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On an (ι, x_0)-Generalized Logistic-Type Function

Seda Karateke [1,†]

[1] *Department of Software Engineering, Faculty of Engineering and Natural Sciences, Istanbul Atlas University, 34408, Istanbul, Turkey*
[†] *seda.karateke@atlas.edu.tr*

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Abstract

In this article, some mathematical properties of (ι, x_0)-generalized logistic-type function are presented. This four-parameter generalized function can be considered as a statistical phenomenon enhancing more vigorous survival analysis models. Moreover, the behaviors of the relevant parametric functions obtained are examined with graphics using computer programming language Python 3.9.

1. Introduction

1.1. Motivation

Crudely put, the logistic and logistic-type functions play an important role in many scientific disciplines including probability and statistics, demography, machine learning, ecology, mathematical psychology and biology [1]. Actually, the logistic function has a long history dating back to the classical statistics and "belief neural networks" [2], [3]. It has a leading role in the logistic regression procedure, especially in terms of its statistical properties that we discuss here.While in early studies this appeared as the solution to a specific differential equation, it was later used as one of many possible smooth, monotonic "squash" functions that mapped real values to a limited range.

Over time, as a result of the increasing interest and need for learning concept and learning algorithms, the probabilistic properties of the logistic function have begun to be studied in depth. This orientation has led to more advanced learning methods. So, it has diversified and strengthened the connections between neural networks (NNs) and statistics.

Methods that preserve the logistic function offer a possibility in this context. So, as alternative methods to contingency table and general regression model; a simple artificial neural network architecture, a more comprehensive generalized additive model, or another flexible "approximate" model in logistic form may be a reason for preference. An example for generalized linear model is the generalization of logistic regression while probabilistic model for multi-class classification problem is a multinomial model. In this models, it is a reasonable approach to use a normalized exponential function as a logistic function, aka "softmax" function, which is defined below, and used intensively in the NNs literature [4], [5], [6].

Now, let $\sigma : \mathbb{R}^N \longmapsto (0,1)^N$ be a function defined by the formula

$$\sigma(j, z_1, z_2, \ldots, z_N) = \frac{e^{z_j}}{\sum\limits_{j=1}^{N} e^{z_j}},$$

for $N \geq 1$. This function σ called as "unit softmax function" employs the classical exponential function to each of the inputs denoted by $z_1, z_2, \ldots, z_N$ and all these values are normalized by being divided by the sum of all the exponentials. The normalization process provides that the sum of the components of the output vector is 1. In addition, the softmax function

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takes as inputs $z_1, z_2, ..., z_N$, and normalizes them into a probability distribution consisting of N probabilities proportional to the exponentials of the input numbers [7]. Moreover, this function takes values between 0 and 1. Here we give an (ι, x_0)-generalized logistic-type function (also can be considered as a parametric generalization of softmax function) and also examine some mathematical properties such as convexity, sub-additivity, and multiplicativity.
The present paper includes four sections. In the following section, the construction of suggested (ι, x_0)-generalized logistic-type function, and its analytical features are presented. After launching a brief introduction related to survival analysis; probability density function of related distribution, parametric exponential survival (PES) and parametric failure (hazard) rate (PFR) functions are given in the third section. We conclude the paper creating "ceteris paribus" graphics of these functions employing the computer programming language Python 3.9. Finally, we also add Python 3.9 codes as in Fig. 9 and Fig 10 at the end of the study to motivate readers to earn/develop her/his programming language ability.

2. Main Results

Let $\iota, \rho > 0$ be the parameters with $\xi > 1$; inspired by [8] and [9], we can consider an (ι, x_0)-generalized logistic-type function as follows:

$$\Psi_{\rho,\iota}(x) = \frac{1}{1+\rho\xi^{-\iota(x-x_0)}} = \frac{\xi^{\iota(x-x_0)}}{\rho+\xi^{\iota(x-x_0)}}, \tag{2.1}$$

where $x, x_0 \in \mathbb{R}$.
The first and second derivatives of the (ι, x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ are given as below:
let $\iota, \rho > 0$ be the parameters, and $\xi > 1$

$$\begin{aligned}
\Psi'_{\rho,\iota}(x) &= \left(\frac{1}{1+\rho\xi^{-\iota(x-x_0)}}\right)' = \rho\iota(\ln\xi)\,\xi^{-\iota(x-x_0)}\left(1+\rho\xi^{-\iota(x-x_0)}\right)^{-2} \\
&= \frac{\rho\iota(\ln\xi)}{\left(1+2\rho\xi^{-\iota(x-x_0)}+\rho^2\xi^{-2\iota(x-x_0)}\right)\xi^{\iota(x-x_0)}} \\
&= \frac{\rho\iota(\ln\xi)}{\left(\xi^{\iota(x-x_0)}+2\rho+\rho^2\xi^{-\iota(x-x_0)}\right)} = \rho\iota(\ln\xi)\left(\xi^{\iota(x-x_0)}+2\rho+\rho^2\xi^{-\iota(x-x_0)}\right)^{-1}
\end{aligned}$$

for all $x, x_0 \in \mathbb{R}$.

