Research Article GU J Sci 38(2): 899-910 (2025) DOI: 10.35378/gujs.1383308

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Trichotomy of Non-oscillatory Solutions for Nonlinear First- order Neutral Difference Equation with Generalized Difference Operators

Yasar BOLAT , Murat GEVGESOGLU*

Kastamonu University, Faculty of Science, Department of Mathematics, 37150, Kastamonu, Türkiye

Highlights

- This paper focuses on trichotomy of non-oscillatory solutions.
- To prove our results discrete analogue of Kiguradze's lemma is used.
- The behavior of the reduced equation was examined.

Article Info

Received: 30 Oct 2023
Accepted: 14 Dec 2024

Keywords

Trichotomy, Oscillation, Nonoscillation, Neutral difference equation, Generalized difference operators

Abstract

In this study, we examine the trichotomy of non-oscillatory solution for the nonlinear first-order neutral difference equation

$$\Delta_a(x_n - a^{n+1}x_{n-1}) + \Delta_a\left(\frac{x_{n-1}}{b_{n-1}}\right) + q_n f(x_{n-\tau}) = 0, n \in \mathbb{N}_{max\{1,\tau\}},$$

where $\Delta_a x_n = x_{n+1} - a x_n$, $\tau \in \mathbb{N}, a \in \mathbb{R}^+$ with $\Delta_a^m \Delta = \Delta_a^{m-1}(\Delta_a)$, a^n is a general term of exponential sequence, (q_n) is real valued sequences; $n - \tau < n$ with $(n-\tau) \to +\infty$ as $n \to +\infty$; under the assumption $\sum_{s=n_0}^{\infty} \frac{1}{b_n} < \infty (= \infty)$, where $b_n = \prod_{i=1}^{n} a^i$. The accuracy of the primary findings is demonstrated by examples.

1. INTRODUCTION

Differential equations and their discrete analogues difference equations have many applications, from social sciences to physical sciences and from health sciences to engineering sciences. For these, readers can look at the references and their cities in [1-4]. The past few years have seen a significant increase in interest in the asymptotic behavior of solutions to difference equations involving generalized difference operators. We assemble here some relevant results.

A solution of a difference equation is sequence (x_n) which satisfies the difference equation for sufficiently large n. In this work, nontrivial solutions are considered for all large n. If the terms of the sequence (x_n) are neither eventually positive nor eventually negative, the solution (x_n) is considered oscillatory. Otherwise, it is referred to as a non-oscillatory solution. If every solution of a difference equation oscillates, the equation is said to be oscillatory. It is non-oscillatory otherwise.

Bolat and Akin, by generalizing the equation that Parhi had considered in [5], gave new results on oscillation in her work in [6]. In this paper, we further generalize the works of Bolat and Akın in [6] and Agata Bezubik and et all. in [7], and we investigate trichotomy of non-oscillatory solutions of difference equation

*Corresponding author, e-mail: mgevgesoglu@kastamonu.edu.tr

$$\Delta_a(x_n - a^{n+1}x_{n-1}) + \Delta_a\left(\frac{x_{n-1}}{b_{n-1}}\right) + q_n f(x_{n-\tau}) = 0, n \in \mathbb{N}_{max\{1,\tau\}}, \quad (1)$$

where $a \in \mathbb{R}^+$, $\tau \in \mathbb{N}$, (a^n) and (q_n) are positive real valued sequences, $b_n = \prod_{i=1}^{n} a^i$, $f: \mathbb{R} \to \mathbb{R}$, $(n-\tau) \to \infty$ as $n \to \infty$.

During the study, we will impose one or both of the assumptions below:

(H_1) $uf(u) > 0$ for $u \neq 0$;

(H_2) $f: \mathbb{R} \to \mathbb{R}$ is continuous.

In this manuscript, we will investigate the three forms of the behaviour of sequential solutions as in unbounded solutions, constant solutions and zero solutions, which is known as trichotomy of non-oscillatory solutions.

Definition 1.

(i) A sequence whose limit is nonzero is called an asymptotically constant sequence.

(ii) A sequence whose limit is zero is called a zero sequence.

(iii) If the sequence $\left(\frac{u_n}{v_n}\right)$ has a non-zero limit then the sequence (u_n) is asymptotically equivalent to sequence (v_n).

