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On a Generalized Mittag-Leffler Function and Fractional Integrals

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Abstract

The object of this paper is to study a generalized Mittag-Leffler function and a modified general class of functions which is reducible to several special functions. convergent conditions of these functions are discussed. Some results pertaining to the generalized Mittag-Leffler function and generating relations involving these functions are derived. Further, fractional integrals involving these functions are achieved. Some illustrative exclusive cases of the results are presented.

1. Introduction

In 1906, Barnes [1] presented a function. In 1940, Wright [2] presented another function. In 1971, Prabhakar [3] studied an extended Mittag-Leffler function which is a particular case of the Wright's function. Recently in 2021, Srivastava [4] presented a more general function which is reducible to the Mittag-Leffler function by giving a suitable value to the general function involved therein. In this paper, two more general functions are presented which are reducible to the Srivastava's function defined by (1.6) and generalized Hurwitz-Lerch zeta function [5].
Now, we present some relevant definitions.

Definition 1.1. *A Swedish Scholar namely Magnus Gustaf "Gösta" Mittag-Leffler introduced his function [6], named after his name, as follows:*

$$E_{\delta}(x)=\sum_{c=0}^{\infty}\frac{x^{c}}{\Gamma(\delta c+1)}, \tag{1.1}$$

where $Re(\delta)>0$.

Definition 1.2. *Another Swedish Scholar namely Anders Wiman [7] presented a more general function as follows (see also [8], [9]):*

$$E_{\delta,\,\rho}(x)=\sum_{c=0}^{\infty}\frac{x^{c}}{\Gamma(\delta c+\rho)}, \tag{1.2}$$

where $\delta,\ \rho\in\mathbb{C}$ *and* $Re(\delta)>0$.

It is obvious that when $\rho=1$ in (1.2), it becomes (1.1).

Definition 1.3. *A British Scholar namely Ernest William Barnes [1] presented his function as follows:*

$$E_{\delta,\,\rho}^{\xi}(s;\,x)=\sum_{c=0}^{\infty}\frac{x^{c}}{(c+\xi)^{s}\,\Gamma(\delta c+\rho)}, \tag{1.3}$$

where $\delta,\ \rho\in\mathbb{C}$ *and* $Re(\delta)>0$

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It is obvious that when $s=0$ in (1.3), it becomes (1.2).

Definition 1.4. *Another British Scholar namely Sir Edward Maitland Wright [2], presented his function as follows:*

$$E_{\delta,\,\rho}(\Phi;\,x)=\sum_{c=0}^{\infty}\frac{\Phi(c)}{\Gamma(\delta c+\rho)}x^{c}, \tag{1.4}$$

where $F(c)$ is a general function and $\delta,\ \rho\in\mathbb{C}$, $Re(\delta)>0$.

It is obvious that when $\Phi(c)=\frac{1}{(c+\xi)^{s}}$ in (1.4), it becomes (1.3).

Definition 1.5. *If we substitute $\Phi(c)=\frac{(\zeta)_c}{c!}$ in (1.4), we achieve the extended Mittag-Leffler function as follows:*

$$E^{\zeta}_{\delta,\,\rho}(x)=\sum_{c=0}^{\infty}\frac{(\zeta)_c\,x^{c}}{\Gamma(\delta c+\rho)\,c!}, \tag{1.5}$$

where $\zeta,\ \delta,\ \rho\in\mathbb{C}$, $Re(\zeta)>0$, $Re(\delta)>0$ and

$$(a)_c=\frac{\Gamma(a+c)}{\Gamma(a)},$$

that is

$$(a)_0=1,\ \ (a)_c=a(a+1)(a+2)...(a+c-1),$$

where $c=1,\ 2,\ 3,\ \dots$.

It is obvious that when $\zeta=1$ in (1.5), it becomes (1.2). Indian Scholar namely Tilak Raj Prabhakar [3] studied (1.5). Some more exclusive cases of (1.4) have been considered and studied, among others, by Kamarujjama et al. [10], Khan and Ahmed [11], [12], Khan and Khan [13], Khan et al. [14], Shukla and Prajapati [15] and Salim [16].

Definition 1.6. *Recently, a Canadian Scholar of Indian origin namely Hari Mohan Srivastava [4], [17] presented his function as follows:*

$$E_{\delta,\,\rho}(\Phi;\,x;\,s,\,\xi)=\sum_{c=0}^{\infty}\frac{\Phi(c)}{(c+\xi)^{s}\,\Gamma(\delta c+\rho)}x^{c}, \tag{1.6}$$

where $\delta,\ \rho\in\mathbb{C}$ and $Re(\delta)>0$.

It is obvious that when $\Phi(c)=\frac{(\zeta)_c}{c!}$, $s=0$ in (1.6), it becomes (1.5). When $s=0$ in (1.6), it becomes (1.4) and when $\Phi(c)=(\zeta)_c,\ \delta=1,\ \rho=1$, it becomes Goyal-Laddha zeta function [18].

Definition 1.7. *Two More general functions are hereby presented as follows:*

$$E_{\alpha,\,\beta,\,\delta}(x,\,s,\,\rho)=\sum_{c=0}^{\infty}\frac{\phi(c)\,x^{c}}{(\delta c+\rho)^{s}\,\Gamma(\alpha c+\beta)}, \tag{1.7}$$

where $\delta,\ \rho,\ s,\ \alpha,\ \beta\in\mathbb{C}$, $Re(\alpha)>0$, $Re(s)\geq 0$,
and

$$E_{\alpha,\,\beta,\,\delta,\,\eta}(x,\,s,\,\rho)=\sum_{c=0}^{\infty}\frac{\phi(c)\,x^{c}}{(\rho+\delta c x^{\eta})^{s}\,\Gamma(\alpha c+\beta)}, \tag{1.8}$$

where $\delta,\ \rho,\ s,\ \alpha,\ \beta\in\mathbb{C}$, $Re(\alpha)>0$, $Re(s)\geq 0$, $\eta\geq 0$.

It is obvious that when $\delta=1$ in (1.7), it becomes (1.6) and when $\eta=0$ in (1.8), it becomes (1.7). If we put $\phi(c)=(\mu)_c,\ \alpha=1$ and $\beta=1$ in (1.8), it becomes the generalized Hurwitz-Lerch zeta function [5].

Definition 1.8. *If we assign*

$$\phi(c)=\frac{(\mu)_c}{c!}$$

in (1.7), we achieve a more generalized Mittag-Leffler function as follows:

$$E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\,s,\,\rho)=\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^{c}}{(\delta c+\rho)^{s}\,\Gamma(\alpha c+\beta)\,c!}, \tag{1.9}$$

where $Re(\mu)>0$, $Re(\delta)>0$, $Re(\rho)>0$, $Re(s)\geq 0$, $Re(\alpha)>0$, $Re(\beta)>0$, $|x|\leq 1$ and $(a)_c$ is defined in (1.5).

It is obvious that when $s=0$ in (1.9), it becomes (1.5).
The function (1.9) is studied in this paper.