Besides, taking the second derivative of (2.1) for $x \in \mathbb{R}$ we get

$$\Psi''_{\rho,\iota}(x) = \rho\iota^2\left(\ln^2\xi\right)\left(\xi^{\iota(x-x_0)}+2\rho+\rho^2\xi^{-\iota(x-x_0)}\right)^{-2}\left(\rho^2\xi^{-\iota(x-x_0)}-\xi^{\iota(x-x_0)}\right)$$

Since

$$\Psi''_{\rho,\iota}(x) > 0 \Longleftrightarrow \left(\rho^2\xi^{-\iota(x-x_0)}-\xi^{\iota(x-x_0)}\right) > 0$$

$$\Leftrightarrow \rho^2\xi^{-\iota(x-x_0)} > \xi^{\iota(x-x_0)}$$

$$\Leftrightarrow \rho^2 > \xi^{2\iota(x-x_0)} \Leftrightarrow |\rho| > \left|\xi^{\iota(x-x_0)}\right| \Leftrightarrow \rho > \xi^{\iota(x-x_0)},$$

and for $\rho > 0, \xi > 1$

$$\log_\xi\rho > \iota(x-x_0) \Leftrightarrow \frac{\log_\xi\rho}{\iota} + x_0 > x$$

is obtained.
Let $x < x_0 + \frac{\log_\xi\rho}{\iota} - 1$, then $x-1 < x+1 < x_0 + \frac{\log_\xi\rho}{\iota}$.
$\Psi'_{\rho,\iota}(x+1) > \Psi'_{\rho,\iota}(x-1)$. Thus $\Psi'_{\rho,\iota}(x)$ is positive and strictly increasing on $\left(-\infty,\ x_0 + \frac{\log_\xi\rho}{\iota}\right)$. Now, let $x > x_0 + \frac{\log_\xi\rho}{\iota} + 1$, then $x+1 > x-1 > x_0 + \frac{\log_\xi\rho}{\iota}$, and $\Psi'_{\rho,\iota}(x+1) < \Psi'_{\rho,\iota}(x-1)$. So $\Psi'_{\rho,\iota}(x)$ is strictly decreasing on $\left(x_0 + \frac{\log_\xi\rho}{\iota}, +\infty\right)$.

Proposition 2.1. *Let $\iota, \rho > 0$ be the parameters, $\xi > 1$, and $\Psi_{\rho,\iota}(x)$ be described as in (2.1). Now, let us take the first derivative:*

$$\begin{aligned}
\Psi'_{\rho,\iota}(x) &= \rho\iota(\ln\xi)\,\xi^{-\iota(x-x_0)}\frac{1}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2} \\
&= \iota(\ln\xi)\,\Psi_{\rho,\iota}(x)\left(1-\Psi_{\rho,\iota}(x)\right).
\end{aligned} \tag{2.2}$$

By then, let's take the second derivative of the function $\Psi_{\rho,\iota}$:

$$\begin{aligned}
\Psi^{''}_{\rho,\iota}(x) &= \iota(\ln\xi)\left(\Psi_{\rho,\iota}(x)-\Psi^{2}_{\rho,\iota}(x)\right)^{'} \\
&= \iota(\ln\xi)\Psi^{'}_{\rho,\iota}(x)\left(1-2\Psi_{\rho,\iota}(x)\right) \\
&= \iota^{2}\left(\ln^{2}\xi\right)\Psi_{\rho,\iota}(x)\left(1-\Psi_{\rho,\iota}(x)\right)\left(1-2\Psi_{\rho,\iota}(x)\right).
\end{aligned}$$

Thus the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ has the following properties:

$$\lim_{x\to+\infty}\Psi_{\rho,\iota}(x)=\lim_{x\to+\infty}\frac{\xi^{\iota(x-x_0)}}{\rho+\xi^{\iota(x-x_0)}}=1,$$

$$\lim_{x\to-\infty}\Psi_{\rho,\iota}(x)=\lim_{x\to-\infty}\frac{\xi^{\iota(x-x_0)}}{\rho+\xi^{\iota(x-x_0)}}=0,$$

$$\lim_{x\to x_0}\Psi_{\rho,\iota}(x)=\lim_{x\to x_0}\frac{\xi^{\iota(x-x_0)}}{\rho+\xi^{\iota(x-x_0)}}=\frac{1}{1+\rho};\rho>0,$$

$$\lim_{x\to x_0}\Psi^{'}_{\rho,\iota}(x)=\lim_{x\to x_0}\frac{\rho\iota(\ln\xi)}{\xi^{\iota(x-x_0)}\left(1+\rho\xi^{-\iota(x-x_0)}\right)^{2}}=\frac{\rho\iota(\ln\xi)}{(1+\rho)^{2}},$$

$$\lim_{x\to-\infty}\Psi^{'}_{\rho,\iota}(x)=\lim_{x\to-\infty}\rho\iota(\ln\xi)\ \frac{1}{\xi^{\iota(x-x_0)}+\rho^{2}\xi^{-\iota(x-x_0)}+2\rho}=0,$$

and

$$\int\Psi_{\rho,\iota}(x)\,dx=\int\frac{\xi^{\iota(x-x_0)}}{\rho+\xi^{\iota(x-x_0)}}dx=\frac{1}{\iota(\ln\xi)}\ln\left(\rho+\xi^{\iota(x-x_0)}\right)+C,\ C\text{ is a constant.}\tag{2.3}$$

Remark 2.2. *Additionally, if $\xi=e$, then (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ acts like an $\iota-$generalization of softplus function (see [1]). The derivative of (2.3) yields the $\iota-$generalized logistic-type function.*

Proposition 2.3. *From (2.2), $\Psi_{\rho,\iota}(x)$ is increasing and positive on $\left(-\infty,\ x_0+\frac{\log_{\xi}\rho}{\iota}\right)$. Furthermore, $l:=\Psi_{\rho,\iota}(x)$ is a solution to the initial value problem*

$$\left\{l^{'}=\iota(\ln\xi)\,l(1-l),\ l(x_0)=\frac{1}{\rho+1};\ \rho>0.\right.$$

Theorem 2.4. *The (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ satisfies the following inequality:*

$$\Psi_{\rho,\iota}(x+y)<\Psi_{\rho,\iota}(x)+\Psi_{\rho,\iota}(y),$$

for $x_0\geq 0$, $\iota,\rho>0$, $x,y\in(-\infty,0)$, and also $x,y\in\left(x_0+\frac{\log_{\xi}\rho}{\iota},+\infty\right)$. In other words, the function $\Psi_{\rho,\iota}$ is sub-additive on $(-\infty,0)\cup\left(x_0+\frac{\log_{\xi}\rho}{\iota},+\infty\right)$.