Definition 2. A subset S of the Banach space B is said to be uniformly Cauchy if for every $\varepsilon > 0$ there exists a positive integer N such that $|x_i - x_j| < \varepsilon$ whenever $i, j > N$ for any $(x_n) \in B$ [8].

Lemma 1. Each bounded and uniformly Cauchy subset of B is relatively compact (Arzela-Ascoli's Theorem [3]).

Theorem 1. Let S be a nonempty, closed, and convex subset of a Banach space B and $T: S \to S$ be a continuous mapping such that $T(S)$ is a relatively compact subset of B. Then T has at least one fixed point in S (Schauder Theorem [3]).

Theorem 2. Let (u_n), (v_n) be two real sequences. Assume that (v_n) is a strictly monotone and divergent sequence, and the following limit exists: $\lim_{n\to\infty} \frac{\Delta u_n}{\Delta v_n} = g$. Then

$\lim_{n\to\infty} \frac{u_n}{v_n} = g$ (Stolz-Cesáro Theorem [2]).

We introduce the notation:

$$Q_n = \sum_{i=n_1}^{n-1} \frac{1}{b_i}. \quad (2)$$

If we denote

$$y_{n+1} = x_n \prod_{i=1}^{n} \frac{1}{a^i} \text{ and } b_n = \prod_{i=1}^{n} a^i, \quad (3)$$

we have $x_n - a^{n+1}x_{n-1} = \prod_{i=1}^{n} a^i \Delta_a y_n$.

Hence using (3) we transform Equation (1) into second- order difference equation

$$\Delta_a(b_n\Delta_a y_n) + \Delta_a y_n + q_n f(b_{n-\tau} y_{n+1-\tau}) = 0, n \in \mathbb{N}_{max\{1,\tau\}}, \tau \in \mathbb{N}. \tag{4}$$

Considering $\Delta_a^m y_n = a^{n+m}\Delta^m\left(\frac{y_n}{a^n}\right), n \in \mathbb{N},$ we have following second-order nonlinear delay ordinary difference equation

$$\Delta(b_n\Delta z_n) + \Delta z_n + {q_n}^* f(b_{n+1-\tau} z_{n+1-\tau}) = 0, n \in \mathbb{N}_{max\{1,\tau\}}, \tau \in \mathbb{N}. \tag{5}$$

where $z_n = \frac{y_n}{a^n}, {q_n}^* = \frac{q_n}{a^{n+2}}$. Assuming

$$f(b_{n+1-\tau} u) = b_n^* g(u), b_n^* > 0, b_n^* = \frac{c_n}{q_n^*} \tag{6}$$

from (5) we reach following second-order nonlinear delay ordinary difference equation

$$\Delta(b_n\Delta z_n) + \Delta z_n + c_n g(z_{n+1-\tau}) = 0, n \in \mathbb{N}_{max\{1,\tau\}}, \tau \in \mathbb{N}, \tag{7}$$

where $c_n = q_n^* b_n^*$.

Similar equations with (7) have been studied in [9-13].

2. MAIN RESULTS

In here, we provide the necessary and sufficient criteria for the existence of non-oscillatory solutions of Equation (1). Firstly, we give following lemmas which will use on our results. First we give the discrete analogue of the Kiguradze's Lemma and the second is its particular case.

Lemma 2. [1] Let (x_n) be defined for $n \geq n_0 \in \mathbb{N}$, and $x_n > 0$ with $\Delta^m x_n$ of constant sign for $n \geq n_0$ and not identically zero. Then, there exists an integer $k, 0 \leq k \leq m$ with $(m+k)$ odd for $\Delta^m x_n \leq 0$ and $(m+k)$ even for $\Delta^m x_n \geq 0$ such that

(i) $k \leq m-1$ implies $(-1)^{m+i}\Delta^i x_n > 0$ for all $n \geq n_0$, $k \leq i \leq m-1$,

(ii) $k \geq 1$ implies $\Delta^i x_n > 0$ for all large $n \geq n_0, 1 \leq i \leq k-1$.

Lemma 3. Assume that following conditions:

(H_1^*) $ug(u) > 0$ for all $u \neq 0$;

(H_2^*) $g: \mathbb{R} \to \mathbb{R}$ is a continuous function;

are satisfied. Assume that (z_n) is an eventually positive solution of Equation (7). Then one of the following hypotheses is always exactly valid:

(i) (z_n)>0, $\Delta z_n > 0$,

(ii) (z_n)>0, $\Delta z_n < 0$

for all sufficiently large n.

