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Research Article

$\phi-$Multiplicative Calculus

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Abstract: In this paper, we present a novel mathematical framework termed "ϕ-multiplicative calculus", which serves as a Golden Fibonacci calculus to fundamental concepts in multiplicative calculus. This innovative calculus introduces a parameter ϕ (Golden ratio), offering a nuanced extension of traditional calculus. Our work encompasses the establishment of ϕ-multiplicative calculus and the demonstration of essential theorems concerning derivatives, integrals, and their operation properties within this mathematical framework. The paper contributes to the academic discourse by providing a comprehensive exploration of the proposed ϕ-multiplicative calculus, presenting a robust foundation for further investigations in this specialized mathematical domain.

Keywords: Fibonacci numbers, multiplicative derivative, Golden calculus, integral.

1. Introduction

The most practical mathematical theory, differential and integral calculus, was developed separately by Isaac Newton and Gottfried Wilhelm Leibnitz in the second part of the 17th century [4]. Then, Leonard Euler redirected calculus by making the idea of function essential, and so created analysis [5].

During the time from 1967 to 1970, Michael Grossman and Robert Katz defined a new type of derivative and integral, reversing the roles of subtraction and addition and establishing a new calculus known as multiplicative calculus. It is also known as an alternate or non-Newtonian calculus at times. Unfortunately, multiplicative calculus isn't as well known as Newton's and Leibnitz's calculus, despite the fact that it completely answers all of the requirements demanded of a calculus theory. Multiplicative calculus has a more limited range of applications than Newton and Leibniz calculus. It indeed covers only positive functions. Therefore, one might question the rationale behind developing a new tool with a restrictive scope when a broader, well-developed tool is already in existence. Bashirov and et al. respond to this question similarly to why

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mathematicians use polar coordinates when there is already a rectangular coordinate system that well describes the points on a plane. Bashirov and et al. believe that multiplicative calculus can be particularly useful as a mathematical tool for economics and finance due to the interpretation given to the multiplicative derivative [1, 10, 21]. Subsequently, the multiplicative finite difference method was developed for computing the numerical solutions of high-order multiplicative limit value problems [19]. Furthermore, applications such as modeling multiplicative differential equations, analyzing multiplicative gradients in noisy images, exploring the impact of multiplicative analysis on biomedical image analysis, and investigating double multiplicative integrals have been seen in the literature [2, 3, 6, 9, 16].

Özvatan develops the Golden Fibonacci calculus in her master's thesis under the supervisor of Pashayev, and numerous applications of this calculus are achieved. The calculus is built around the Golden derivative as a finite difference operator with Golden and Silver ratio bases, allowing us to introduce Golden polynomials and Taylor expansions in terms of these polynomials. The Golden binomial and its expansion in terms of Fibonomial coefficients is derived. They demonstrated that Golden binomials correspond to Carlitz' characteristic polynomials. The Golden-heat and Golden-wave equations are introduced and solved using Golden Fibonacci exponential functions and associated whole functions. They build the higher order Golden Fibonacci calculus by presenting higher order Golden Fibonacci derivatives that are connected to powers of golden ratio. This calculus has higher order Fibonacci numbers, higher Golden periodic functions, and higher Fibonomials. They present the generating function for a new form of polynomial, the Bernoulli-Fibonacci polynomials, and investigate their characteristics using the Golden Fibonacci exponential function [17].

As mentioned above, valuable studies have been conducted on the ϕ-multiplicative calculus. As is known, the classical derivative method relies on the definition of a limit for the derivation of a function. Therefore, the classical differentiation method can introduce errors. However, since the Golden Fibonacci calculation is not dependent on the limit definition and performs algebraic operations, there is no margin for error. Similarly, in the golden product calculation, no error is incurred. ϕ-multiplicative calculus exhibits a narrower domain of application in comparison to Golden Fibonacci calculus. However, although the use of the natural logarithm in the ϕ-multiplicative calculation is expected to increase the time, it completes solutions more quickly in examples involving exponential functions, often demonstrating a more effective solution to specific problems. Thus, it provides an alternative approach to problem-solving. Against this backdrop, the current study aims to spotlight ϕ-multiplicative calculus within the realm of analysis and elucidate its applications.