Proof. We need to prove the cases $x,y\in(-\infty,0)$ and $x,y\in\left(x_0+\frac{\log_{\xi}\rho}{\iota},+\infty\right)$, respectively.
The case $x=y=0$ is straightforward.
For any fixed y: we obtain

$$\begin{aligned}
\varphi_{\rho,\iota}(x,y)\ &:= \Psi_{\rho,\iota}(x+y)-\Psi_{\rho,\iota}(x)-\Psi_{\rho,\iota}(y) \\
&= \frac{1}{1+\rho\xi^{-\iota(x+y-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(x-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(y-x_0)}} \\
&= \frac{\xi^{\iota(x+y-x_0)}}{\xi^{\iota(x+y-x_0)}+\rho}-\frac{\xi^{\iota(x-x_0)}}{\xi^{\iota(x-x_0)}+\rho}-\frac{\xi^{\iota(y-x_0)}}{\xi^{\iota(y-x_0)}+\rho},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial x}\varphi_{\rho,\iota}(x,y) &= \frac{\partial}{\partial x}\left(\frac{1}{1+\rho\xi^{-\iota(x+y-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(x-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(y-x_0)}}\right)\\
&= \frac{-\left(\rho\xi^{-\iota(x+y-x_0)}(-\iota)(\ln\xi)\right)}{\left(1+\rho\xi^{-\iota(x+y-x_0)}\right)^2}-\left(\frac{-\left(\rho\xi^{-\iota(x-x_0)}(-\iota)(\ln\xi)\right)}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2}\right)\\
&= \frac{\iota\rho(\ln\xi)\xi^{-\iota(x+y-x_0)}}{\left(1+\rho\xi^{-\iota(x+y-x_0)}\right)^2}-\frac{\iota\rho(\ln\xi)\xi^{-\iota(x-x_0)}}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2}.
\end{aligned}$$

For $\Psi'_{\rho,\iota}(x)$ is decreasing on $(x_0,+\infty)$, hence $\Psi_{\rho,\iota}(x)$ is decreasing on the same interval. Then for $x,y\in(x_0,+\infty)$, we can have

$$\begin{aligned}
\varphi_{\rho,\iota}(x,y) &< \varphi_{\rho,\iota}\left(x,x_0+\frac{\log_\xi\rho}{\iota}\right)\\
&= \lim_{x\to x_0+\frac{\log_\xi\rho}{\iota}}\varphi_{\rho,\iota}\left(x,x_0+\frac{\log_\xi\rho}{\iota}\right)\\
&= \varphi_{\rho,\iota}\left(x_0+\frac{\log_\xi\rho}{\iota},x_0+\frac{\log_\xi\rho}{\iota}\right)\\
&= \Psi_{\rho,\iota}\left(2x_0+\frac{2\log_\xi\rho}{\iota}\right)-\Psi_{\rho,\iota}\left(x_0+\frac{\log_\xi\rho}{\iota}\right)-\Psi_{\rho,\iota}\left(x_0+\frac{\log_\xi\rho}{\iota}\right)\\
&= \frac{1}{1+\rho\xi^{-\iota\left(2x_0+\frac{2\log_\xi\rho}{\iota}-x_0\right)}}-\frac{2}{1+\rho\xi^{-\iota\left(x_0+\frac{\log_\xi\rho}{\iota}-x_0\right)}}\\
&= \frac{1}{1+\rho\xi^{-\iota\left(x_0+\frac{2\log_\xi\rho}{\iota}\right)}}-\frac{2}{1+\rho\frac{1}{\rho}}=\frac{1}{1+\rho\xi^{-\iota\left(x_0+\frac{2\log_\xi\rho}{\iota}\right)}}-1\\
&= -\frac{\rho\xi^{-\iota\left(x_0+\frac{2\log_\xi\rho}{\iota}\right)}}{1+\rho\xi^{-\iota\left(x_0+\frac{2\log_\xi\rho}{\iota}\right)}}<0.
\end{aligned}$$

Thus $\varphi_{\rho,\iota}$ is increasing on $\left(-\infty,x_0+\frac{\log_\xi\rho}{\iota}\right)$.
We have

$$\begin{aligned}
\varphi_{\rho,\iota}(x,y) &< \varphi_{\rho,\iota}(x,0)=\lim_{x\to 0}\varphi_{\rho,\iota}(x,0)\\
&= \lim_{x\to 0}\left(\Psi_{\rho,\iota}(x+0)-\Psi_{\rho,\iota}(x)-\Psi_{\rho,\iota}(0)\right)\\
&= \lim_{x\to 0}\left\{\frac{1}{1+\rho\xi^{-\iota(x-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(x-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(-x_0)}}\right\}\\
&= \lim_{x\to 0}-\frac{1}{1+\rho\xi^{\iota x_0}}<0.
\end{aligned}$$

□

Remark 2.5. *In Theorem 2.4; if we take $x_0=0,\iota>0,\xi=e$, and $\rho=1$ then $\varphi_{\rho,\iota}(x,y)$ becomes sub-additive on $(-\infty,+\infty)$.*

For $\iota>0$, $x_0\in(-\infty,+\infty)$, and $y\in(0,+\infty)$ the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ fulfills the followings:

(i)

$$1<\frac{\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)}<\xi^{\iota y},\ \forall x\in(-\infty,+\infty),$$

(ii)

$$\frac{2\xi^{\iota y}}{1+\xi^{\iota y}}<\frac{\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)}<\xi^{\iota y},\forall x\in\left(-\infty,x_0+\frac{\log_\xi\rho}{\iota}\right),$$

(iii)

$$1<\frac{\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)}<\frac{2\xi^{\iota y}}{1+\xi^{\iota y}},\forall x\in\left(x_0+\frac{\log_\xi\rho}{\iota},+\infty\right).$$