Remark 1.9. *Other generalized Mittag-Leffler functions and generalized Hurwitz-Lerch zeta functions can be achieved by assigning suitable values to $\phi(c)$ in (1.7) and (1.8). Three such examples are given here.*
(i) If we assign $s=0$, $\phi(c)=\frac{\Gamma(\zeta c+\mu)}{\Gamma(\delta c+\rho)}$ in (1.7), we achieve the following generalized Mittag-Leffler function:

$$E_{\alpha,\,\beta,\,\delta,\,\rho}^{\zeta,\,\mu}(x)=\sum_{c=0}^{\infty}\frac{\Gamma(\zeta c+\mu)\,x^{c}}{\Gamma(\alpha c+\beta)\,\Gamma(\delta c+\rho)},\tag{1.10}$$

where $Re(\mu)>0$, $Re(\alpha)>0$, $Re(\beta)>0$, $Re(\delta)>0$, $Re(\rho>0$.
It is obvious that when $\zeta=\delta$, $\mu=\rho$ in (1.10), it becomes (1.2). Using (2.2), it may be ascertained that the series in (1.10) is absolutely convergent when $|\alpha+\delta|>|\zeta|$ and $|x|\leq 1$.
(ii) If we assign $\phi(c)=\Gamma(\zeta c+\mu)$ in (1.7) and (1.8), we achieve the following generalized Hurwitz-Lerch zeta functions:

$$\phi_{\zeta,\,\mu}^{\delta,\,\alpha,\,\beta}(x,\,s,\,\rho)=\sum_{c=0}^{\infty}\frac{\Gamma(\zeta c+\mu)\,x^{c}}{(\rho+\delta c)^{s}\,\Gamma(\alpha c+\beta)},\tag{1.11}$$

where $\zeta,\,\mu,\,\delta,\,\rho,\alpha,\,\beta,\,s\in\mathbb{C}$, $Re(\zeta)>0$, $Re(\delta)>0$, $Re(\alpha)>0$, $Re(s)\geq 0$,
and

$$\phi_{\zeta,\,\mu}^{\delta,\,\eta,\,\alpha,\,\beta}(x,\,s,\,\rho)=\sum_{c=0}^{\infty}\frac{\Gamma(\zeta c+\mu)\,x^{c}}{(\rho+\delta c x^{\eta})^{s}\,\Gamma(\alpha c+\beta)},\tag{1.12}$$

where $\zeta,\,\mu,\,\alpha,\,\beta,\,\delta,\,\rho,\,s\in\mathbb{C}$, $Re(\zeta)>0$, $Re(\alpha)>0$, $Re(s)\geq 0$, $\eta\geq 0$.
It is obvious that when $\zeta=\alpha$, $\mu=\beta$, $\delta=1$ in (1.11), it becomes the Hurwitz-Lerch zeta function [19], (p. 27, Eq. (1)) and when $\eta=0$ in (1.12), it becomes (1.11). Using (2.2), it may be ascertained that series in (1.11) and (1.12) are absolutely convergent when $|\alpha|>|\zeta|$ and $|x|\leq 1$.

Lemma 1.10. *The function $E_{\alpha,\,\beta,\,\delta}^{\mu}(x,\,s,\rho)$ expressed by (1.9) is represented as an integral as follows:*

$$E_{\alpha,\,\beta,\,\delta}^{\mu}(x,\,s,\rho)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}\,e^{-\rho t}\,E_{\alpha,\,\beta}^{\mu}(xe^{-\delta t})dt,\tag{1.13}$$

where $E_{\alpha,\,\beta}^{\mu}(xe^{-\delta t})$ is expressed by (1.5) and $Re(\mu)>0$, $Re(\alpha)>0$, $Re(\beta)>0$, $Re(\delta)>0$, $Re(s)>0$, $|x|\leq 1$.

Proof. Assigning $p=(\delta c+\rho)$ in [19], (p. 1, Eq. (5))

$$p^{-s}=\frac{1}{\Gamma(s)}\int_{0}^{\infty}e^{-pt}\,t^{s-1}dt,\ \ Re(s)>0,$$

we achieve

$$(\delta c+\rho)^{-s}=\frac{1}{\Gamma(s)}\int_{0}^{\infty}e^{-(\delta c+\rho)t}\,t^{s-1}dt,\ \ Re(s)>0.$$

Now, from (1.9), we achieve

$$E_{\alpha,\,\beta,\,\delta}^{\mu}(x,\,s,\rho)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}\,e^{-\rho t}\left\{\sum_{c=0}^{\infty}\frac{(\mu)_{c}(xe^{-\delta t})^{c}}{\Gamma(\alpha c+\beta)\,c!}\right\}dt$$

and applying (1.5), we easily procure (1.13). □

Remark 1.11. *If we assign $\alpha=1$ in (1.13), we procure the expression:*

$$E_{\beta,\,\delta}^{\mu}(x,\,s,\rho)=\frac{1}{\Gamma(s)\,\Gamma(\beta)}\int_{0}^{\infty}t^{s-1}\,e^{-\rho t}\,{}_{1}F_{1}(\mu;\,\beta;\,xe^{-\delta t})dt,$$

where ${}_{1}F_{1}(\mu;\,\beta;\,xe^{-\delta t})$ is the confluent hypergeometric function [19] and $\mathrm{Re}(\mu)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\delta)>0$, $\mathrm{Re}(s)>0$, $|x|\leq 1$.

Definition 1.12. *Riemann-Liouville's fractional integral of order ω of $f(t)$ is given as follows [20]:*

$$I_x^{\omega}\{f(t)\} = \frac{1}{\Gamma(\omega)} \int_0^x (x-t)^{\omega-1} f(t)dt, \tag{1.14}$$

where ω, x are complex variables and $Re(\omega) > 0$.

Definition 1.13. *A modified general class of functions is hereby presented as follows:*

$$\begin{aligned} V_n^{\lambda}(z) &= V_n^{\lambda,\, h_m,\, c,\, d,\, g_j}[p,\ \tau,\ k,\ w,\ q,\ \rho_m,\ k_m,\ \gamma_j,\ a_j,\ b_r,\ \eta,\ \alpha,\ \beta,\ \delta;\ z] \\ &= \lambda \sum_{n=0}^{\infty} \frac{(p)^n \prod_{m=1}^{t} \left[(h_m)_{n\rho_m+k_m}\right] (c+\eta n+\beta)^{-\tau} (z/2)^{nk+dw+q}}{\prod_{j=1}^{s} \left[(g_j)_{n\gamma_j+a_j}\right] \prod_{r=1}^{u} \left[(d)_{\alpha n\delta+b_r}\right]}, \end{aligned} \tag{1.15}$$

where

(i) $p,\ k,\ w$ and $q \in \mathbb{R}$.
(ii) $t,\ s$ and $u \in \mathbb{N}$.
(iii) $h_m,\ \rho_m,\ k_m,\ c,\ d,\ g_j,\ \gamma_j,\ a_j,\ \eta,\ \alpha,\ \beta,\ \delta,\ b_r$ and $\tau \in \mathbb{C}$. d may be considered as real or complex.
(iv) $Re(h_m) > 0$, $Re(\rho_m) > 0$, $Re(g_j) > 0$, $Re(\gamma_j) > 0$, $Re(d) > 0$, z being a variable and λ being an arbitrary constant.
(v) The series in (1.15) is absolutely convergent when $|\alpha\delta+\gamma_j| > |\rho_m|$ and $|p(z/2)^k| \le 1$.

Remark 1.14. *On substituting $\rho_m = 1$, $c = d$, $\eta = \alpha$ and $\gamma_j = 1$ in (1.15), it becomes the general class of functions defined in [?], [5].*

Remark 1.15. *If we assign $p = 2,\ k = 1,\ c = d = 1,\ \tau = 1,\ w = 0,\ q = 0,\ \alpha = \eta = 1,\ \beta = -1,\ \delta = 1,\ b_1 = -1,\ r = 1,\ k_m = 0,\ a_j = 0$ and $\lambda = \frac{\prod_{m=1}^{t}\Gamma(h_m)}{\prod_{j=1}^{s}\Gamma(g_j)}$ in (1.15), it becomes the Wright's generalized hypergeometric function as follows [19], (p. 183):*

$${}_t\Psi_s\left[\begin{matrix}(h_m,\ \rho_m)_{1,\ t} \\ (g_j,\ \gamma_j)_{1,\ s}\end{matrix};\ z\right] = \sum_{n=0}^{\infty} \frac{\prod_{m=1}^{t}\Gamma(h_m+n\rho_m)}{\prod_{j=1}^{s}\Gamma(g_j+n\gamma_j)} \frac{z^n}{n!} \tag{1.16}$$

2. Convergence conditions of (1.9) and (1.15)

Here convergence conditions of the series in (1.9) and (1.15) are discussed.

