{a^{1-\nu} b^{\nu}} \le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right]. \tag{2.29}$$

If $t,\ s \in [m,M] \subset (0,\infty)$, then by (2.29) we get

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right] t^{1-\nu} s^{\nu} \le (1-\nu)t + \nu s \le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right] t^{1-\nu} s^{\nu}. \tag{2.30}$$

Now, if we take the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$ in (2.30), we derive the desired result (2.27). □

Corollary 2.10. *With the assumptions of Theorem 2.9, we have the inequalities for Hadamard product*

$$A^{1-\nu} \circ B^{\nu} \le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right] A^{1-\nu} \circ B^{\nu} \le (1-\nu)A + \nu B \le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right] A^{1-\nu} \circ B^{\nu} \tag{2.31}$$

for all $\nu \in [0,1]$.

In particular,

$$A^{1/2} \circ B^{1/2} \le \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] A^{1/2} \circ B^{1/2} \le \frac{A+B}{2} \circ 1 \le \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] A^{1/2} \circ B^{1/2}. \tag{2.32}$$

The proof of this corollary is similar to the one of Corollary 2.2 by utilizing Theorem 2.9.

If we take $B = A$ in Corollary 2.10, then we get the following inequalities for one operator A satisfying the condition $0 < m \leq A \leq M$,

$$\begin{aligned} A^{1-\nu} \circ A^{\nu} &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right] A^{1-\nu} \circ A^{\nu} \\ &\leq A \circ 1 \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right] A^{1-\nu} \circ A^{\nu} \end{aligned} \tag{2.33}$$

for all $\nu \in [0,1]$.

In particular,

$$\begin{aligned} A^{1/2} \circ A^{1/2} &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] A^{1/2} \circ A^{1/2} \\ &\leq A \circ 1 \\ &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] A^{1/2} \circ A^{1/2}. \end{aligned} \tag{2.34}$$

3. Inequalities for Sums

We also have the following inequalities for sums of operators:

Proposition 3.1. *Assume that $0 < m \leq A_i, B_j \leq M$ and $p_i, q_j \geq 0$ for $i \in \{1, ..., n\}$, $j \in \{1, ..., k\}$, and put $P_n := \sum_{i=1}^n p_i$, $Q_k := \sum_{j=1}^k q_j$. Then*

$$\begin{aligned} 0 &\leq \frac{m}{2M^2}\nu(1-\nu)\left[Q_k\left(\sum_{i=1}^n p_i A_i^2\right) \otimes 1 + P_n 1 \otimes \left(\sum_{j=1}^k q_j B_j^2\right) - 2\left(\sum_{i=1}^n p_i A_i\right) \otimes \left(\sum_{j=1}^k q_j B_j\right)\right] \\ &\leq (1-\nu) Q_k \left(\sum_{i=1}^n p_i A_i\right) \otimes 1 + \nu P_n 1 \otimes \left(\sum_{j=1}^k q_j B_j\right) - \left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^k q_j B_j^{\nu}\right) \\ &\leq \frac{M}{2m^2}\nu(1-\nu)\left[Q_k\left(\sum_{i=1}^n p_i A_i^2\right) \otimes 1 + P_n 1 \otimes \left(\sum_{j=1}^k q_j B_j^2\right) - 2\left(\sum_{i=1}^n p_i A_i\right) \otimes \left(\sum_{j=1}^k q_j B_j\right)\right] \\ &\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 P_n Q_k \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} 0 &\leq (1-\nu) Q_k \left(\sum_{i=1}^n p_i A_i\right) \otimes 1 + \nu P_n 1 \otimes \left(\sum_{j=1}^k q_j B_j\right) - \left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^k q_j B_j^{\nu}\right) \\ &\leq M\nu(1-\nu) \times \left[\frac{\left(\sum_{i=1}^n p_i A^{-1}\right) \otimes \left(\sum_{j=1}^k q_j B\right) + \left(\sum_{i=1}^n p_i A\right) \otimes \left(\sum_{j=1}^k q_j B^{-1}\right)}{2} - P_n Q_k\right]. \end{aligned} \tag{3.2}$$

Proof. From (2.9) we get

$$\begin{aligned} 0 &\leq \frac{m}{2M^2}\nu(1-\nu)\left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right) \\ &\leq (1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu} \\ &\leq \frac{M}{2m^2}\nu(1-\nu)\left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right) \\ &\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 \end{aligned}$$

for all for $i \in \{1,...,n\}$, $j \in \{1,...,k\}$ and $\nu \in [0,1]$.

If we multiply by $p_i q_j \geq 0$ and sum, then we get

$$
\begin{aligned}
0 &\leq \frac{m}{2M^2}\nu(1-\nu)\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right) \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left[(1-\nu)A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu}\right] \\
&\leq \frac{M}{2m^2}\nu(1-\nu)\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right) \\
&\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i .
\end{aligned} \tag{3.3}
$$

Observe that

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i^2 \otimes 1 + \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i 1 \otimes B_j^2 - 2\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i \otimes B_j \\
&= Q_k\left(\sum_{i=1}^{n} p_i A_i^2\right) \otimes 1 + P_n 1 \otimes \left(\sum_{j=1}^{k} q_j B_j^2\right) - 2\left(\sum_{i=1}^{n} p_i A_i\right) \otimes \left(\sum_{j=1}^{k} q_j B_j\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left[(1-\nu)A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu}\right] &= (1-\nu)\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i \otimes 1 + \nu\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i 1 \otimes B_j \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i^{1-\nu} \otimes B_j^{\nu} \\
&= (1-\nu)Q_k\left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu P_n 1 \otimes \left(\sum_{j=1}^{k} q_j B_j\right) \\
&\quad - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right).
\end{aligned}
$$

By (3.3) we then get the desired result (3.1).