Proof. Since for all $x\in(-\infty,+\infty)$,

$$\begin{aligned}\left(\frac{\Psi'_{\rho,\iota}(x)}{\Psi_{\rho,\iota}(x)}\right)' &= \left(\iota(\ln\xi)\left(1-\Psi_{\rho,\iota}(x)\right)\right)'=\iota(\ln\xi)\left(1-\Psi_{\rho,\iota}(x)\right)' \\ &= -\frac{\rho\iota^2\left(\ln^2\xi\right)}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2\xi^{\iota(x-x_0)}}<0.\end{aligned}$$

Then

$$\left(\frac{\Psi'_{\rho,\iota}(x)}{\Psi_{\rho,\iota}(x)}\right)'<-\frac{\rho\iota^2\left(\ln^2\xi\right)}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2}<0,\forall x\in(-\infty,+\infty).$$

Hence, the function $\frac{\Psi'_{\rho,\iota}(x)}{\Psi_{\rho,\iota}(x)}$ is decreasing on $(-\infty,+\infty)$.
Let

$$\aleph(x):=\frac{\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)},x\in(-\infty,+\infty),$$

and

$$\nu(x)=\log_e\aleph(x)=\ln\aleph(x).$$

So

$$\begin{aligned}\aleph'(x) &= \frac{\Psi'_{\rho,\iota}(x+y)\Psi_{\rho,\iota}(x)-\Psi'_{\rho,\iota}(x)\Psi_{\rho,\iota}(x+y)}{\Psi^2_{\rho,\iota}(x)} \\ &= \frac{\Psi'_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)}-\frac{\Psi'_{\rho,\iota}(x)\Psi_{\rho,\iota}(x+y)}{\Psi^2_{\rho,\iota}(x)},\end{aligned}$$

and also one has

$$\begin{aligned}\nu'(x) &= \frac{\Psi'_{\rho,\iota}(x+y)\Psi_{\rho,\iota}(x)-\Psi'_{\rho,\iota}(x)\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(x+y)} \\ &= \frac{\Psi'_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x+y)}-\frac{\Psi'_{\rho,\iota}(x)}{\Psi_{\rho,\iota}(x)}<0.\end{aligned}$$

Therefore, $\nu(x)$ and $\aleph(x)$ are both decreasing.
Accordingly,

$$\begin{aligned}\lim_{x\to+\infty}\aleph(x) &= \lim_{x\to+\infty}\frac{\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)} \\ &= \lim_{x\to+\infty}\left(\frac{1+\rho\xi^{-\iota(x-x_0)}}{1+\rho\xi^{-\iota(x+y-x_0)}}\right)\frac{\xi^{\iota(x-x_0)}}{\xi^{\iota(x-x_0)}} \\ &= \lim_{x\to+\infty}\frac{\xi^{\iota(x-x_0)}+\rho}{\xi^{\iota(x-x_0)}+\rho\xi^{-\iota y}}=1,\end{aligned}$$

and

$$\lim_{x\to-\infty}\aleph(x)=\lim_{x\to-\infty}\frac{1+\rho\xi^{-\iota(x-x_0)}}{1+\rho\xi^{-\iota(x+y-x_0)}}=\xi^{\iota y},$$

$$1=\lim_{x\to+\infty}\aleph(x)<\aleph(x)<\lim_{x\to-\infty}\aleph(x)=\xi^{\iota y},x\in(-\infty,+\infty),$$

and also

$$\lim_{x\to x_0+\frac{\log_\xi\rho}{\iota}}\frac{1+\rho\xi^{-\iota(x-x_0)}}{1+\rho\xi^{-\iota(x+y-x_0)}}=\frac{2\xi^{\iota y}}{1+\xi^{\iota y}},$$

$$1=\lim_{x\to+\infty}\aleph(x)<\aleph(x)<\lim_{x\to x_0+\frac{\log_\xi\rho}{\iota}}\aleph(x)=\frac{2\xi^{\iota y}}{1+\xi^{\iota y}},x\in\left(x_0+\frac{\log_\xi\rho}{\iota},+\infty\right).$$

Corollary 2.6. *For $\iota>0$ and $x_0\in(-\infty,+\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ yields inequalities below:*

$$1<\frac{\Psi_{\rho,\iota}\left(x+\frac{1}{\iota}\right)}{\Psi_{\rho,\iota}(x)}<\xi;x\in(-\infty,+\infty),$$

$$\frac{2\rho\xi}{1+\rho\xi}<\frac{\Psi_{\rho,\iota}\left(x+\frac{1}{\iota}\right)}{\Psi_{\rho,\iota}(x)}<\xi;\ x\in(-\infty,x_0),$$

and also

$$1<\frac{\Psi_{\rho,\iota}\left(x+\frac{1}{\iota}\right)}{\Psi_{\rho,\iota}(x)}<\frac{2\rho\xi}{1+\rho\xi}.$$

□

Corollary 2.7. *(see [10], [11], [12]) Let S be an open subinterval of $(0,\infty)$, and let $g:S\longrightarrow(0,\infty)$ be differentiable. g is AH-convex (concave)$\Longleftrightarrow\frac{g'(x)}{g^2(x)}$ is increasing (decreasing).*

Theorem 2.8. *For $\iota>0$ and $x_0\in[0,\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ is AH-concave on $(x_0,+\infty)$. Namely,*

$$\Psi_{\rho,\iota}\left(\frac{x+y}{2}\right)\geq\frac{2\Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y)}{\Psi_{\rho,\iota}(x)+\Psi_{\rho,\iota}(y)},x\in(x_0,+\infty).$$

Proof. Let us take

$$\Psi'_{\rho,\iota}(x)=\frac{\iota\rho(\ln\xi)\xi^{-\iota(x-x_0)}}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2},$$

and

$$\Psi^2_{\rho,\iota}(x)=\left(1+\rho\xi^{-\iota(x-x_0)}\right)^{-2}.$$

Then

$$\begin{aligned}\left(\frac{\Psi'_{\rho,\iota}(x)}{\Psi^2_{\rho,\iota}(x)}\right)' &= \left(\iota\rho(\ln\xi)\xi^{-\iota(x-x_0)}\right)' \\ &= -\iota^2\left(\ln^2\xi\right)\xi^{-\iota(x-x_0)}<0.\end{aligned}$$