Proof. The proof of Lemma 3 is obvious from Lemma 2.

Lemma 4. Suppose that $0 < a \leq 1$. Then

$$\sum_{i=1}^{\infty} \frac{1}{b_i} = \infty \tag{8}$$

where $b_n = \prod_{i=1}^{n} a^i$.

Lemma 5. Assume that $1 < a \leq 1.549856$. Then

$$1 \leq \sum_{i=1}^{\infty} \frac{1}{b_i} < \infty \tag{9}$$

where $b_n = \prod_{i=1}^{n} a^i$.

Lemma 6. Assume that $a > 1.549856$. Then

$$0 < \sum_{i=1}^{\infty} \frac{1}{b_i} < 1 \tag{10}$$

where $b_n = \prod_{i=1}^{n} a^i$.

For the proofs of Lemma $4-6$, the number 1.549856 can be checked with the help of computer.

So $a = 1.549856$ is a boundary value for $\sum_{i=1}^{\infty} \frac{1}{b_i}$ to be less or greater then 1.

Lemma 7. Let $0 < a \leq 1.549856$, and (8) and (9) be satisfied. If (x_n) is a non-oscillatory (positive) solution of Equation (1), then one of the following hypotheses is always exactly valid:

(I) $\lim_{n\to\infty} \frac{x_n}{b_{n+1}} = 0$;

(II) there are positive constants C_1, C_2 and a positive integer n_0 such that $C_1 b_{n+1} \leq x_n \leq C_2 b_{n+1} Q_{n+1}$ for $n \geq n_0$ where $b_n = \prod_{i=1}^{n} a^i$ and Q_{n+1} is as in (2).

Proof. Assume that (z_n) be an eventually positive solution of Equation (7). In this case, by Lemma 3, there are two possibilities: There exists a positive constant C_1 such that $z_{n+1} \geq C_1$ or $\lim_{n\to\infty} z_{n+1} = 0$. If $\lim_{n\to\infty} z_{n+1} = 0$ then $\lim_{n\to\infty} \frac{x_n}{b_{n+1}} = 0$ is satisfied. Assume that $z_{n+1} \geq C_1$. Then by (2) we have $z_{n+1} = \frac{y_{n+1}}{a^{n+1}} = \frac{1}{a^{n+1}} x_n \prod_{i=1}^{n} \frac{1}{a^i} = \frac{x_n}{b_{n+1}} \geq C_1$, that is, in the case (II) the inequality $C_1 b_{n+1} \leq x_n$ is satisfied. Now we show that in the case (II) the inequality $x_n \leq C_2 b_{n+1} Q_{n+1}$ is also satisfied for $n \geq n_0$.

We can find a positive integer n_1 and write from Equation (7)

$$\Delta(b_n \Delta z_n) + \Delta z_n < 0 \text{ for n} \geq n_1. \tag{11}$$

Summing up (11) from n_1 to $n-1$, we obtain

$$\Delta z_n + \frac{1}{b_n} z_n < \frac{M_1}{b_n} \text{ for } n \geq n_1.$$

Since z_n is decreasing and $\frac{1}{b_n} z_n > 0$, $\Delta z_n < \Delta z_n + \frac{1}{b_n} z_n$. Therefore from (11), we can write

$$\Delta z_n < \Delta z_n + \frac{1}{b_n} z_n < \frac{M_1}{b_n} \text{ for } n \geq n_1, \tag{12}$$

where $M_1 = max\{z_{n_1}, b_{n_1}\Delta z_{n_1}\}$ is nonnegative constant. Next summing up (12) from n_1 to $n-1$, we have

$$z_n < M_1 \sum_{i=n_1}^{n-1} \frac{1}{b_i} + M_2 \text{ for } n \geq n_1, \tag{13}$$

where $M_2 = z_{n_1}$ is positive constant. By Lemma 4 and Lemma 5, we have

$$1 \leq \sum_{i=n_1}^{n-1} \frac{1}{b_i} \text{ for } n \geq n_1.$$

Therefore from (13), we have

$$\begin{aligned} z_n &< M_1 \sum_{i=n_1}^{n-1} \frac{1}{b_i} + M_2 \sum_{i=n_1}^{n-1} \frac{1}{b_i} \\ &< max\{M_1, M_2\} \sum_{i=n_1}^{n-1} \frac{1}{b_i} \\ &= C_2 Q_n \text{ for } n \geq n_1, \end{aligned}$$

where C_2=$2max\{M_1, M_2\}$. Last inequality implies that

$$z_{n+1} \leq C_2 Q_{n+1} \text{ for } n \geq n_1.$$

Hence, we obtain

$$x_n \leq C_2 b_{n+1} Q_{n+1}$$

The proof is completed.