Theorem 2.1. *If $Re(\mu) > 0$, $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\delta) > 0$, $Re(\rho) > 0$, $Re(s) \ge 0$ and $|x| \le 1$, then series in (1.9) is absolutely convergent.*

Proof. D' Alembert's ratio test is applied to prove the theorem. Taking

$$U_c(x) = \frac{(\mu)_c\, x^c}{(\delta c+\rho)^s\, \Gamma(\alpha c+\beta)\, c!}.$$

Then

$$U_{c+1}(x) = \frac{(\mu)_{c+1}\, x^{c+1}}{(\delta c+\rho+\delta)^s\, \Gamma(\alpha c+\beta+\alpha)\, (c+1)!}$$

and on simplification

$$\left|\frac{U_{c+1}(x)}{U_c(x)}\right| = \left|\frac{\mu+c}{c+1} \frac{(\delta c+\rho)^s}{(\delta c+\delta+\rho)^s} \frac{\Gamma(\alpha c+\beta)}{\Gamma(\alpha c+\beta+\alpha)} x\right|. \tag{2.1}$$

Applying in (2.1), the result [19], (p. 5, Eq. (2)):

$$\frac{\Gamma(a)}{\Gamma(a+b)} = e^{\gamma\, b} \prod_{n=0}^{\infty} \left(1+\frac{b}{a+n}\right) e^{\frac{-b}{1+n}}, \tag{2.2}$$

where $\gamma(=0.58)$ is the Euler constant, (2.1) becomes

$$\left|\frac{U_{c+1}(x)}{U_c(x)}\right| = \left|\frac{\mu+c}{c+1} \frac{(\delta c+\rho)^s}{(\delta c+\delta+\rho)^s}\, e^{\gamma\, \alpha} \prod_{n=0}^{\infty} \left(\frac{n+\alpha c+\beta+\alpha}{n+\alpha c+\beta}\right) e^{\frac{-\alpha}{1+n}}\, x\right|.$$

On simplification we procure

$$\left|\frac{U_{c+1}(x)}{U_c(x)}\right|=\left|\frac{\frac{\mu}{c}+1}{1+\frac{1}{c}}\,\frac{(1+\frac{\rho}{\delta c})^s}{(1+\frac{1}{c}+\frac{\rho}{\delta c})^s}e^{\gamma\,\alpha}\prod_{n=0}^{\infty}\left(\frac{1+\frac{\beta}{\alpha c}+\frac{n}{\alpha c}+\frac{1}{c}}{1+\frac{\beta}{\alpha c}+\frac{n}{\alpha c}}\right)e^{\frac{-\alpha}{1+n}}\,x\right|.$$

Now, it is observed that

$$\lim_{c\to\infty}\left|\frac{U_{c+1}(x)}{U_c(x)}\right|=\left|\,x\,e^{\gamma\,\alpha}\prod_{n=0}^{\infty}e^{\frac{-\alpha}{1+n}}\right|.$$

Therefore, series in (1.9) converges absolutely when

$$\left|x\,e^{\gamma\,\alpha}\prod_{n=0}^{\infty}e^{\frac{-\alpha}{1+n}}\right|<1.$$

Or

$$|x|\leq 1,$$

since

$$\left|e^{\gamma\,\alpha}\prod_{n=0}^{\infty}e^{\frac{-\alpha}{1+n}}\right|<1,$$

provided that $\mathrm{Re}(\mu)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\delta)>0$, $\mathrm{Re}(\rho)>0$ and $\mathrm{Re}(s)\geq 0$. □

Theorem 2.2. *If $Re(\alpha)>0$, $Re(\delta)>0$, $|\alpha\delta+\gamma_j|>|\rho_m|$ and $|p(x/2)^k|\leq 1$, then series in* (1.15) *converges absolutely.*

Proof. Taking

$$U_n(z)=\lambda\,\frac{(p)^n\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+k_m}\right](c+\eta n+\beta)^{-\tau}(z/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}.$$

Then

$$U_{n+1}(z)=\lambda\,\frac{(p)^{n+1}\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+\rho_m+k_m}\right](c+\eta n+\eta+\beta)^{-\tau}(z/2)^{nk+k+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+\alpha\delta+b_r}\right]}$$

and on simplification we procure

$$\begin{aligned}\left|\frac{U_{n+1}(x)}{U_n(x)}\right|=&\left|\prod_{m=1}^{t}\left\{\frac{\Gamma(h_m+n\rho_m+k_m)}{\Gamma(h_m+n\rho_m+k_m+\rho_m)}\right\}^{-1}\prod_{j=1}^{s}\left\{\frac{\Gamma(g_j+n\gamma_j+a_j)}{\Gamma(g_j+n\gamma_j+a_j+\gamma_j)}\right\}\right.\\&\left.\times\prod_{r=1}^{u}\left\{\frac{\Gamma(d+n\alpha\delta+b_r)}{\Gamma(d+n\alpha\delta+b_r+\alpha\delta)}\right\}\left\{\frac{(c+n\eta+\beta)}{(c+n\eta+\beta+\eta)}\right\}^{\tau}p(z/2)^k\right|.\end{aligned}\tag{2.3}$$

On applying (2.2), (2.3) becomes

$$\begin{aligned}\left|\frac{U_{n+1}(x)}{U_n(x)}\right|=&\left|\prod_{m=1}^{t}\left\{e^{\gamma\rho_m}\prod_{l=0}^{\infty}\left(\frac{\frac{h_m}{n\rho_m}+1+\frac{k_m}{n\rho_m}+\frac{l}{n\rho_m}+\frac{1}{n}}{\frac{h_m}{n\rho_m}+1+\frac{k_m}{n\rho_m}+\frac{l}{n\rho_m}}\right)e^{-\frac{\rho_m}{l+1}}\right\}^{-1}\prod_{j=1}^{s}e^{\gamma\gamma_j}\prod_{\rho=0}^{\infty}\left(\frac{\frac{g_j}{n\gamma_j}+1+\frac{a_j}{n\gamma_j}+\frac{\rho}{n\gamma_j}+\frac{1}{n}}{\frac{g_j}{n\gamma_j}+1+\frac{a_j}{n\gamma_j}+\frac{\rho}{n\gamma_j}}\right)e^{-\frac{\gamma_j}{\rho+1}}\right.\\&\left.\times\prod_{r=1}^{u}e^{\gamma\alpha\delta}\prod_{\varepsilon=0}^{\infty}\left(\frac{\frac{d}{n\alpha\delta}+1+\frac{b_r}{n\alpha\delta}+\frac{\gamma}{n\alpha\delta}+\frac{1}{n}}{\frac{d}{n\alpha\delta}+1+\frac{b_r}{n\alpha\delta}+\frac{\gamma}{n\alpha\delta}}\right)e^{-\frac{\alpha\delta}{\varepsilon+1}}\left(\frac{\frac{c}{n\eta}+1+\frac{\beta}{n\eta}}{\frac{c}{n\eta}+1+\frac{\beta}{n\eta}+\frac{1}{n}}\right)^{\tau}p(z/2)^k\right|,\end{aligned}$$

where γ is given with (2.2).
Now, it is observed that

$$\lim_{n\to\infty}\left|\frac{U_{n+1}(x)}{U_n(x)}\right|=\left|\prod_{j=1}^{s}\prod_{m=1}^{t}\left\{e^{\gamma(\alpha\delta+\gamma_j-\rho_m)}\prod_{\varepsilon=0}^{\infty}\prod_{\rho=0}^{\infty}\prod_{l=0}^{\infty}e^{-\left(\frac{\alpha\delta}{\varepsilon+1}+\frac{\gamma_j}{\rho+1}-\frac{\rho_m}{l+1}\right)}\right\}p(z/2)^k\right|$$

Therefore, series in (1.15) converges absolutely when

$$\left|\prod_{j=1}^{s}\prod_{m=1}^{t}\left\{e^{\gamma(\alpha\delta+\gamma_j-\rho_m)}\prod_{\varepsilon=0}^{\infty}\prod_{\rho=0}^{\infty}\prod_{l=0}^{\infty}e^{-\left(\frac{\alpha\delta}{\varepsilon+1}+\frac{\gamma_j}{\rho+1}-\frac{\rho_m}{l+1}\right)}\right\}p(z/2)^k\right|<1.$$

Or

$$|p(z/2)^k|\leq 1,$$

since

$$\left|\prod_{j=1}^{s}\prod_{m=1}^{t}\left\{e^{\gamma(\alpha\delta+\gamma_j-\rho_m)}\prod_{\varepsilon=0}^{\infty}\prod_{\rho=0}^{\infty}\prod_{l=0}^{\infty}e^{-\left(\frac{\alpha\delta}{\varepsilon+1}+\frac{\gamma_j}{\rho+1}-\frac{\rho_m}{l+1}\right)}\right\}\right|<1,$$

provided $|\alpha\delta+\gamma_j|>|\rho_m|$. □

3. Generating relations

Here we drive some generating relations pertaining to (1.9) and (1.15).