Now, we examine the necessary information about Golden Fibonacci calculus.

The Fibonacci numbers satisfy the recursion relation

$$F_{n+2} = F_{n+1} + F_n$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. First few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13.

The Binet formule of the Fibonacci sequence is

$$F_n = \frac{\phi^n - \hat{\phi}^n}{\phi - \hat{\phi}},$$

where

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1,618033 \quad \text{and} \quad \hat{\phi} = \frac{1-\sqrt{5}}{2} = -\frac{1}{\phi}$$

are roots of quadratic equation of the Fibonacci sequence

$$x^2 - x - 1 = 0.$$

The number ϕ is known as the Golden ratio. The Golden ratio has been applied in an extensive variety, from natural occurrences to architecture and music. Some authors have satisfied and studied many generalizations of Fibonacci numbers. More information can be found in [12].

Let's define a Fibonomial Calculus for the sequence $\{F_n\}_{n\geq 0}$ in order to explain what it is:

- $F-$factorial: $F_n! = F_n F_{n-1} F_{n-2} \ldots F_2 F_1$, $F_0! = 1$.
- Fibonomial coefficients: $\binom{n}{k}_F = \dfrac{F_n!}{F_{n-k}! F_k!}$, $\binom{n}{0}_F = 1$.

  So, the following identity is valid:

$$\binom{n}{k}_F = \binom{n}{n-k}_F.$$

- The binomial theorem for the $F-$analog given by

$$(x +_F y)^n = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k}.$$

- The $F-$exponential function e_F^x defined by

$$e_F^x = \sum_{n=0}^{\infty} \frac{x^n}{F_n!}.$$

  Further insights can be found in the works of [13, 14, 17].

Let $f:\mathbb{R}-\{0\}\to\mathbb{R}$. The Golden derivative operator D_F of $f(x)$ is given as

$$D_F[f(x)]=\frac{f(\phi x)-f(\hat{\phi}x)}{(\phi-\hat{\phi})x}=\frac{f(\phi x)-f(\hat{\phi}x)}{x\sqrt{5}}. \tag{1}$$

The Golden derivative operator is a linear operator because the following conditions apply for any pair of functions f and g and scalar α;

$$D_F[f(x)+g(x)]=D_F[f(x)]+D_F[g(x)],$$

$$D_F[\alpha f(x)]=\alpha D_F[f(x)].$$

The Golden derivative operator D_F on x^n yields

$$D_F[x^n]=F_n x^{n-1}.$$

This, $F-$exponential function under Golden derivative are given by

$$D_F[e_F^{ax}]=ae_F^{ax} \quad (a \quad \text{any constant}).$$

The $F-$analogues of the sine and cosine functions can be described in terms of the exponential function by analogy with their well-known Euler formulas:

$$\sin_F x=\frac{e_F^{ix}-e_F^{-ix}}{2i} \quad \text{and} \quad \cos_F x=\frac{e_F^{ix}+e_F^{-ix}}{2},$$

and the Golden derivatives of these equations are as follows:

$$D_F[\sin_F \lambda x]=\lambda\cos_F \lambda x \quad \text{and} \quad D_F[\cos_F \lambda x]=-\lambda\sin_F \lambda x.$$

The Golden Leibnitz rule using the Golden derivative operator D_F is derived

$$D_F[f(x)g(x)]=D_F[f(x)]g(\phi x)+f(\hat{\phi}x)D_F[g(x)].$$

The Golden derivative of the quotient of $f(x)$ and $g(x)$ may now be computed

$$D_F\left[\frac{f(x)}{g(x)}\right]=\frac{D_F[f(x)]g(\phi x)-f(\phi x)D_F[g(x)]}{g(\phi x)g(\hat{\phi}x)}.$$

Consult prior studies for further details [13, 17, 18].