One has the desired result by Corollary 2.7. □

Theorem 2.9. *For $\iota>0$ and $x_0\in(-\infty,+\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ is logarithmically concave on $(-\infty,+\infty)$. Namely, for all $x,y\in(-\infty,+\infty)$; $z,p>1$ and $\frac{1}{z}+\frac{1}{p}=1$, the following inequality holds:*

$$\Psi_{\rho,\iota}\left(\frac{x}{z}+\frac{y}{p}\right)\geq\left[\Psi_{\rho,\iota}(x)\right]^{\frac{1}{z}}\left[\Psi_{\rho,\iota}(y)\right]^{\frac{1}{p}}. \tag{2.4}$$

Proof. Let

$$D_{\rho,\iota}\ (x) := \ln \Psi_{\rho,\iota}(x) = \log_e \Psi_{\rho,\iota}(x) = \log_e \left(\frac{1}{1+\rho\xi^{-\iota(x-x_0)}}\right),$$

$$\ln\left(\frac{1}{1+\rho\xi^{-\iota(x-x_0)}}\right) = \ln 1 - \ln\left(1+\rho\xi^{-\iota(x-x_0)}\right),$$

thus

$$D_{\rho,\iota}\ (x) = -\ln\left(1+\rho\xi^{-\iota(x-x_0)}\right).$$

Now take the first derivative of $D_{\rho,\iota}$,

$$D'_{\rho,\iota}\ (x) = \iota\rho\,(\ln\xi)\,\frac{\xi^{-\iota(x-x_0)}}{1+\rho\xi^{-\iota(x-x_0)}},$$

and also the second derivative of $D_{\rho,\iota}$ yields the following:

$$D''_{\rho,\iota}\ (x) = -\iota^2\rho\left(\ln^2\xi\right)\frac{\xi^{-\iota(x-x_0)}}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2} < 0,$$

which indicates the inequality in (2.4). □

Theorem 2.10. *For $\iota > 0$ and $x_0 \in (-\infty,+\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ verifies the following inequalities:*

$$\Psi^2_{\rho,\iota}(x+y) \geq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y);\ x,y \in [0,+\infty),$$

and

$$\Psi^2_{\rho,\iota}(x+y) \leq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y);\ x,y \in (-\infty,0].$$

Furthermore, for $x = y = 0$, equality is satisfied.

Proof. For $\iota > 0$, $x,y \in [0,+\infty)$; $x+y \geq x$ and $x+y \geq y$ are valid. Since $\Psi_{\rho,\iota}(x)$ is increasing,

$$\Psi_{\rho,\iota}(x+y) \geq \Psi_{\rho,\iota}(x), \tag{2.5}$$

and

$$\Psi_{\rho,\iota}(x+y) \geq \Psi_{\rho,\iota}(y). \tag{2.6}$$

So, the product of (2.5) and (2.6) demonstrates the first inequality. Using the similar mindset, the second one may be proved. □

Theorem 2.11. *For $\iota > 0$ and $x_0 \in (-\infty,+\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ satisfies the inequalities below:*

$$\Psi^2_{\rho,\iota}(xy) \leq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y); x,y \in (0,1],$$

and

$$\Psi^2_{\rho,\iota}(xy) \geq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y);\ x,y \in [1,+\infty).$$

Proof. For $x,y \in (0,1]$, $xy \leq x$ and $xy \leq y$ are true.
As $\Psi_{\rho,\iota}(x)$ is increasing,

$$\Psi_{\rho,\iota}(x) \geq \Psi_{\rho,\iota}(xy) > 0,$$

and

$$\Psi_{\rho,\iota}(y) \geq \Psi_{\rho,\iota}(xy) > 0.$$

are satisfied. Furthermore, product of these two inequalities yields:

$$\Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y) \geq \Psi^2_{\rho,\iota}(xy).$$

Namely,

$$\Psi^2_{\rho,\iota}(xy) \leq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y)$$

is obtained.
Since for $x,y \in [1,+\infty)$, there exist $xy \geq x, xy \geq y$ and $\Psi_{\rho,\iota}(x)$ is increasing. Then

$$\Psi_{\rho,\iota}(x) \leq \Psi_{\rho,\iota}(xy),$$

and

$$\Psi_{\rho,\iota}(y) \leq \Psi_{\rho,\iota}(xy).$$

Multiplication of the last two inequalities gives the following:

$$\Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y) \leq \Psi^2_{\rho,\iota}(xy).$$

Below, one has the desired inequality:

$$\Psi^2_{\rho,\iota}(xy) \geq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y).$$

□

Theorem 2.12. *For $\iota > 0$ and $x_0 \in (-\infty,+\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ is supermultiplicative on $(1,+\infty)$.*

$$\Psi_{\rho,\iota}(xy) > \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y);\ x,y \in (-\infty,+\infty)$$

holds.

Proof. For $0 < \Psi_{\rho,\iota}(u) < 1$, then

$$\Psi^2_{\rho,\iota}(u) < \Psi_{\rho,\iota}(u)$$

for $u \in (-\infty,+\infty)$. Since $\Psi_{\rho,\iota}$ is increasing, and $xy \geq x$, $xy > y$ on $(1,+\infty)$,

$$\Psi_{\rho,\iota}(xy) > \Psi^2_{\rho,\iota}(xy) > \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y)$$

is true. □

Presently, some sharp inequalities related to the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ (the (ι,x_0)-generalized softplus activation function) are studied:

Theorem 2.13. *For $\iota > 0$ and $x_0 \in (-\infty,+\infty)$, the following inequalities are satisfied:*

$$\frac{\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}} < \ln\left(+\rho\xi^{\iota(x-x_0)}\right) < \ln(1+\rho) - \frac{1}{1+\rho} + \frac{\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}},$$

$$\ln(1+\rho) - \frac{1}{1+\rho} + \frac{\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}} < \ln\left(1+\rho\xi^{\iota(x-x_0)}\right),\ x \in (x_0,+\infty),$$

$$\frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}} < \ln\left(1+\rho\xi^{\iota(x-x_0)}\right),\ x \in (-\infty,+\infty). \tag{2.7}$$