Theorem 3.1. *If* $|t|<|\rho|$, $Re(\mu)>0$ *along with conditions associated with* (1.9)*, we procure the generating relation as follows:*

$$\sum_{n=0}^{\infty}(\xi)_n\,E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\,\xi+n,\,\rho)\frac{t^n}{n!}=E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\,\xi,\,\rho-t). \tag{3.1}$$

Proof. Applying (1.9), we find

$$\begin{aligned}\sum_{n=0}^{\infty}(\xi)_n\,E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\,\xi+n,\,\rho)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}(\xi)_n\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^c}{(\delta c+\rho)^{\xi+n}\,\Gamma(\alpha c+\beta)\,c!}\frac{t^n}{n!}\\&=\sum_{c=0}^{\infty}\frac{1}{\Gamma(\alpha c+\beta)(\delta c+\rho)^{\xi}}\left[\sum_{n=0}^{\infty}(\xi)_n\left(\frac{t}{\delta c+\rho}\right)^n\frac{1}{n!}\right](\mu)_c\frac{x^c}{c!}.\end{aligned} \tag{3.2}$$

Applying in (3.2), the result

$$\sum_{n=0}^{\infty}\frac{(\xi)_n\,x^n}{n!}=(1-x)^{-\xi},\quad |x|<1, \tag{3.3}$$

we procure

$$\begin{aligned}\sum_{n=0}^{\infty}(\xi)_n\,E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\,\xi+n,\,\rho)\frac{t^n}{n!}&=\sum_{c=0}^{\infty}\frac{1}{\Gamma(\alpha c+\beta)(\delta c+\rho)^{\xi}}\left(1-\frac{t}{\delta c+\rho}\right)^{-\xi}(\mu)_c\frac{x^c}{c!}\\&=\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^c}{(\delta c+\rho-t)^{\xi}\,\Gamma(\alpha c+\beta)c!}.\end{aligned}$$

Using (1.9), (3.1) is arrived at, provided $|t|<|\rho|$. □

Theorem 3.2. *If* $|\rho|>|t|$, $Re(u+\xi)>Re(v)>0$ *along with conditions associated with* (1.9)*, we procure the bilateral generating function as follows:*

$$\sum_{n=0}^{\infty}\frac{(\xi)_n(u)_n}{(v)_n}E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\,\xi+u-v+n,\,\rho)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{(\mu)_n\,x^n}{(\delta n+\rho)^{\xi+u-v}\,\Gamma(\alpha n+\beta)\,n!}\,{}_2F_1\left(\xi,\,u;\,v;\,\frac{t}{\delta n+\rho}\right),$$

where ${}_2F_1\,(a,\,b;\,v;\,z)$ *represents the hypergeometric function [19] (p. 56, Eq. (2)).*

Proof. applying (1.9), we procure

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\xi)_n(u)_n}{(v)_n}E^{\mu}_{\alpha,\,\beta,\,\delta}(z,\,\xi+u-v+n,\,\rho)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\frac{(\xi)_n(u)_n}{(v)_n}\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^c}{(\delta c+\rho)^{\xi+u-v+n}\,\Gamma(\alpha c+\beta)\,c!}\frac{t^n}{n!}\\&=\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^c}{(\delta c+\rho)^{\xi+u-v}\,\Gamma(\alpha c+\beta)\,c!}\left[\sum_{n=0}^{\infty}\frac{(\xi)_n(u)_n}{(v)_n}\left(\frac{t}{\delta c+\rho}\right)^n\frac{1}{n!}\right]\\&=\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^c}{(\delta c+\rho)^{\xi+u-v}\,\Gamma(\alpha c+\beta)\,c!}\,{}_2F_1\left(\xi,\,u;\,v;\,\frac{t}{\delta c+\rho}\right)\\&=\sum_{n=0}^{\infty}\frac{(\mu)_n\,x^n}{(\delta n+\rho)^{\xi+u-v}\,\Gamma(\alpha n+\beta)\,n!}\,{}_2F_1\left(\xi,\,u;\,v;\,\frac{t}{\delta n+\rho}\right),\end{aligned}$$

provided $|\rho|>|t|$ and $\mathrm{Re}(u+\xi)>\mathrm{Re}(v)>0$. □

Theorem 3.3. *If conditions associated with (1.9) and (1.15) are satisfied, we procure the bilateral generating function as follows:*

$$\sum_{n=0}^{\infty}(\varepsilon)_n E^{\mu}_{\gamma,\,\sigma,\,\eta}(x,\,\varepsilon+n,\,\rho)\,V^{\lambda}_n(y)\,\frac{t^n}{n!}=E^{\mu}_{\gamma,\,\sigma,\,\eta}\left(x,\,\varepsilon,\,\rho-pt\left(\frac{y}{2}\right)^k\right)\left(\frac{y}{2}\right)^{dw+q}\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}. \tag{3.4}$$

Proof. Applying (1.15), it is procured that

$$\begin{aligned}\sum_{n=0}^{\infty}(\varepsilon)_n E^{\mu}_{\gamma,\,\sigma,\,\eta}(x,\,\varepsilon+n,\,\rho)\,V^{\lambda}_n(y)\,\frac{t^n}{n!}&=\sum_{n=0}^{\infty}(\varepsilon)_n E^{\mu}_{\gamma,\,\sigma,\,\eta}(x,\,\varepsilon+n,\,\rho)\,\frac{t^n}{n!}\\&\times\lambda\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\left(\frac{y}{2}\right)^{nk+dw+q}.\end{aligned} \tag{3.5}$$

Applying (1.9) in (3.5), we find

$$\begin{aligned}\sum_{n=0}^{\infty}(\varepsilon)_n E^{\mu}_{\gamma,\,\sigma,\,\eta}(x,\,\varepsilon+n,\,\rho)\,V^{\lambda}_n(y)\,\frac{t^n}{n!}&=\lambda\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\left(\frac{y}{2}\right)^{nk+dw+q}\\&\times\left[\sum_{n=0}^{\infty}(\varepsilon)_n\left\{\sum_{\omega=0}^{\infty}\frac{(\mu)_\omega\,x^\omega}{(\eta\omega+\rho)^{\varepsilon+n}\,\Gamma(\gamma\omega+\sigma)\,\omega!}\right\}\frac{t^n}{n!}\right]\\&=\left(\frac{y}{2}\right)^{dw+q}\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\\&\times\sum_{\omega=0}^{\infty}\frac{(\mu)_\omega\,x^\omega}{(\eta\omega+\rho)^{\varepsilon}\,\Gamma(\gamma\omega+\sigma)\,\omega!}\left[\sum_{n=0}^{\infty}(\varepsilon)_n\left\{\frac{pt(\frac{y}{2})^k}{\eta\omega+\rho}\right\}^n\frac{1}{n!}\right].\end{aligned} \tag{3.6}$$

Applying (3.3) in (3.6), we find

$$\begin{aligned}\sum_{n=0}^{\infty}(\varepsilon)_n E^{\mu}_{\gamma,\,\sigma,\,\eta}(x,\,\varepsilon+n,\,\rho)\,V^{\lambda}_n(y)\,\frac{t^n}{n!}&=\left(\frac{y}{2}\right)^{dw+q}\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\\&\times\sum_{\omega=0}^{\infty}\frac{(\mu)_\omega\,x^\omega}{(\eta\omega+\rho)^{\varepsilon}\,\Gamma(\gamma\omega+\sigma)\,\omega!}\left\{1-\frac{pt(\frac{y}{2})^k}{\eta\omega+\rho}\right\}^{-\varepsilon}\\&=\left(\frac{y}{2}\right)^{dw+q}\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\\&\times\sum_{\omega=0}^{\infty}\frac{(\mu)_\omega\,x^\omega}{\left(\eta\omega+\rho-pt(\frac{y}{2})^k\right)^{\varepsilon}\,\Gamma(\gamma\omega+\sigma)\,\omega!}.\end{aligned} \tag{3.7}$$

On applying (1.9), (3.7) easily approaches to (3.4). □

4. Special cases of the generating relation (3.4)

Here some special cases of (3.4) are achieved.