The function $H(x)$ is called the Golden antiderivative of any function $h(x)$ if $D_F[H(x)]=h(x)$. It is indicated by

$$H(x)+C=\int h(x)d_F x,$$

where C is the constant term. Let a be a real number. The Jackson integral of $h(x)$ is defined to be the series

$$\int_b^a h(x)d_F x = \int_0^a h(x)d_F x - \int_0^b h(x)d_F x,$$

where

$$H(x) = \int_0^y h(x)d_F x = y\sqrt{5}\sum_{k=0}^{\infty}\frac{\hat{\phi}^k}{\phi^{k+1}}f\left(\frac{\hat{\phi}^k}{\phi^{k+1}}y\right).$$

If $H(x)$ is an antiderivative of $h(x)$ and $H(x)$ is continuous at $x = 0$, we have

$$\int_b^a D_F[h(x)]d_F x = h(a) - h(b),$$

where $0 \leq b < a \leq \infty$. Refer to the works of [11, 18, 20].

Now, we will look at the information about multiplicative derivatives that are required.

A real function f is said to be differentiable at a point $x \in \mathbb{R}$ if and only if f is defined on some open interval I containing x and

$$D[f(x)] = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \tag{2}$$

exists. In that case, $f(x)$ is referred to as the derivative of f at x.

Here, first we write $f(x+h)/f(x)$ instead of $f(x+h) - f(x)$ in the (2) equation. Then, if $1/h$ is substituted for h, the multiplicative derivative is obtained as follows:

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, the multiplicative derivative $D^*[f(x)]$ of f at $x \in A$ is defined

$$\begin{aligned} D^*[f(x)] &= \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}} \\ &= \lim_{h\to 0} e^{\ln\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}} \\ &= e^{\lim_{h\to 0}\frac{\ln(f(x+h)) - \ln(f(x))}{h}} \\ &= e^{D[\ln(f(x))]} = e^{\frac{D[f(x)]}{f(x)}}. \end{aligned}$$

The multiplicative integral of f is represented by the symbol

$$*\int f(x)^{dx}.$$

If f is a positive function and $\ln f$ on $[a,b]$ is integrable, then f on $[a,b]$ has a multiplicative integral, which is defined by

$$*\int_b^a f(x)^{dx} = e^{\int_b^a \ln f(x)dx}, \quad 0 < b < a.$$

Reference previous works for more insights [7, 8, 15, 22].

2. ϕ-Multiplicative Calculus

2.1. Golden Multiplicative Derivative

Subtraction and division are the operations in the difference quotient in (1). The multiplicative derivative of a function g, on the other hand, is based on the ratio (3). This is as follows similar to the difference quotient (1) with subtraction of $f(\phi x) - f(\hat{\phi} x)$ replaced with division by $f(\phi x)/f(\hat{\phi} x)$ and division by $x\sqrt{5}$ replaced with taking an $1/(x\sqrt{5})$ power:

$$\left(\frac{f(\phi x)}{f(\hat{\phi} x)}\right)^{\frac{1}{x\sqrt{5}}}. \tag{3}$$

Definition 2.1 *Assume that the function f is Golden differentiable and positive ($f(\phi x) > 0$ for all x). The Golden multiplicative derivative of f is defined as follows:*

$$D_F^*[f(x)] = \left(\frac{f(\phi x)}{f(\hat{\phi} x)}\right)^{\frac{1}{x\sqrt{5}}}. \tag{4}$$

Theorem 2.2 *If a positive function f is D_F-differentiable at x, then it is also D_F^*-differentiable at x, and*

$$D_F^*[f(x)] = e_F^{D_F[\ln(f(x))]} \tag{5}$$

Proof We may compute the derivative in equation (4) using what we know about the Golden derivative of f:

$$\begin{aligned} D_F^*[f(x)] = \left(\frac{f(\phi x)}{f(\hat{\phi} x)}\right)^{\frac{1}{x\sqrt{5}}} &= e_F^{\ln\left(\frac{f(\phi x)}{f(\hat{\phi} x)}\right)^{\frac{1}{x\sqrt{5}}}} \\ &= e_F^{\frac{\ln(f(\phi x)) - \ln(f(\hat{\phi} x))}{x\sqrt{5}}} \\ &= e_F^{D_F[\ln(f(x))]}, \end{aligned}$$

where $\ln f(x) = (\ln o f)(x)$. □

Higher-order Golden multiplicative derivatives are simply the Golden multiplicative derivative of a Golden multiplicative derivative. You would use the same Golden multiplicative derivative rules that you learned for finding the first Golden multiplicative derivative of a function.