Corollary 1. Let $0 < a \leq 1.549856$. Assume that (8) and (9) are satisfied and conditions (H_1) and (H_2) are hold. If (z_n) is an eventually positive solution to (7), then

(c_1) $\lim_{n\to\infty} z_n = 0$;

(c_2) there exist positive constants C_1, C_2, and sufficiently large n_2 such that $C_1 \leq z_n \leq C_2 Q_n$ for $n \geq n_2$, where (Q_n) is defined by (2).

Theorem 3. Suppose that $a > 1$ and conditions (H_1), (H_2) and (6) are hold. The condition

$$\sum_{i=n_6-1}^{\infty} c_i \sum_{j=n_6}^{i} \frac{1}{b_j} < \infty \tag{14}$$

is a necessary condition for Equation (7) to has an asymptotically constant solution.

Proof. Suppose that (z_n) is an asymptotically constant solution of Equation (7). Then, the solution (z_n) becomes a non-oscillatory sequence. We suppose that (z_n) is an eventually positive solution (Without loss of generality). Then according to Lemma 3, (z_n) is either of type (i) or type (ii). Each solution in the type (i) tends to infinity. But this contradicts our assumption to be an asymptotically constant solution. Therefore (z_n) is of type (ii). Assume that

$$\lim_{n\to\infty} z_n = \alpha > 0 \tag{15}$$

Then we have an integer n_2 such that $C_3 \le z_{n+1-\sigma} \le C_4$ for $n \ge n_2$ where C_3 and C_4 are positive constants. By (H_1) and (H_2), we can find a positive constant $C_5 = \min_{u\in[C_3,C_4]}\{g(u)\}$ and $n_3 \ge n_2$ that for $n \ge n_3$ we have

$$C_5 \le g(z_{n+1-\sigma}) \tag{16}$$

Hence, we can find an $n_4 \ge n_3$ that (16) and (ii) are satisfied for $n \ge n_4$. We can write (6) in the form

$$-\Delta(b_i\Delta z_i + z_i) = c_i g(z_{i+1-\sigma}).$$

Multiplying this equation by $\sum_{j=n_4}^{i}\frac{1}{b_j}$ and summing up both sides of it from $i = n_4 - 1$ to $n-1$, we have

$$-\sum_{j=n_4-1}^{n-1}\left(\sum_{j=n_4}^{i}\frac{1}{b_j}\right)\Delta(b_i\Delta z_i + z_i) = \sum_{j=n_4-1}^{n-1}\left(\sum_{j=n_4}^{i}\frac{1}{b_j}\right)c_i g(z_{i+1-\sigma}). \tag{17}$$

By (16), we can write following inequality

$$\sum_{j=n_4-1}^{n-1} c_i g(z_{i+1-\sigma})\left(\sum_{j=n_4}^{i}\frac{1}{b_j}\right) \ge C_5\sum_{j=n_4-1}^{n-1} c_i\left(\sum_{j=n_4}^{i}\frac{1}{b_j}\right). \tag{18}$$

By the partial sum formula $\sum_{j=n_4-1}^{n-1} y_i\Delta x_i = y_{n-1}x_{n-1} - y_{n_4-1}x_{n_4-1} - \sum_{j=n_4-1}^{n-1} x_{i+1}\Delta y_i$, from the left side of (19) we write

$$-\sum_{j=n_4-1}^{n-1}\left(\sum_{j=n_4}^{i}\frac{1}{b_j}\right)\Delta(b_i\Delta z_i + z_i) = -\left[\sum_{j=n_4}^{i}\frac{1}{b_j}(b_i\Delta z_i + z_i)\right]_{i=n_4-1}^{n-1} + \sum_{i=n_4-1}^{n-1}\left(\Delta z_{i+1} + \frac{1}{b_{i+1}}z_{i+1}\right). \tag{19}$$