Proof. Let us define

$$\Delta(x) := \ln\left(1+\rho\xi^{\iota(x-x_0)}\right) - \frac{\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}},\ x \in (-\infty,+\infty),$$

$$\Delta'(x) = \frac{\iota(\ln\xi)\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}}\left(\rho - \frac{1}{1+\rho\xi^{\iota(x-x_0)}}\right) > 0,\ x \in (-\infty,+\infty).$$

So, $\Delta(x)$ is increasing on $(-\infty,+\infty)$.
For $x\in(-\infty,x_0)$,

$$0=\lim_{x\to-\infty}\Delta(x)<\Delta(x)<\lim_{x\to x_0}\Delta(x)=\ln(1+\rho)-\frac{1}{1+\rho},$$

which yields that the first inequality is valid.
For $x\in(x_0,+\infty)$,

$$\ln(1+\rho)-\frac{1}{1+\rho}=\lim_{x\to x_0}\Delta(x)<\Delta(x)<\lim_{x\to+\infty}\Delta(x)<+\infty,$$

which indicates that the second inequality is held.
Also, for $x\in(-\infty,+\infty)$,

$$0=\lim_{x\to-\infty}\Delta(x)<\Delta(x)<\lim_{x\to+\infty}\Delta(x)<+\infty,$$

which demonstrates that the third one is also satisfied. $\square$

Theorem 2.14. *For $\iota,\rho>0$, $x_0\in(-\infty,+\infty)$ and $x\in(-\infty,+\infty)$, the inequality*

$$\rho\xi^{\iota(x-x_0)}-\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)>0 \tag{2.8}$$

is provided.

Proof. Let

$$\Xi(x):=\rho\xi^{\iota(x-x_0)}-\ln\left(1+\rho\xi^{\iota(x-x_0)}\right),\ x\in(-\infty,+\infty),$$

and

$$\Xi'(x)=\rho\iota(\ln\xi)\xi^{\iota(x-x_0)}\left(\frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}}\right)>0,$$

that indicates that $\Xi(x)$ is increasing on $(-\infty,+\infty)$. Hence, we get

$$\lim_{x\to-\infty}\Xi(x)=\lim_{x\to-\infty}\left(\rho\xi^{\iota(x-x_0)}-\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right)=0.$$

So

$$0=\lim_{x\to-\infty}\Xi(x)<\Xi(x)<\lim_{x\to-\infty}\left(\rho\xi^{\iota(x-x_0)}-\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right),$$

which verifies the last inequality, is valid. $\square$

Theorem 2.15. *For $\iota,\rho>0$, $\xi>1$; $x,x_0\in(-\infty,+\infty)$, let $m(x)=\left(1+\rho\xi^{\iota(x-x_0)}\right)^{\frac{1}{\rho\xi^{\iota(x-x_0)}}}$ and $k(x)=\left(1+\rho\xi^{\iota(x-x_0)}\right)^{1+\frac{1}{\rho\xi^{\iota(x-x_0)}}}$ be decreasing and increasing, respectively. Then the following inequalities hold:*

$$(1+\rho)\ln(1+\rho)\xi^{\iota(x-x_0)}<\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)<\rho\xi^{\iota(x-x_0)};\ x\in(-\infty,x_0),$$

$$\rho\xi^{\iota(x-x_0)}<\left(1+\rho\xi^{\iota(x-x_0)}\right)\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)<(1+\rho)\ln(1+\rho)\xi^{\iota(x-x_0)};x\in(-\infty,x_0),$$

and

$$\rho\xi^{\iota(x-x_0)}<\left(1+\rho\xi^{\iota(x-x_0)}\right)\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)<\left(1+\rho\xi^{\iota(x-x_0)}\right)\rho\xi^{\iota(x-x_0)};x\in(-\infty,+\infty).$$

Proof. For $x\in(-\infty,+\infty)$, $\iota,\rho>0$, and $\xi>1$,
let

$$\begin{aligned} M(x) &:= \ln(m(x)) \\ &= \ln\left(\left(1+\rho\xi^{\iota(x-x_0)}\right)^{\frac{1}{\rho\xi^{\iota(x-x_0)}}}\right) \\ &= \frac{\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)}{\rho\xi^{\iota(x-x_0)}}, \end{aligned}$$

and

$$\begin{aligned} K(x) &:= \ln(k(x)) \\ &= \ln\left(\left(1+\rho\xi^{\iota(x-x_0)}\right)^{1+\frac{1}{\rho\xi^{\iota(x-x_0)}}}\right) \\ &= \left(1+\frac{1}{\rho\xi^{\iota(x-x_0)}}\right)\ln\left(1+\rho\xi^{\iota(x-x_0)}\right). \end{aligned}$$

Now, taking the derivatives,

$$M'(x)=\frac{\iota(\ln\xi)}{\rho\xi^{\iota(x-x_0)}}\left(\frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}}-\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right)<0.$$

Thus using (2.7), we conclude that $M(x)$ is decreasing and, accordingly, $m(x)$ is also decreasing.
In like manner,

$$K'(x)=\frac{\iota(\ln\xi)}{\rho\xi^{\iota(x-x_0)}}\left(\rho\xi^{\iota(x-x_0)}-\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right)>0,$$

employing (2.8), it is obvious that $K(x)$ is increasing and so is $k(x)$.
Besides, let us take

$$\begin{aligned} M(x_0) &= \frac{\ln(1+\rho)}{\rho};\ K(x_0)=\left(1+\frac{1}{\rho}\right)\ln(1+\rho), \\ \lim_{x\to-\infty}M(x) &= 1;\ \lim_{x\to+\infty}M(x)=0, \\ \lim_{x\to-\infty}K(x) &= 1;\ \lim_{x\to+\infty}K(x)=+\infty. \end{aligned}$$