Corollary 2.3 *Let f positive function. The nth Golden multiplicative derivative of f is given by*

$$D_F^{*(n)}[f(x)] = e_F^{D_F^{(n)}[\ln(f(x))]}, \quad n = 0, 1, 2, \ldots.$$

Proof Utilizing equality (5), we get second Golden multiplicative derivative as

$$\begin{aligned} D_F^{**}[f(x)] &= e_F^{D_F[\ln(e_F^{D_F[\ln(f(x))]})]} \\ &= e_F^{D_F[D_F[\ln(f(x))]]} \\ &= e_F^{D_F^2[\ln(f(x))]}. \end{aligned}$$

To identify further high-level derivations, we continue with the same transaction:

$$\begin{aligned} D_F^{*(3)}[f(x)] &= e_F^{D_F[\ln(e_F^{D_F^2[\ln(f(x))]})]} \\ &= e_F^{D_F[D_F^2[\ln(f(x))]]} \\ &= e_F^{D_F^3[\ln(f(x))]}. \end{aligned}$$

By induction method, if the nth Golden multiplicative derivative of f exists at x, then it is acquired by

$$D_F^{*(n)}[f(x)] = e_F^{D_F^{(n)}[\ln(f(x))]}.$$

□

2.2. The Operation Properties of the Golden Multiplicative Derivative

Here are a few guidelines that are supported by Definition 2.1. Assume that, λ, μ is a positive constant and that f and g are D_F^*−differentiable. The following list may thus be displayed with ease:

- $D_F^*[\lambda f(x)] = D_F^*[f(x)]$ (Constant rule),
- $D_F^*[f(x)g(x)] = D_F^*[f(x)]D_F^*[g(x)]$ (Product rule),
- $D_F^*[f(x)/g(x)] = D_F^*[f(x)]/D_F^*[g(x)]$ (Quotient rule),

- $D_F^*[fog(x)] = D_{F,g}^*[f(g(x))]^{D_F[g(x)]}$ (Chain rule),
- $D_F^*[f(x)^{g(x)}] = (D_F^*[f(x)])^{g(\phi x)} \left(f(\hat{\phi}x)\right)^{D_F[g(x)]}$ (Power rule).

The proofs of the rules are shown as follows:

- The proof of the constant rule is

$$\begin{aligned} D_F^*[\lambda f(x)] &= \left(\frac{\lambda f(\phi x)}{\lambda f(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= \left(\frac{f(\phi x)}{f(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= D_F^*[f(x)]. \end{aligned}$$

- The proof of the product rule is

$$\begin{aligned} D_F^*[f(x)g(x)] &= \left(\frac{f(\phi x)g(\phi x)}{f(\hat{\phi}x)g(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= \left(\frac{f(\phi x)}{f(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \left(\frac{g(\phi x)}{g(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= D_F^*[f(x)]D_F^*[g(x)]. \end{aligned}$$

- The proof of the quotient rule is

$$\begin{aligned} D_F^*[f(x)/g(x)] &= \left(\frac{f(\phi x)/g(\phi x)}{f(\hat{\phi}x)/g(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= \left(\frac{f(\phi x)}{f(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} / \left(\frac{g(\phi x)}{g(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= D_F^*[f(x)]/D_F^*[g(x)] \end{aligned}$$