(i) On taking $p=-2,\ t=1,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ g_1=1,\ g_2=1,\ \gamma_1=1,\ \gamma_2=1,\ c=d,\ \tau=1,\ k=1,\ w=0,\ q=0,\ k_1=0,\ a_1=0,\ a_2=0,\ \eta=\alpha,\ \beta=0,\ \delta=1,\ b_1=0$ and $\lambda=\frac{1}{\Gamma(d)}$ in (3.4), the modified general class of functions takes the form of Wright's generalized Bessel function [21] and we procure the generating relation as follows:

$$\sum_{n=0}^{\infty}(\varepsilon)_n E^{\mu}_{\gamma,\,\sigma,\,\alpha}(x,\,\varepsilon+n,\,\rho)\,J^{\alpha}_d(y)\,\frac{t^n}{n!}=\frac{1}{\Gamma(1+\alpha n+d)\,n!}E^{\mu}_{\gamma,\,\sigma,\,\alpha}(x,\,\varepsilon,\,\rho+ty),$$

where $J^{\alpha}_d(y)$ represents the Wright's generalized Bessel function.

(ii) On taking $p=-1,\ t=1,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ k_1=0,\ g_1=1,\ g_2=1,\ \gamma_1=1,\ \gamma_2=1,\ c=d=\frac{1}{2},\ \tau=1,\ k=2,\ w=2,\ q=0,\ a_1=0,\ a_2=-1,\ \beta=-\frac{1}{2},\ \delta=1,\ b_1=1,\ \eta=\alpha=1$ and $\lambda=1$ in (3.4), the modified general class of functions takes the form of sine function and we procure the generating relation as follows:

$$\sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma}(x,\ \varepsilon+n,\ \rho)\ sin\ y\ \frac{t^n}{n!}=\frac{y\sqrt{\pi}}{2\,\Gamma(\frac{3}{2}+n)\,n!}\ E^{\mu}_{\gamma,\,\sigma}\left(x,\ \varepsilon,\ \rho+t\left(\frac{y}{2}\right)^2\right).$$

(iii) On taking $p=-1,\ t=1,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ k_1=0,\ g_1=1,\ g_2=1,\ \gamma_1=1,\ \gamma_2=1,\ c=d=\frac{1}{2},\ \tau=1,\ k=2,\ w=0,\ q=0,\ a_1=0,\ a_2=-1,\ \beta=-\frac{1}{2},\ \delta=1,\ b_1=0,\ \eta=\alpha=1$ and $\lambda=1$ in (3.4), the modified general class of functions takes the form of cosine function and we procure the generating relation as follows:

$$\sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma}(x,\ \varepsilon+n,\ \rho)\ cos\ y\ \frac{t^n}{n!}=\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}+n)\,n!}\ E^{\mu}_{\gamma,\,\sigma}\left(x,\ \varepsilon,\ \rho+t\left(\frac{y}{2}\right)^2\right).$$

(iv) On taking $p=-2,\ u=1,\ c=d=1,\ \rho_m=1,\ \gamma_j=1,\ t=P,\ s=Q,\ \tau=1,\ k=1,\ w=0,\ q=0,\ k_m=0,\ a_j=0,\ b_1=-1,\ \eta=\alpha=1,\ \beta=-1,\ \delta=1$ and $\lambda=\frac{\prod_{m=1}^{P}\Gamma(h_m)}{\prod_{j=1}^{Q}\Gamma(g_j)}$ in (3.4), the general class of functions takes the form of MacRobert's $E-$ function [19], (p. 203, Eq. (1)) and we procure the generating relation as follows:

$$\sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma}(x,\ \varepsilon+n,\ \rho)\ E\left[P;\ (h_P);\ Q;\ (g_Q);\ \frac{1}{y}\right]\ \frac{t^n}{n!}=\frac{\prod_{m=1}^{P}\Gamma(h_m)}{\prod_{j=1}^{Q}\Gamma(g_j)}\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{P}(h_m)_n}{\prod_{j=1}^{Q}(g_j)_n}\ E^{\mu}_{\gamma,\,\sigma}(x,\ \varepsilon,\ \rho+yt).$$

where $E\,[P;\ (h_P);\ Q;\ (g_Q);\ z]$ represents the MacRobert's $E-$ function.

Remark 4.1. *Other spacial cases of the generating relation* (3.4) *may be procured using the substitutions of section* 7.

5. Some results pertaining to (1.9)

Here we establish some results pertaining to (1.9) associated with differentiation and integration.

Theorem 5.1. *If* $Re(\mu)>0$, $Re(\varepsilon)>0$ *along with conditions associated with* (1.9)*, the result procured is as follows:*

$$\frac{1}{\Gamma(\varepsilon)}\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}\int_0^1 t^{\alpha c+\beta-1}(1-t)^{\varepsilon-1}dt=E^{\mu}_{\alpha,\,\beta+\varepsilon,\,\delta}(x,\ s,\ \rho). \tag{5.1}$$

Proof. (5.1) may easily be proved using Beta integral. □

Assigning $s=0$ in (5.1), it becomes a result procured by Prabhakar [3].

Theorem 5.2. *If* $a>0$ *along with conditions associated with* (1.9)*, the result procured is as follows:*

$$E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\alpha},\ s,\ \rho)=x\,\frac{d}{dx}E^{\mu}_{\alpha,\,\beta+1,\,\delta}(ax^{\alpha},\ s,\ \rho)+\beta\,E^{\mu}_{\alpha,\,\beta+1,\,\delta}(ax^{\alpha},\ s,\ \rho). \tag{5.2}$$

Proof. Applying (1.9), we procure

$$\begin{aligned}\frac{d}{dx}E^{\mu}_{\alpha,\,\beta+1,\,\delta}(ax^{\alpha},\ s,\ \rho)&=\sum_{c=0}^{\infty}\frac{(\mu)_c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta+1)\,c!}\frac{d}{dx}(ax^{\alpha})^c\\&=\frac{1}{x}\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c(\alpha c)\,x^{\alpha c}}{(\delta c+\rho)^s\,(\alpha c+\beta)\,\Gamma(\alpha c+\beta)\,c!}\\&=\frac{1}{x}\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c\{(\alpha c+\beta)-\beta\}\,x^{\alpha c}}{(\delta c+\rho)^s\,(\alpha c+\beta)\,\Gamma(\alpha c+\beta)\,c!}\\&=\frac{1}{x}\left\{\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c(\alpha c+\beta)\,x^{\alpha c}}{(\delta c+\rho)^s\,(\alpha c+\beta)\,\Gamma(\alpha c+\beta)\,c!}-\beta\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c\,x^{\alpha c}}{(\delta c+\rho)^s\,(\alpha c+\beta)\,\Gamma(\alpha c+\beta)\,c!}\right\}\\&=\frac{1}{x}\left\{\sum_{c=0}^{\infty}\frac{(\mu)_c\,(ax^{\alpha})^c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}-\beta\sum_{c=0}^{\infty}\frac{(\mu)_c\,(ax^{\alpha})^c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta+1)\,c!}\right\}\end{aligned} \tag{5.3}$$

On applying (1.9), (5.3) arrives at

$$x\,\frac{d}{dx}E^{\mu}_{\alpha,\,\beta+1,\,\delta}(ax^{\alpha},\ s,\ \rho)=E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\alpha},\ s,\ \rho)-\beta\,E^{\mu}_{\alpha,\,\beta+1,\,\delta}(ax^{\alpha},\ s,\ \rho).$$