From (7), we have

$$\Delta(b_n\Delta z_n) + \Delta z_n = -c_n g(z_{n+1-\tau}) < 0. \tag{20}$$

By (20), we can find an $n_5 \ge n_4$ such that

$$\Delta(b_n\Delta z_n) + \Delta z_n < -A \text{ for } n \ge n_5, \tag{21}$$

where A is positive constant. By (20), we have two cases: $b_n\Delta z_n + z_n < 0$ or $b_n\Delta z_n + z_n > 0$. Suppose that $b_n\Delta z_n + z_n < 0$. Summing up (20) from n_5 to $n-1$ and considering $C_3 \le z_{n+1-\sigma} \le C_4$, we obtain

$$b_n\Delta z_n + C_3 \le b_n\Delta z_n + z_n < -A_1(n - n_5) + A_2, \tag{22}$$

where A_1 and $A_2 = max\{z_{n_5}, b_{n_5}\Delta z_{n_5}\}$ are positive costants. We have

$$b_n\Delta z_n \le -A_1(n-n_5)+A_2-C_3. \tag{23}$$

Thus we can find an $n_6 \ge n_5$ such that $-A_1(n-n_5)+A_2-C_3<0$ for $n \ge n_6$. Hence from (23) we can write

$$b_n\Delta z_n < -M \text{ for } n \ge n_6, \tag{24}$$

where M is a positive constant. Summing up (24) from n_6 to $n-1$ and considering (8), we obtain

$$z_n < -M\sum_{j=n_6}^{n-1}\frac{1}{b_j}+z_{n_6}=-\infty,$$

as $n\to\infty$ which contradicts with z_n to be positive. In that case

$$b_n\Delta z_n + z_n > 0. \tag{25}$$

Hence considering $b_n\Delta z_n+z_n$ to be decreasing, from (19), we have

$$\begin{aligned}
-\sum_{j=n_6-1}^{n-1}\left(\sum_{j=n_6}^{i}\frac{1}{b_j}\right)\Delta(b_i\Delta z_i+z_i) &= -\left[\sum_{j=n_6}^{i}\frac{1}{b_j}(b_i\Delta z_i+z_i)\right]_{i=n_6-1}^{n-1}+\sum_{i=n_6-1}^{n-1}\left(\Delta z_{i+1}+\frac{1}{b_{i+1}}z_{i+1}\right) \\
&< \sum_{i=n_6}^{n-1}\frac{1}{b_{i+1}}z_{i+1} \\
&< C_4\sum_{i=n_6}^{n-1}\frac{1}{b_{i+1}} \\
&= C_4Q_{n+1}.
\end{aligned} \tag{26}$$

Thus by Lemma 5 and Lemma 6, from (18) and (26), we have

$$C_5\sum_{i=n_6-1}^{n-1}C_i\left(\sum_{j=n_6}^{i}\frac{1}{b_j}\right) < C_4Q_{n+1} < \infty, \tag{27}$$

that is, the condition (14) is obtained

$$\sum_{i=n_6-1}^{n-1}C_i\left(\sum_{j=n_6}^{i}\frac{1}{b_j}\right) < \infty.$$

So the proof is completed.

Following examples show that the condition (14) is not a necessary condition for Equation (7) to has an asymptotically zero solution.

Example 1. Consider

$$\Delta(b_n\Delta z_n)+\Delta z_n + c_n z_n = 0 \tag{28}$$

which is the linear form of Equation (7), where $b_n=\prod_{i=1}^{n}a^i=1, c_n=\frac{1}{4}$ and $\tau=1$. One can see that

$$\sum_{j=n_4-1}^{\infty} c_i \left(\sum_{j=n_4}^{i} \frac{1}{b_j} \right) = \sum_{i=n_4-1}^{\infty} \frac{1}{4}(i-n_4) = \infty.$$

Therefore, although Equation (7) has an asymptotically zero solution, the condition (14) of Theorem 3 is not satisfied. Such a solution is $z_n = \frac{1}{2^n}$.

Example 2. Consider

$$\Delta(b_n \Delta z_n) + \Delta z_n + c_n z_n = 0 \,, \tag{29}$$

which is in the form (7), where $b_n = \prod_{i=1}^{n} \frac{1}{4^i} = \frac{1}{2^{n(n+1)}}, c_n = \frac{12.2^{n(n+1)}+16(2^{2(n+1)}-1)}{2^{2(n+1)}(2^{n(n+1)}+1)}$, $g(u) = u$, and $\tau = 1$. It is observed that the condition (14) is satisfied and Equation (29) has a solution $z_n = 1 + \frac{1}{2^{n(n+1)}}$ which is asymptotically constant.