- The proof of the chain rule is

$$\begin{aligned} D_F^*[fog(x)] &= \left(\frac{f(g(\phi x))}{f(g(\hat{\phi}x))}\right)^{\frac{1}{x\sqrt{5}}} \\ &= \left(\frac{f(g(\phi x))}{f(g(\hat{\phi}x))}\right)^{\frac{1}{g(\phi x)-g(\hat{\phi}x)} \frac{g(\phi x)-g(\hat{\phi}x)}{x\sqrt{5}}} \\ &= D_{F,g}^*[f(g(x))]^{D_F[g(x)]}. \end{aligned}$$

- The proof of the power rule is

$$D_F^*[f(x)^{g(x)}] = \left(\frac{f(\phi x)^{g(\phi x)}}{f(\hat{\phi}x)^{g(\hat{\phi}x)}}\right)^{\frac{1}{x\sqrt{5}}}$$

$$= \left(\frac{f(\phi x)^{g(\phi x)}}{f(\hat{\phi}x)^{g(\phi x)}}\frac{f(\hat{\phi}x)^{g(\phi x)}}{f(\hat{\phi}x)^{g(\hat{\phi}x)}}\right)^{\frac{1}{x\sqrt{5}}}$$

$$= \left(\frac{f(\phi x)}{f(\hat{\phi}x)}\right)^{\frac{g(\phi x)}{x\sqrt{5}}}\left(f(\hat{\phi}x)\right)^{\frac{g(\phi x)-g(\hat{\phi}x)}{x\sqrt{5}}}$$

$$= \left(D_F^*[f(x)]\right)^{g(\phi x)}\left(f(\hat{\phi}x)\right)^{D_F[g(x)]}.$$

Theorem 2.4 *Let $f(x)$ and $g(x)$ be two functions, for $n \in \mathbb{Z}^+$, then*

$$D_F^{*(n)}[f(x)^{g(x)}] = e_F^{\sum_{k=0}^{n}\binom{n}{k}_F D_F^k[\ln f(\phi^{n-k}x)]D_F^{n-k}[g(x)]}.$$

Proof We can prove it by induction method. It is obviously true for $n = 1$. Let's assume it is true for $n = m$. Let's show that it is true for $n = m+1$.

$$D_F^{*(m+1)}[f(x)^{g(x)}] = D_F^*[D_F^{*(m)}[f(x)^{g(x)}]]$$

$$= D_F^*\left[e_F^{\sum_{k=0}^{m}\binom{m}{k}_F D_F^k[\ln f(\phi^{m-k}x)]D_F^{m-k}[g(x)]}\right]$$

$$= e_F^{D_F[\sum_{k=0}^{m}\binom{m}{k}_F D_F^k[\ln f(\phi^{m-k}x)]D_F^{m-k}[g(x)]]}$$

$$= e_F^{\sum_{k=0}^{m+1}\binom{m+1}{k}_F D_F^k[\ln f(\phi^{m-k+1}x)]D_F^{m-k+1}[g(x)]}.$$

□

Example 2.5 *Find the golden multiplicative derivative of the exponential function $f(x) = \alpha^x$ with $\alpha > 0$.*

$$D_F^*[f(x)] = D_F^*[\alpha^x] = e_F^{D_F[\ln \alpha^x]} = e_F^{\ln \alpha D_F[x]} = \alpha.$$

In the Table 1, we compare the classical derivative, multiplicative derivative, Golden derivative, and Golden multiplicative derivative of of an arbitrary function.

2.3. Golden Multiplicative Antiderivative (Integral)

This part aims to explore the fundamental principles of the Golden multiplicative integral, shedding light on its diverse applications in various mathematical contexts.