For $x\in(-\infty,x_0)$, we obtain

$$\frac{\ln(1+\rho)}{\rho}=M(x_0)<M(x)<\lim_{x\to-\infty}M(x)=1,$$

so that the first inequality is satisfied.
Again for $x\in(-\infty,x_0)$, we have

$$1=\lim_{x\to-\infty}K(x)<K(x)<K(x_0)=\left(1+\frac{1}{\rho}\right)\ln(1+\rho),$$

which yields the second one.
Lastly, for $x\in(-\infty,+\infty)$, one has

$$0=\lim_{x\to+\infty}M(x)<M(x)<\lim_{x\to-\infty}M(x)=1,$$

which verifies the following

$$\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)<\rho\xi^{\iota(x-x_0)}.$$

As well,

$$1=\lim_{x\to-\infty}K(x)<K(x)<\lim_{x\to+\infty}K(x)=+\infty,$$

which again implies

$$\frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}}<\ln\left(1+\rho\xi^{\iota(x-x_0)}\right). \tag{2.9}$$

By the last two inequalities, the desired third inequality is proved. □

Theorem 2.16. *For $\iota > 0$ and $x_0 \in (-\infty, +\infty)$, set*

$$F(x) = \frac{\rho\xi^{\iota(x-x_0)} \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)}{\rho\xi^{\iota(x-x_0)} - \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)},\ x \in (-\infty, x_0).$$

It follows that, $F(x)$ is increasing and satisfies the inequality below:

$$2 < F(x) < \frac{\rho \ln(1+\rho)}{\rho - \ln(1+\rho)}.$$

Proof. First of all,

$$\lim_{x\to x_0} F(x) = \frac{\ln(2)}{\rho - \ln(2)},$$

and

$$\begin{aligned}
\lim_{x\to-\infty} F(x) &= \lim_{x\to-\infty} \frac{\ln\left(1+\rho\xi^{\iota(x-x_0)}\right) + \frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}}}{1 - \frac{1}{1+\rho\xi^{\iota(x-x_0)}}} \\
&= \lim_{x\to-\infty} \left(1 + \frac{\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\left(1+\rho\xi^{\iota(x-x_0)}\right)}{\rho\xi^{\iota(x-x_0)}}\right) \\
&= 2.
\end{aligned}$$

Furthermore, let

$$\Lambda(x) := \rho\xi^{\iota(x-x_0)} \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)$$

and

$$\Upsilon(x) := \rho\xi^{\iota(x-x_0)} - \ln\left(1+\rho\xi^{\iota(x-x_0)}\right).$$

Hence

$$\begin{aligned}
\Lambda(-\infty) &= \lim_{x\to-\infty} \Lambda(x) \\
&= \lim_{x\to-\infty} \left(\rho\xi^{\iota(x-x_0)} \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon(-\infty) &= \lim_{x\to-\infty} \Upsilon(x) \\
&= \lim_{x\to-\infty} \left(\rho\xi^{\iota(x-x_0)} - \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right) \\
&= 0.
\end{aligned}$$

So,

$$\Lambda'(x) = \iota\rho(\ln\xi)\,\xi^{\iota(x-x_0)} \left(\ln\left(1+\rho\xi^{\iota(x-x_0)}\right) + \frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}}\right) > 0,$$

and

$$\Upsilon'(x) = \iota\rho(\ln\xi)\,\xi^{\iota(x-x_0)} \left(1 - \frac{1}{1+\rho\xi^{\iota(x-x_0)}}\right) > 0.$$

Additionally,

$$\begin{aligned}
\left(\frac{\Lambda'(x)}{\Upsilon'(x)}\right)' &= \left(\frac{\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\left(1+\rho\xi^{\iota(x-x_0)}\right)}{\rho\xi^{\iota(x-x_0)}}\right)' \\
&= \frac{\iota(\ln\xi)}{\rho\xi^{\iota(x-x_0)}} \left(\rho\xi^{\iota(x-x_0)} - \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right) > 0.
\end{aligned}$$

In this manner, $\frac{\Lambda'(x)}{\Upsilon'(x)}$ is increasing, which elucidates that $\frac{\Lambda(x)}{\Upsilon(x)} = F(x)$ is increasing, too (by the corollary 1.2 of [13]).

Ultimately, for $x \in (-\infty, x_0)$,

$$2 = \lim_{x\to-\infty} F(x) < F(x) < \lim_{x\to x_0} F(x) = \frac{\rho \ln(1+\rho)}{\rho - \ln(1+\rho)}. \tag{2.10}$$

□

Proposition 2.17. *Let*

$$\xi := \frac{\rho \ln(1+\rho)}{\rho - \ln(1+\rho)},$$

the inequality (2.10) *can be written as*

$$\ln\left(1+\rho\xi^{\iota(x-x_0)}\right) < \frac{\xi\rho\xi^{\iota(x-x_0)}}{\xi+\rho\xi^{\iota(x-x_0)}},$$

and by inequality (2.9)

$$\frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}} < \ln\left(1+\rho\xi^{\iota(x-x_0)}\right) < \frac{\xi\rho\xi^{\iota(x-x_0)}}{\xi+\rho\xi^{\iota(x-x_0)}}$$

is observed.

3. A statistical interpretation of the (ι, x_0)-generalized logistic-type function in survival analysis

Survival Analysis is a subfield of statistics used to describe and measure data on the time until an event occurs. For example, it analyzes the expected time until failure in mechanical systems and death in biological organisms [14]. ”Time-to-event processes” are especially common in medical research because they provide more information than whether an event occurred or not [15]. In addition, ”reliability theory” or ”reliability analysis” are other names used for this area in engineering sciences. In this section, the (ι, x_0)-generalized logistic-type function

$$F(x,\gamma) := \Psi_{\rho,\iota}(x) = \frac{\xi^{\iota(x-x_0)}}{\rho+\xi^{\iota(x-x_0)}},\ \iota,\rho > 0;\ x_0 \in (-\infty,+\infty);\ \xi > 1,$$

is considered as a distribution function and the probability density function of the suggested distribution is

$$f(x,\gamma) = \Psi'_{\rho,\iota}(x) = \frac{\iota\rho(\ln\xi)\,\xi^{\iota(x-x_0)}}{\left(\rho+\xi^{\iota(x-x_0)}\right)^2},$$

where, $\gamma = (\iota, x_0)$ is the parameter set.