Theorem 4. Suppose that (H_1), (H_2), (6) and (14) are satisfied. Then $\forall c \in \mathbb{R}$, Equation (1) has a solution (x_n) such that $\lim_{n\to\infty} x_n = c$.

Proof. According to Theorem 3, the condition (14) is the necessary condition for Equation (7) to have an asymptotically constant solution. Using (4), (5), (6) and (3), we see that Equation (1) has an asymptotically constant solution x_n such that

$$\lim_{n\to\infty} x_n = c.$$

Corollary 2. Let conditions (H_1), (H_2) and (6) be hold. Then the condition

$$\sum_{j=n_6-1}^{\infty} c_i \left(\sum_{j=n_6}^{i} \frac{1}{b_j} \right) = \infty \tag{30}$$

implies that (7) has no asymptotically constant solution.

The proof can be made directly from Theorem 3.

Theorem 5. Assume that (H_1) and (H_2) are satisfied. Then a necessary and sufficient condition for Equation (1) to has a solution (x_n) which is asymptotically equivalent to the sequence ($b_n = \prod_{i=1}^{n} a^i$) is

$$\sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{1}{a_l} \sum_{i=1}^{\infty} c_i < \infty. \tag{31}$$

Proof. From Theorems 3 and 4, we can directly prove the theorem by using the notation (3) and in the condition (14).

Theorem 6. Assume that (H_1) and (H_2) are satisfied. Let g be a monotonic function. Then a necessary and sufficient condition for Equation (7) to has a solution (z_n) satisfying

$$\lim_{n\to\infty} \frac{z_n}{Q_n} \neq 0 \tag{32}$$

is

$$\sum_{i=1}^{\infty} d_i |CQ_{i+1-\sigma}| < \infty, \tag{33}$$

where $C \neq 0$ is a constant.

Proof. *Necessity*. Suppose that (z_n) is a non-oscillatory solution to Equation (7) which satisfies (32). Without loss of generality, we can assume that

$$\lim_{n\to\infty} \frac{z_n}{Q_n} = \lambda > 0.$$

Then we can find positive constants C_7 and C_8 such that $C_7 Q_n \leq z_n \leq C_8 Q_n$ for sufficiently large n. In that case we can find a positive integer n_4 such that $C_7 Q_{n+1-\sigma} \leq z_{n+1-\sigma} \leq C_8 Q_{n+1-\sigma}$ for $n \geq n_4$. Since g is a monotonic function,

$$g(z_{n+1-\sigma} \geq g(C_9 Q_{i+1-\sigma}) \tag{34}$$

where $C_9 = C_7$, if the function g is nondecreasing and $C_9 = C_8$, if the function g is nonincreasing. By (H_1), $g(C_9 Q_{i+1-\sigma})$ is positive. On the other hand, if we sum Equation (7) form $n_5 = n_4 + \sigma$ to $n-1$, and consider Lemma 3, we have

$$0 < b_n \Delta z_n + z_n = b_{n_5} \Delta z_{n_5} + z_{n_5} - \sum_{i=n_5}^{n-1} c_n g(z_{n+1-\sigma}). \tag{35}$$

(35) implies that

$$\sum_{i=n_5}^{n-1} c_n g(z_{n+1-\sigma}) \leq b_{n_5} \Delta z_{n_5} + z_{n_5} < \infty.$$

Hence by (34), we obtain

$$\sum_{i=n_5}^{\infty} c_n g(C_9 Q_{i+1-\sigma}) < \infty.$$

Sufficiency. Let consider C_{10} is a positive constant. Set $I_n = \left[\frac{C_{10}}{2} Q_n, C_{10} Q_n\right]$. According to the property of g and the assumption (H_1), g has a maximum value on the interval I_n, which we denote if the function g is nonincreasing, as the point $C_{11} Q_n$ with $C_{11} = \frac{C_{10}}{2}$, and if the function g is nondecreasing with $C_{11} = C_{10}$. Then we have

$$g(z_n) \leq g(C_{11} Q_n) \text{ for } n \in I_n. \tag{36}$$

Assume that (34) holds for $C = C_{11}$. Thus we can find a positive integer n_6 such that

$$\sum_{i=n_6}^{\infty} c_n g(C_9 Q_{i+1-\sigma}) \leq \frac{C_{10}}{2}. \tag{37}$$