Table 1: Some Golden multiplicative derivative

$f(x)$	$D[f(x)]$	$D^*[f(x)]$	$D_F[f(x)]$	$D_F^*[f(x)]$
t	0	1	0	1
te^{nx}	nte^{nx}	e^n	$t\sum_{m=0}^{\infty}\frac{F_m(nx)^m}{m!}$	e_F^n
ta^x	$ta^x\ln a$	a	$t\dfrac{a^{\phi x}-a^{\hat{\phi}x}}{x\sqrt{5}}$	a
tx^n	tnx^{n-1}	$e^{\frac{n}{x}}$	tF_nx^{n-1}	$e_F^{nD_F[\ln x^n]}$
$te^{h(x)}$	$tD[h(x)]e^{h(x)}$	$e^{D[h(x)]}$	$t\sum_{m=0}^{\infty}\frac{F_m(h(x))^m}{m!}$	$e_F^{D_F[h(x)]}$
$\frac{1}{h(x)}$	$\frac{-D[h(x)]}{h^2(x)}$	$e^{\frac{-D[h(x)]}{h^3(x)}}$	$\frac{-D_F[h(x)]}{h(\phi x)h(\hat{\phi}x)}$	$e_F^{D_F[-\ln(h(x))]}$
$t(h(x))^n$	$ntD[h(x)](h(x))^{n-1}$	$e^{\frac{nD[h(x)]}{h(x)}}$	$t\dfrac{(h(\phi x))^n-(h(\hat{\phi}x))^n}{x\sqrt{5}}$	$e_F^{nD_F[\ln(h(x))]}$
$e^{\sin_F bx}$	$b\cos bxe^{\sin_F bx}$	$e^{b\cos_F bx}$	$\dfrac{e^{\sin_F b\phi x}-e^{\sin_F b\hat{\phi}x}}{x\sqrt{5}}$	$e_F^{D_F[\sin_F bx]}$
$e^{\cos_F bx}$	$-b\sin_F bxe^{\cos_F bx}$	$e^{-b\sin_F bx}$	$\dfrac{e^{\cos_F b\phi x}-e^{\cos_F b\hat{\phi}x}}{x\sqrt{5}}$	$e_F^{D_F[\cos_F bx]}$

Definition 2.6 *Let h be a positive, bounded function on the range $0 < a < b$: The Golden multiplicative integral, the $F-$analogue of the multiplicative integral, may be defined by*

$$*\int h(x)^{d_F x} = e_F^{\int \ln h(x) d_F x}$$

and the definition of the definite golden multiplicative integral is

$$*\int_b^a h(x)^{d_F x} = e_F^{\int_0^a \ln h(x) d_F(x) - \int_0^b \ln h(x) d_F(x)}.$$

2.4. The Operation Properties of Golden Multiplicative Integral

If f and g are $F-$integrable on $[a,b]$, we can then simply demonstrate the following rules of $F-$integral:

- $*\int_b^a (f(x)^k)^{d_F x} = *\int_b^a (f(x)^{d_F x})^k$ (Constant rule),
- $*\int_b^a (f(x)g(x))^{d_F x} = *\int_b^a f(x)^{d_F x} * \int_b^a g(x)^{d_F x}$ (Product rule),
- $*\int_b^a (f(x)/g(x))^{d_F x} = *\int_b^a f(x)^{d_F x} / *\int_b^a g(x)^{d_F x}$ (Quotient rule),
- $*\int_b^a f(x)^{d_F x} = *\int_b^c f(x)^{d_F x} * \int_c^a f(x)^{d_F x}, \quad b \le c \le a.$

The proofs of the rules are shown as follows:

- The proof of the constant rule is

$$
\begin{aligned}
*\int_b^a (f(x)^k)^{d_F x} &= e_F^{\int_b^a \ln f(x)^k d_F x} \\
&= e_F^{k\int_b^a \ln f(x) d_F x} \\
&= \left(e_F^{\int_b^a \ln f(x) d_F x}\right)^k \\
&= *\int_b^a (f(x)^{d_F x})^k.
\end{aligned}
$$

- The proof of the product rule is

$$
\begin{aligned}
*\int_b^a (f(x)g(x))^{d_F x} &= e_F^{\int_b^a \ln(f(x)g(x)) d_F x} \\
&= e_F^{\int_b^a \ln f(x) d_F x + \int_b^a \ln g(x) d_F x} \\
&= e_F^{\int_b^a \ln f(x) d_F x} e_F^{\int_b^a \ln g(x) d_F x} \\
&= *\int_b^a f(x)^{d_F x} * \int_b^a g(x)^{d_F x}.
\end{aligned}
$$