On simplification, (5.2) is arrived at. □

Corollary 5.3. *Assigning $\beta = \beta + \varepsilon$ in (5.2), the result obtained is as follows:*

$$E^{\mu}_{\alpha,\,\beta+\varepsilon,\,\delta}(ax^{\alpha},\,s,\,\rho) = x\,\frac{d}{dx}E^{\mu}_{\alpha,\,\beta+\varepsilon+1,\,\delta}(ax^{\alpha},\,s,\,\rho) + (\beta+\varepsilon)\,E^{\mu}_{\alpha,\,\beta+\varepsilon+1,\,\delta}(ax^{\alpha},\,s,\,\rho). \tag{5.4}$$

Theorem 5.4. *If $a > 0$ along with conditions associated with (1.9), the result procured is as follows:*

$$E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\delta},\,s-1,\,\rho) = x\frac{d}{dx}E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\delta},\,s,\,\rho) + \rho\,E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\alpha},\,s,\,\rho). \tag{5.5}$$

Proof. Applying (1.9), we procure

$$\begin{aligned}
\frac{d}{dx}E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\delta},\,s,\,\rho) &= \sum_{c=0}^{\infty}\frac{(\mu)_c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}\,\frac{d}{dx}(ax^{\delta})^c \\
&= \frac{1}{x}\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c(\delta c)\,x^{\delta c}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!} \\
&= \frac{1}{x}\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c\{(\delta c+\rho)-\rho\}\,x^{\delta c}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!} \\
&= \frac{1}{x}\left\{\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c(\delta c+\rho)\,x^{\delta c}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!} - \rho\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c\,x^{\delta c}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}\right\} \\
&= \frac{1}{x}\left\{\sum_{c=0}^{\infty}\frac{(\mu)_c\,(ax^{\delta})^c}{(\delta c+\rho)^{s-1}\,\Gamma(\alpha c+\beta)\,c!} - \rho\sum_{c=0}^{\infty}\frac{(\mu)_c\,(ax^{\delta})^c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}\right\}
\end{aligned} \tag{5.6}$$

On applying (1.9), (5.6) becomes

$$x\,\frac{d}{dx}E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\delta},\,s,\,\rho) = E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\delta},\,s-1,\,\rho) - \rho\,E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\delta},\,s,\,\rho).$$

On simplification, (5.5) is easily arrived at. □

Theorem 5.5. *Along with conditions associated with (1.9), for any $n \in \mathbb{N}$ the result procured is as follows:*

$$\left(\frac{d}{dx}\right)^n\left\{x^{\beta-1}E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\alpha},\,s,\,\rho)\right\} = x^{\beta-n-1}\,E^{\mu}_{\alpha,\,\beta-n,\,\delta}(ax^{\alpha},\,s,\,\rho). \tag{5.7}$$

Proof. we find

$$\begin{aligned}
\frac{d}{dx}\left\{x^{\beta-1}E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\alpha},\,s,\,\rho)\right\} &= \sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}\,\frac{d}{dx}x^{\alpha c+\beta-1} \\
&= \sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c(\alpha c+\beta-1)\,x^{\alpha c+\beta-2}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}.
\end{aligned} \tag{5.8}$$

Applying $\Gamma(x) = (x-1)\Gamma(x-1)$ in (5.8), we procure

$$\begin{aligned}
\frac{d}{dx}\left\{x^{\beta-1}E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\alpha},\,s,\,\rho)\right\} &= \sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c(\alpha c+\beta-1)\,x^{\alpha c+\beta-2}}{(\delta c+\rho)^s\,(\alpha c+\beta-1)\,\Gamma(\alpha c+\beta-1)\,c!} \\
&= \sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c\,x^{\alpha c+\beta-2}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta-1)\,c!} \\
&= x^{\beta-2}\sum_{c=0}^{\infty}\frac{(\mu)_c\,(ax^{\alpha})^c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta-1)\,c!} \\
&= x^{\beta-2}\,E^{\mu}_{\alpha,\,\beta-1,\,\delta}(ax^{\alpha},\,s,\,\rho)
\end{aligned}$$

Similarly, we get

$$\left(\frac{d}{dx}\right)^2\left\{x^{\beta-1}E^{\mu}_{\alpha,\,\beta,\,\delta}(ax^{\alpha},\,s,\,\rho)\right\} = x^{\beta-3}\,E^{\mu}_{\alpha,\,\beta-2,\,\delta}(ax^{\alpha},\,s,\,\rho).$$

Following the same process we procure (5.7). □

Assigning $s = 0$ in (5.7), a result of Kilbas, Saigo and Saxena [8] is procured.

6. Applications

Here fractional integral (1.14) is applied to procure images of (1.9) and (1.15), and finally to gain integrals involving special functions.

Theorem 6.1. *If* $Re(\varepsilon)>0$ *along with conditions associated with* (1.9)*, the result procured is as follows:*

$$I_x^{\varepsilon}\left\{t^{\beta-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(at^{\alpha},\,s,\,\rho)\right\}=x^{\varepsilon+\beta-1}\,E^{\mu}_{\alpha,\,\beta+\varepsilon,\,\delta}(ax^{\alpha},\,s,\,\rho). \tag{6.1}$$

Proof. On applying (1.14), it is procured that

$$I_x^{\varepsilon}\left\{t^{\beta-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(at^{\alpha},\,s,\,\rho)\right\}=\frac{1}{\Gamma(\varepsilon)}\int_0^x t^{\beta-1}(x-t)^{\varepsilon-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(at^{\alpha},\,s,\,\rho)dt. \tag{6.2}$$

Use of (1.9) in (6.2) gives

$$I_x^{\varepsilon}\left\{t^{\beta-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(at^{\alpha},\,s,\,\rho)\right\}=\frac{1}{\Gamma(\varepsilon)}\int_0^x t^{\beta-1}(x-t)^{\varepsilon-1}\,\sum_{k=0}^{\infty}\frac{(\mu)_k\,(at^{\alpha})^k}{(\delta k+\rho)^s\,\Gamma(\alpha k+\beta)\,k!}dt.$$

Conditions associated with (1.9) permit to interchange the order of integration and summation and it is done to gain

$$I_x^{\varepsilon}\left\{t^{\beta-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(at^{\alpha},\,s,\,\rho)\right\}=\frac{1}{\Gamma(\varepsilon)}\sum_{k=0}^{\infty}\frac{(\mu)_k\,a^k}{(\delta k+\rho)^s\,\Gamma(\alpha k+\beta)\,k!}\int_0^x t^{\alpha k+\beta-1}(x-t)^{\varepsilon-1}dt. \tag{6.3}$$

Applying in (6.3), the result [20], (p. 185, Eq. (7))

$$\int_0^x y^{b-1}(x-y)^{a-1}dy=x^{a+b-1}\,\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}, \tag{6.4}$$

where $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, to gain

$$I_x^{\varepsilon}\left\{t^{\beta-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(at^{\alpha},\,s,\,\rho)\right\}=\frac{1}{\Gamma(\varepsilon)}\sum_{k=0}^{\infty}\frac{(\mu)_k\,a^k}{(\delta k+\rho)^s\,\Gamma(\alpha k+\beta)\,k!}x^{\varepsilon+\alpha k+\beta-1}\,\frac{\Gamma(\varepsilon)\,\Gamma(\alpha k+\beta)}{\Gamma(\varepsilon+\alpha k+\beta)}.$$

On simplifying, it is procured that

$$I_x^{\varepsilon}\left\{t^{\beta-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(at^{\alpha},\,s,\,\rho)\right\}=x^{\varepsilon+\beta-1}\sum_{k=0}^{\infty}\frac{(\mu)_k\,(ax^{\alpha})^k}{(\delta k+\rho)^s\,\Gamma(\alpha k+\beta+\varepsilon)\,k!}. \tag{6.5}$$

Now, use of (1.9) in (6.5) completes the proof. □

Corollary 6.2. *From* (6.1) *and* (6.2)*, it is found that*

$$\frac{1}{\Gamma(\varepsilon)}\int_0^x t^{\beta-1}(x-t)^{\varepsilon-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(at^{\alpha},\,s,\,\rho)dt=x^{\varepsilon+\beta-1}\,E^{\mu}_{\alpha,\,\beta+\varepsilon,\,\delta}(ax^{\alpha},\,s,\,\rho). \tag{6.6}$$