Now let consider the Banach space B of all real $z = (z_n)$ such that

$$\|z_n\| = \sup_{n \geq n_7} \frac{z_n}{Q_n^2} < \infty,$$

where $n_7 = n_6 + \sigma - 1$. Set

$$S = \left\{(z_n) \in B: z_n = \frac{C_{10}}{2} \text{ for } n < n_7,\ z_n \in I_n \text{ for } n \geq n_7\right\}.$$

It is easy to see that S is a subset of B which is convex, closed and bounded. We define an operator $T: S \to B$ as in the follow

$$(Tz)_n = \begin{cases} \dfrac{C_{10}}{2Q_n} & \text{for } n < n_7, \\ \dfrac{C_{10}}{2Q_n} + \displaystyle\sum_{i=n_7}^{n-1} \frac{1}{b_i} \sum_{k=i}^{\infty} c_k g(z_{k+1-\sigma}) & \text{for } n \geq n_7. \end{cases} \tag{38}$$

First, we show that $T(S) \subset S$. Indeed, if $z \in S$, it is clear from (38) that $(Tz)_n \geq \frac{C_{10}}{2Q_n}$ for $n \geq 1$. Furthermore, by (38) for $n \in I_n$, we have

$$\begin{aligned} (Tz)_n &< \frac{C_{10}}{2Q_n} + \sum_{i=n_7}^{n-1} \frac{1}{b_i} \sum_{k=i}^{\infty} c_k g(z_{k+1-\sigma}) \\ &< \frac{C_{10}}{2Q_n} + \sum_{i=1}^{n-1} \frac{1}{b_i} \sum_{k=n_7}^{\infty} c_k g(z_{k+1-\sigma}) \\ &< \frac{C_{10}}{2Q_n} + \frac{1}{Q_n} \sum_{k=n_7}^{\infty} c_k g(C_9 Q_{k+1-\sigma}) \\ &< \frac{C_{10}}{2Q_n} + \frac{1}{Q_n} \frac{C_{10}}{2} = \frac{C_{10}}{Q_n}. \end{aligned}$$

Hence T is an operator which maps S into itself.

Next we will prove that T is a continuous operator. Assume that $(z^{(m)})$ is a sequence in S such that $z^{(m)} \to z$ as $m \to \infty$. Since S is a closed subset, $z \in S$. Let get

$$\left|\left(Tz^{(m)}\right)_n - (Tz)_n\right| \leq \frac{1}{Q_n} \sum_{k=n_7}^{\infty} c_k \left|g\left(z_{k+1-\sigma}^{(m)}\right) - g(z_{k+1-\sigma})\right|.$$

From this we can write

$$\left\|\left(Tz^{(m)}\right)_n - (Tz)_n\right\| \leq \frac{1}{Q_n} \sum_{k=n_7}^{\infty} C_k \left|g\left(z_{k+1-\sigma}^{(m)}\right) - g(z_{k+1-\sigma})\right|.$$

By (H_1) and (H_2) we have $\lim_{n\to\infty} Q_n = \infty$ and considering (34) and (38), from last inequality we reach

$$\left\|\left(Tz^{(m)}\right)_n - (Tz)_n\right\| \leq \frac{2}{Q_n} \sum_{k=n_7}^{\infty} c_k g(C_9 Q_{k+1-\sigma}) \to 0.$$

Thus, we show that T is a continuous transformation.

Now, we will prove that $T(S)$ is uniformly Cauchy. For this, we need to prove that, given any $\varepsilon > 0$, we can find an integer n_8 such that for $m > n > n_8$. We get

$$\left\| \frac{(Tz)_m}{Q_m^2} - \frac{(Tz)_n}{Q_n^2} \right\| < \varepsilon$$

for any $z \in S$. Indeed, we have

$$\left\| \frac{(Tz)_m}{Q_m^2} - \frac{(Tz)_n}{Q_n^2} \right\| \le \frac{2}{Q_n} \sum_{k=1}^{\infty} c_k g(z_{k+1-\sigma}) \le \frac{C}{Q_n} \to 0.$$