- The proof of the quotient rule is

$$
\begin{aligned}
*\int_b^a (f(x)/g(x))^{d_F x} &= e_F^{\int_b^a \ln(f(x)/g(x)) d_F x} \\
&= e_F^{\int_b^a \ln f(x) d_F x - \int_b^a \ln g(x) d_F x} \\
&= \frac{e_F^{\int_b^a \ln f(x) d_F x}}{e_F^{\int_b^a \ln g(x) d_F x}} \\
&= *\int_b^a f(x)^{d_F x} / *\int_b^a g(x)^{d_F x}.
\end{aligned}
$$

-

$$
\begin{aligned}
*\int_b^a f(x)^{d_F x} &= e_F^{\int_0^a \ln f(x) d_F x - \int_0^b \ln f(x) d_F x} \\
&= e_F^{\int_0^c \ln f(x) d_F x - \int_0^b \ln f(x) d_F x + \int_0^a \ln f(x) d_F x - \int_0^c \ln f(x) d_F x} \\
&= e_F^{\int_b^c \ln f(x) d_F x + \int_c^a \ln f(x) d_F x} \\
&= *\int_b^c f(x)^{d_F x} * \int_c^a f(x)^{d_F x}.
\end{aligned}
$$

Example 2.7 *Let $f(x) = e_F^{\cos_F(\lambda x)}$, where λ is a constant. Then Golden multiplicative integral* *of $f(x)$ is obtained by*

$$\begin{aligned}
*\int (e_F^{\cos_F(\lambda x)})^{d_F x} &= e_F^{\int \ln e_F^{\cos_F(\lambda x)} d_F x} \\
&= e_F^{\int \cos_F(\lambda x) d_F x} \\
&= e_F^{\int \cos_F(\lambda x) d_F x} \\
&= e_F^{\frac{1}{\lambda}\sin_F(\lambda x)}.
\end{aligned}$$

Example 2.8 *Let $f(x) = e_F^{\lambda x}$, where $\lambda \in \mathbb{Z}^+$. Then Golden multiplicative integral of $f(x)$ is obtained by*

$$\begin{aligned}
*\int (e_F^{\lambda x})^{d_F x} &= e_F^{\int \ln e_F^{\lambda x} d_F x} \\
&= e_F^{\int \lambda x d_F x} \\
&= e_F^{c} e_F^{\frac{\lambda}{F_2} x^2}.
\end{aligned}$$

In the Table 2, we compare the classical integral, multiplicative integral, Golden integral, and Golden multiplicative integral of of an arbitrary function.

Table 2: Some Golden Multiplicative Integral

$f(x)$	$\int f(x)dx$	$*\int f(x)^{dx}$	$\int f(x)d_F x$	$*\int f(x)^{d_F x}$
1	x	e^c	x	e_F^c
t	tx	$e^c t^x$	tx	$e_F^c t^x$
e^{nx}	$\frac{e^{nx}}{n}$	$e^c e^{\frac{nx^2}{2}}$	$\frac{e_F^{nx}}{n}$	$e_F^c e_F^{\frac{nx^2}{F_2}}$
$e^{\cos x}$	$-$	$e^c e^{\sin x}$	$-$	$e_F^c e_F^{\sin x}$
$e^{\sin x}$	$-$	$e^c e^{-\cos x}$	$-$	$e_F^c e_F^{-\cos x}$

3. Conclusion

In conclusion, our paper introduces and establishes the ϕ-multiplicative calculus, a novel mathematical framework that extends fundamental concepts in multiplicative calculus by incorporating the Golden ratio (ϕ) as a key parameter. We have successfully demonstrated the essential theorems pertaining to derivatives, integrals, and operational properties within the ϕ-multiplicative calculus. This work contributes significantly to the academic discourse by providing a comprehensive exploration of this specialized mathematical domain, thereby laying a solid foundation for future research and investigations in the field.

Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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