Corollary 6.3. $x=1$ *in* (6.6) *gives the Eulerian integral as follows:*

$$\frac{1}{\Gamma(\varepsilon)}\int_0^1 t^{\beta-1}(1-t)^{\varepsilon-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(at^{\alpha},\,s,\,\rho)dt=E^{\mu}_{\alpha,\,\beta+\varepsilon,\,\delta}(a,\,s,\,\rho). \tag{6.7}$$

Further, on assigning $t=\frac{x-u}{y-u}$ *and* $a=\lambda(y-u)^{\alpha}$ *in* (6.7)*, an interesting integral is procured as follows:*

$$\frac{1}{\Gamma(\varepsilon)}\int_u^y (x-u)^{\beta-1}(y-x)^{\varepsilon-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}\{\lambda(x-u)^{\alpha},\,s,\,\rho\}dx=(y-u)^{\beta+\varepsilon-1}\,E^{\mu}_{\alpha,\,\beta+\varepsilon,\,\delta}\{\lambda(y-u)^{\alpha},\,s,\,\rho\}$$

and on assigning $t=\frac{y-x}{y-u}$ *and* $a=\lambda(y-u)^{\alpha}$ *in* (6.7)*, an interesting integral is procured as follows:*

$$\frac{1}{\Gamma(\varepsilon)}\int_u^y (y-x)^{\beta-1}(x-u)^{\varepsilon-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}\{\lambda(y-x)^{\alpha},\,s,\,\rho\}dx=(y-u)^{\beta+\varepsilon-1}\,E^{\mu}_{\alpha,\,\beta+\varepsilon,\,\delta}\{\lambda(y-u)^{\alpha},\,s,\,\rho\}.$$

On assigning $s=0$ in these integrals, Prabhakar's [3] integrals are achieved.

Corollary 6.4. *If* $a>0$, $Re(\varepsilon)>0$ *along with the conditions associated with* (1.9)*, the result is procured as follows:*

$$I_x^{\varepsilon}\left\{t^{\beta-1}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(at^{\alpha},\,s,\,\rho);\,x\right\}=x^{\varepsilon+\beta-1}\left\{x\frac{d}{dx}E^{\mu}_{\alpha,\,\beta+\varepsilon+1,\,\delta}(ax^{\alpha},\,s,\,\rho)+(\beta+\varepsilon)\,E^{\mu}_{\alpha,\,\beta+\varepsilon+1,\,\delta}(ax^{\alpha},\,s,\,\rho)\right\}. \tag{6.8}$$

Putting the value of $E^{\mu}_{\alpha,\ \beta+\varepsilon,\ \delta}(ax^{\alpha},\ s,\ \rho)$ from (5.4) in (6.1), (6.8) is proved.

Theorem 6.5. *If $Re(\varepsilon)>0$ along with conditions associated with (1.9) and (1.15), the result procured is as follows:*

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(az^{\sigma},\ y,\ \rho)\,V_n^{\lambda}\{b(x-z)^{\sigma}\}\right]=\frac{\lambda}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+k_m}\right](c+\eta n+\beta)^{-\tau}(b/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}$$
$$\times\Gamma\{\sigma(nk+dw+q)+\varepsilon\}\,x^{\sigma(nk+dw+q)+\varepsilon+\gamma-1}$$
$$\times E^{\mu}_{\sigma,\ \gamma+\sigma(nk+dw+q)+\varepsilon,\ \omega}(ax^{\sigma},\ y,\ \rho). \tag{6.9}$$

Proof. Applying (1.14), it is procured that

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(az^{\sigma},\ y,\ \rho)\,V_n^{\lambda}\{b(x-z)^{\sigma}\}\right]=\frac{1}{\Gamma(\varepsilon)}\int_0^x z^{\gamma-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(az^{\sigma},\ y,\ \rho)\,V_n^{\lambda}\{b(x-z)^{\sigma}\}(x-z)^{\varepsilon-1}dz. \tag{6.10}$$

Use of (1.9) and (1.15) in (6.10), and interchange of the order of integration and summations permitted by the conditions associated therein, give the l.h.s of (6.10) (supposing L) as follows:

$$L=\lambda\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+k_m}\right](c+\eta n+\beta)^{-\tau}(b/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}\sum_{v=0}^{\infty}\frac{(\mu)_v\,a^v}{(\omega v+\rho)^y\,\Gamma(\sigma v+\gamma)\,v!}$$
$$\times\frac{1}{\Gamma(\varepsilon)}\int_0^x z^{\sigma v+\gamma-1}(x-z)^{\sigma(nk+dw+q)+\varepsilon-1}dz. \tag{6.11}$$

On evaluation of z-integral in (6.11) using (6.4), it is procured that

$$L=\lambda\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+k_m}\right](c+\eta n+\beta)^{-\tau}(b/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}\sum_{v=0}^{\infty}\frac{(\mu)_v\,a^v}{(\omega v+\rho)^y\,\Gamma(\sigma v+\gamma)\,v!}$$
$$\times\frac{1}{\Gamma(\varepsilon)}x^{\sigma(nk+dw+q)+\varepsilon+\sigma v+\gamma-1}\,\frac{\Gamma(\sigma v+\gamma)\Gamma\{\sigma(nk+dw+q)+\varepsilon\}}{\Gamma\{\sigma v+\gamma+\sigma(nk+dw+q)+\varepsilon\}}. \tag{6.12}$$

Now, use of (1.9) in (6.12) completes the proof. □

Corollary 6.6. *From (6.9) and (6.10), it is found that*

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(az^{\sigma},\ y,\ \rho)\,V_n^{\lambda}\{b(x-z)^{\sigma}\}dz=\lambda\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+k_m}\right](c+\eta n+\beta)^{-\tau}(b/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}$$
$$\times\Gamma\{\sigma(nk+dw+q)+\varepsilon\}\,x^{\sigma(nk+dw+q)+\varepsilon+\gamma-1}$$
$$\times E^{\mu}_{\sigma,\ \gamma+\sigma(nk+dw+q)+\varepsilon,\ \omega}(ax^{\sigma},\ y,\ \rho).$$

7. Special cases of (6.9)

Here some special cases of (6.9) are achieved.

(i) On assigning $p=-1,\ t=1,\ \beta=0,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ k_1=0,\ \tau=1,\ g_1=1,\ g_2=1,\ \gamma_1=1,\ \gamma_2=1,\ k=2,\eta=\alpha=1,\ a_1=0,\ a_2=0,\ q=0,\ c=d,\ w=1,\ b_1=0,\ \delta=1$ and $\lambda=1/\Gamma(d)$ in (6.9), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(az^{\sigma},\ y,\ \rho)\,J_d\{b(x-z)^{\sigma}\}\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+d}\,\Gamma\{\sigma(2n+d)+\varepsilon\}}{\Gamma(1+d+n)\,n!}$$
$$\times x^{\sigma(2n+d)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\ \gamma+\sigma(2n+d)+\varepsilon,\ \omega}(ax^{\sigma},\ y,\ \rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(at^{\sigma},\ y,\ \rho)\,J_d\{b(x-z)^{\sigma}\}dz=\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+d}\,\Gamma\{\sigma(2n+d)+\varepsilon\}}{\Gamma(1+d+n)\,n!}$$
$$\times x^{\sigma(2n+d)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\ \gamma+\sigma(2n+d)+\varepsilon,\ \omega}(ax^{\sigma},\ y,\ \rho),$$

where $J_d(z)$ represents the Bessel function of the first kind [22], (p. 4, Eq. (2)).