The inequality (3.2) follows in a similar way from (2.18). □

Corollary 3.2. *With the assumptions of Proposition 3.1, we have the Hadamard product inequalities*

$$
\begin{aligned}
0 &\leq \frac{m}{2M^2}\nu(1-\nu)\left[\left(Q_k\left(\sum_{i=1}^{n} p_i A_i^2\right) + P_n\left(\sum_{j=1}^{k} q_j B_j^2\right)\right) \circ 1 \; - 2\left(\sum_{i=1}^{n} p_i A_i\right) \circ \left(\sum_{j=1}^{k} q_j B_j\right)\right] \\
&\leq \left[(1-\nu)Q_k\left(\sum_{i=1}^{n} p_i A_i\right) + \nu P_n\left(\sum_{j=1}^{k} q_j B_j\right)\right] \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right) \\
&\leq \frac{M}{2m^2}\nu(1-\nu)\left[\left(Q_k\left(\sum_{i=1}^{n} p_i A_i^2\right) + P_n\left(\sum_{j=1}^{k} q_j B_j^2\right)\right) \circ 1 \; - 2\left(\sum_{i=1}^{n} p_i A_i\right) \circ \left(\sum_{j=1}^{k} q_j B_j\right)\right] \\
&\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 P_n Q_k
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
0 &\leq \left[(1-\nu) Q_k \left(\sum_{i=1}^{n} p_i A_i\right) + \nu P_n \left(\sum_{j=1}^{k} q_j B_j\right)\right] \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right) \qquad (3.5)\\
&\leq M\nu(1-\nu) \times \left[\frac{\left(\sum_{i=1}^{n} p_i A^{-1}\right) \circ \left(\sum_{j=1}^{k} q_j B\right) + \left(\sum_{i=1}^{n} p_i A\right) \circ \left(\sum_{j=1}^{k} q_j B^{-1}\right)}{2} - P_n Q_k\right].
\end{aligned}
$$

If we take $k = n$, $p_i = q_i$ and $B_i = A_i$, then we get the simpler inequalities

$$
\begin{aligned}
0 &\leq \frac{m}{M^2}\nu(1-\nu) \times \left[P_n \left(\sum_{i=1}^{n} p_i A_i^2\right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i\right) \circ \left(\sum_{i=1}^{n} p_i A_i\right)\right] \qquad (3.6)\\
&\leq P_n \left(\sum_{i=1}^{n} p_i A_i\right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right)\\
&\leq \frac{M}{2m^2}\nu(1-\nu) \times \left[P_n \left(\sum_{i=1}^{n} p_i A_i^2\right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i\right) \circ \left(\sum_{i=1}^{n} p_i A_i\right)\right]\\
&\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 P_n^2
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\leq P_n \left(\sum_{i=1}^{n} p_i A_i\right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right) \qquad (3.7)\\
&\leq M\nu(1-\nu)\left[\left(\sum_{i=1}^{n} p_i A^{-1}\right) \circ \left(\sum_{i=1}^{n} p_i A\right) - P_n^2\right],
\end{aligned}
$$

for all $\nu \in [0,1]$, provided that $0 < m \leq A_i \leq M$ and $p_i \geq 0$ for $i \in \{1,...,n\}$.

We also have the multiplicative inequalities:

Proposition 3.3. *With the assumptions of Proposition 3.3,*

$$
\begin{aligned}
\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right) &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right] \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right) \qquad (3.8)\\
&\leq (1-\nu) Q_k \left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu P_n 1 \otimes \left(\sum_{j=1}^{k} q_j B_j\right)\\
&\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right] \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right) &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right] \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right) \qquad (3.9)\\
&\leq (1-\nu) Q_k \left(\sum_{i=1}^{n} p_i A_i\right) \circ 1 + \nu P_n 1 \circ \left(\sum_{j=1}^{k} q_j B_j\right)\\
&\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right] \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right),
\end{aligned}
$$

for all $\nu \in [0,1]$.

If we take $k=n$, $p_i=q_i$ and $B_i=A_i$ in (3.9), then we get the simpler inequalities

$$\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right)\circ\left(\sum_{i=1}^{n} p_i A_i^{\nu}\right)\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right]\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right)\circ\left(\sum_{j=1}^{k} q_j B_j^{\nu}\right) \tag{3.10}$$
$$\le P_n\left(\sum_{i=1}^{n} p_i A_i\right)\circ 1$$
$$\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right]\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right)\circ\left(\sum_{i=1}^{n} p_i A_i^{\nu}\right),$$

for all $\nu\in[0,1]$, provided that $0<m\le A_i\le M$ and $p_i\ge 0$ for $i\in\{1,\dots,n\}$.

4. Conclusion

In this paper, by utilizing some recent refinements and reverses of scalar Young's inequality, we provided some upper and lower bounds for the Young differences

$$(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}$$

and

$$[(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}$$

for $\nu\in[0,1]$ and $A,\ B>0$. The case of weighted sums for sequences of operators were also investigated.

Article Information

Acknowledgements: The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: The article has a single author. The author has read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

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