Thus, by Schauder Theorem, there exists $z \in S$ such that $z_n = (Tz)_n$ for $n \ge n_8$. Furthermore, by Theorem 2 and definition of Q_n we have

$$\lim_{n\to\infty} \frac{z_n}{Q_n} = \lim_{n\to\infty} \frac{\Delta z_n}{\Delta Q_n} = \lim_{n\to\infty} (b_n \Delta z_n).$$

Hence

$$\lim_{n\to\infty} \frac{z_n}{Q_n} \le \lim_{n\to\infty} \left(\frac{C}{2} + \sum_{k=1}^{\infty} c_k g(z_{k+1-\sigma}) \right)$$
$$\le \lim_{n\to\infty} \left(\frac{C}{2} + \sum_{k=1}^{\infty} c_k g(C_{10} Q_{k+1-\sigma}) \right) = \frac{C}{2}.$$

Hence we complete the proof.

Example 3. Consider the equation

$$\Delta_a (x_n - a^{n+1} x_{n-1}) + \Delta_a \left(\frac{x_{n-1}}{b_{n-1}} \right) + q_n f(x_{n-\tau}) = 0, \qquad (39)$$

where $a = 4, b_n = 2^{n(n+1)}$ and $q_n = 3 + \frac{7}{2} 2^{-n} - 2^{n+4} - \frac{2^n+1}{2^n 2^{(n+1)(n+2)}} - \frac{2^{n+1}+4}{2^{n-1} 2^{n(n+1)}}$.

$x_n = 1 + \frac{1}{2^n}$ is an asymptotically constant solution of Equation (39).

3. CONCLUSION

In this paper, a new generalized equation was considered by expanding the equation containing the generalized difference operator previously considered by Bolat and Akın [6] and Agata Bezubik and et all. in [7], and this equation was reduced to an equation with a normal operator. The behavior of the solutions of this reduced equation was examined by making use of previous [9-13] studies new results were given.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Agarwal, R. P., Wong, Y. J., "Advanced Topics in Difference Equations", Kluwer, Dordrecht, (1997).

[2] Agarwal, R. P., "Difference Equations and Inequalities, Theory, Methods and Applications", Second Edition, Marcel Dekker, New York, (2000).

[3] Elaydi, S., "An Introduction to Difference Equations", Third Edition, Springer, New York, (2005).

[4] Gyori, I., Ladas, G., "Oscillation Theory of Delay Differential Equations with Applications", Clarendon, Oxford, (1991).

[5] Parhi, N., "Oscillation and non-oscillation of solutions of second order difference equations involving generalized difference", Applied Mathematics and Computation, 218: 458-468, (2011). DOI: 10.1016/j.amc.2011.05.086

[6] Bolat, Y., Akın, Ö., "Oscillation criteria for higher-order half-linear delay difference equations involving generalized difference", Mathematica Slovaca, 66(3): 677-686, (2016). DOI: 10.1515/ms-2015-0169

[7] Bezubik, A., Migda, M., Nockowska-Rosiak, M., Schmeidel, E., "Trichotomy of nonoscillatory solutions to second-order neutral difference equation with quasi-difference", Advances in Difference Equations, 2015: 192, (2015). DOI: 10.1186/s13662-015-0531-6

[8] Cheng, S. S., Patula, W. T., "An existence theorem for a nonlinear difference equation", Nonlinear Analysis, 20: 193-203, (1993).

[9] Saker, S. H., "Oscillation criteria of second-order half-linear delay difference equations", Kyungpook Mathematical Journal, 45: 579-594, (2005).

[10] Saker, S. H., "Oscillation of second-order nonlinear difference equations", Bulletin of the Korean Mathematical Society, 40(3): 489-501, (2003). DOI: 10.4134/BKMS.2003.40.3.489

[11] Vidhyaa, K., Thandapani, E., Alzabut, J., Özbekler, A., "Oscillation criteria for non-canonical second order nonlinear delay difference equations with a superlinear neutral term", Electronic Journal of Differential Equations, 2023(45): 1-12, (2023). DOI: 10.58997/ejde.2023.45

[12] Tan, M., Yang, E., "Oscillation and nonoscillation theorems for second order nonlinear difference equations", Journal of Mathematical Analysis and Applications, 276: 239-247, (2002). DOI: 10.1016/S0022-247X(02)00435-3

[13] Zhang, Z., Bi, P., Chen, J., "Oscillation of second order nonlinear difference equation with continuous variable", Journal of Mathematical Analysis and Applications, 255: 349–357, (2001).