(ii) On assigning $p=-1,\ t=1,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ k_1=0,\ \tau=1,\ g_1=3/2,\ g_2=1,\ \gamma_1=1,\ \gamma_2=1,\ k=2,\ a_1=0,\ a_2=0,\ w=1,\ c=d,\ q=1,\ b_1=1/2,\ \eta=\alpha=1,\ \beta=1/2,\ \delta=1$ and $\lambda=1/\{\Gamma(d)\Gamma(3/2)\}$ in (6.9), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,H_d\{b(x-z)^{\sigma}\}\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+d+1}\,\Gamma\{\sigma(2n+d+1)+\varepsilon\}}{\Gamma(n+\frac{3}{2})\,\Gamma(d+n+\frac{3}{2})}\times x^{\sigma(2n+d+1)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\,\gamma+\sigma(2n+d+1)+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,H_d\{b(x-z)^{\sigma}\}dz=\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+d+1}\,\Gamma\{\sigma(2n+d+1)+\varepsilon\}}{\Gamma(n+\frac{3}{2})\,\Gamma(d+n+\frac{3}{2})}\times x^{\sigma(2n+d+1)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\,\gamma+\sigma(2n+d+1)+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where $H_d(z)$ represents the Struve's function [22], (p. 38, Eq. (55)).

(iii) On assigning $p=-1,\ t=1,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ \eta=\alpha=1,\ k_1=0,\ \beta=-1,\ k=2,\ g_1=(\zeta+\xi+3)/2,\ g_2=(\zeta-\xi+3)/2,\gamma_1=1,\ \gamma_2=1,\ c=d=1,\ \tau=1,\ w=\zeta,\ q=1,\ a_1=0,\ a_2=0,\ b_1=-1,\ \delta=1$ and $\lambda=2^{\zeta+1}/(\zeta\pm\xi+1)$ in (6.9), the result is procured as follow:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,s_{\zeta,\,\xi}\{b(x-z)^{\sigma}\}\right]=\frac{2^{\zeta+1}}{(\zeta\pm\xi+1)}\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+\zeta+1}\,\Gamma\{\sigma(2n+\zeta+1)+\varepsilon\}}{\left(\frac{\zeta\pm\xi+3}{2}\right)_n}\times x^{\sigma(2n+\zeta+1)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\,\gamma+\sigma(2n+\zeta+1)+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,s_{\zeta,\,\xi}\{b(x-z)^{\sigma}\}dz=\frac{2^{\zeta+1}}{(\zeta\pm\xi+1)}\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+\zeta+1}\,\Gamma\{\sigma(2n+\zeta+1)+\varepsilon\}}{\left(\frac{\zeta\pm\xi+3}{2}\right)_n}\times x^{\sigma(2n+\zeta+1)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\,\gamma+\sigma(2n+\zeta+1)+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where $s_{\zeta,\,\xi}(z)$ represents the Lommel's function [22], (p. 40, Eq. (69)).

(iv) On assigning $p=2,\ t=1,\ s=1,\ u=1,\ h_1=h,\ \rho_1=1,\ k_1=0,\ g_1=1,\ \gamma_1=1,\ \eta=\alpha,\ a_1=0,\ c=d,\ \beta=-1,\ \tau=1,\ k=1,\ w=0,\ q=0,\ \delta=1,\ b_1=-1$ and $\lambda=1/\Gamma(d)$ in (3.1), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,E^{h}_{\alpha,\,d}\{b(x-z)^{\sigma}\}\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(h)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{\Gamma(\alpha n+d)\,n!}x^{\sigma n+\varepsilon+\gamma-1}E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,E^{h}_{\alpha,\,d}\{b(x-z)^{\sigma}\}dz=\sum_{n=0}^{\infty}\frac{(h)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{\Gamma(\alpha n+d)\,n!}x^{\sigma n+\varepsilon+\gamma-1}E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where $E^{h}_{\alpha,\,d}(z)$ represents (1.5).

(v) On assigning $p=2,\ t=1,\ s=1,\ u=1,\ h_1=h,\ \rho_1=1,\ \eta=\alpha=1,\ k_1=0,\ c=d,\ \beta=0,\ k=1,\ w=0,\ q=0,\ g_1=1,\ \gamma_1=1,\ a_1=0,\ b_1=0,\ \delta=0$ and $\lambda=1$ in (6.9), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,\phi_h\{b(x-z)^{\sigma},\,\tau,\,d\}\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(h)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{(d+n)^{\tau}\,n!}x^{\sigma n+\varepsilon+\gamma-1}\times E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,\phi_h\{b(x-z)^{\sigma},\,\tau,\,d\}dz=\sum_{n=0}^{\infty}\frac{(h)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{(d+n)^{\tau}\,n!}x^{\sigma n+\varepsilon+\gamma-1}\times E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where $\phi_h\{z,\,\tau,\,d\}$ represents for the Goyal-Laddha zeta function [18].

(vi) On assigning $p=2,\ u=1,\ \rho_m=1,\ c=d=1,\ \eta=\alpha=1,\ \tau=1,\ \beta=-1,\ k=1,\ w=0,\ q=0,\ k_m=0,\ \gamma_j=1,\ a_j=0,\ b_1=-1,\ \delta=1$ and $\lambda=1$ in (6.9), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\;{}_tF_s\left(h_t;\;g_s;\;\;b(x-z)^{\sigma}\right)\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{\prod\limits_{m=1}^{P}(h_m)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{\prod\limits_{j=1}^{Q}(g_j)_n\,n!}x^{\sigma n+\varepsilon+\gamma-1}\times E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\;{}_PF_Q\left(h_P;\;g_Q;\;\;b(x-z)^{\sigma}\right)dz=\sum_{n=0}^{\infty}\frac{\prod\limits_{m=1}^{P}(h_m)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{\prod\limits_{j=1}^{Q}(g_j)_n\,n!}x^{\sigma n+\varepsilon+\gamma-1}\times E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where ${}_tF_s\left(h_t;\;g_s;\;z\right)$ represents the generalized hypergeometric function [19], (p. 182, Eq. (1)).

(vii) On assigning $p=2,\ k=1,\ c=d=1,\ \tau=1,\ w=0,\ q=0,\ \alpha=\eta=1,\ \beta=-1,\ \delta=1,\ b_1=-1,\ r=1,\ k_m=0,\ a_j=0$ and $\lambda=\frac{\prod_{m=1}^{t}\Gamma(h_m)}{\prod_{j=1}^{s}\Gamma(g_j)}$ in (6.9), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\;{}_t\Psi_s\left[{(h_m,\,\rho_m)_{1,\,t} \atop (g_j,\,\gamma_j)_{1,\,s}};\;b(x-z)^{\sigma}\right]\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{\prod\limits_{m=1}^{t}\Gamma(h_m+n\rho_m)\,b^n\,\Gamma(\sigma n+\varepsilon)}{\prod\limits_{j=1}^{s}\Gamma(g_j+n\gamma_j)\,n!}x^{\sigma n+\varepsilon+\gamma-1}E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\;{}_t\Psi_s\left[{(h_m,\,\rho_m)_{1,\,t} \atop (g_j,\,\gamma_j)_{1,\,s}};\;b(x-z)^{\sigma}\right]dz=\sum_{n=0}^{\infty}\frac{\prod\limits_{m=1}^{t}\Gamma(h_m+n\rho_m)\,b^n\,\Gamma(\sigma n+\varepsilon)}{\prod\limits_{j=1}^{s}\Gamma(g_j+n\gamma_j)\,n!}x^{\sigma n+\varepsilon+\gamma-1}E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where ${}_t\Psi_s\left[{(h_m,\,\rho_m)_{1,\,t} \atop (g_j,\,\gamma_j)_{1,\,s}};\;z\right]$ represents the Wright's generalized hypergeometric function (1.16) [19], (p. 183).

Remark 7.1. *Other special cases of* (6.9) *can be obtained using the substitutions of section* 4*.*

8. Conclusion

Two general functions reducible to Mittag-Leffler function and Riemann-zeta function, and a modified general class of functions reducible to several special functions have been represented and defined in this paper, and their convergence conditions have been discussed. Generating relations and fractional integrals involving new defined functions have been achieved. Some particular cases of the results have been achieved. Similar results may be obtained involving (1.10). A further study of the fractional integral operator defined by (1.14) may be carried out with the generalized Mittag-Leffler function defined by (1.9) in the kernel and its integral transforms may be studied. Moreover, composition relations between the fractional integral operator defined by (1.14) and integral operator with the generalized Mittag-Leffler function defined by (1.9) in the kernel may be obtained.

Declarations

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