One of the common terms used in survival analysis is ”survival (reliability) function”. Primarily, we are deeply interested in parametric exponential version of this function and its graphical results in the sense of behavior of the function with respect to arbitrary chosen parameters: ξ, ι, x_0, and ρ. The distribution of survival times can be better predicted by a function such as the exponential function, which, create parametric survival models.

Now, we define parametric exponential survival (PES) and parametric failure (hazard) rate (PFR) functions, respectively as seen below:

$$\begin{aligned}\overline{\Psi}(x,\gamma) &= 1 - F(x,\gamma) = \frac{\rho}{\rho+\xi^{\iota(x-x_0)}},\\ h(x,\gamma) &= \frac{f(x,\gamma)}{\overline{\Psi}(x,\gamma)} = \frac{\iota(\ln\xi)\,\xi^{\iota(x-x_0)}}{\rho+\xi^{\iota(x-x_0)}}.\end{aligned}$$

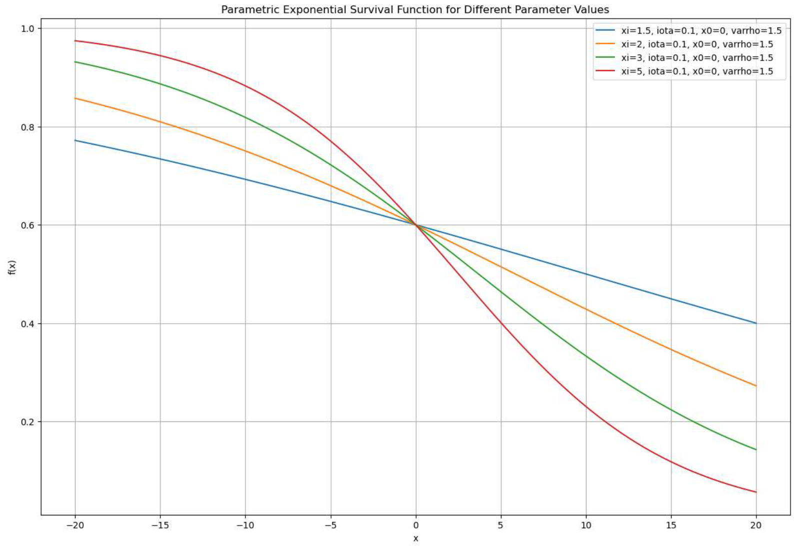

Figure 1: Behaviour of PES with respect to distinct parameter values of ξ.

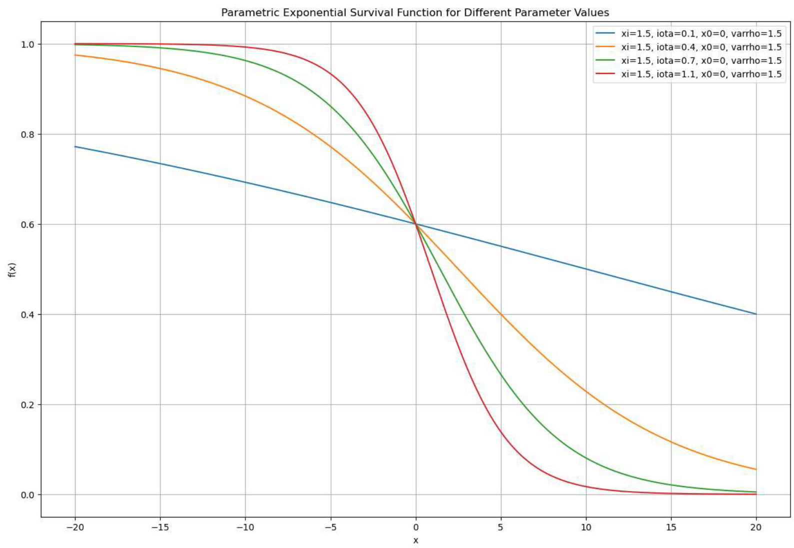

Figure 2: Behaviour of PES with respect to distinct parameter values of ι.

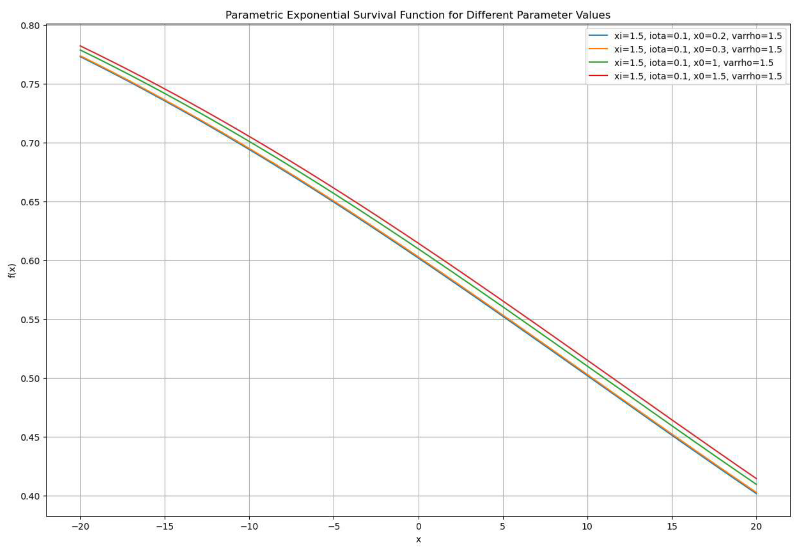

Figure 3: Behaviour of PES with respect to distinct parameter values of x_0.

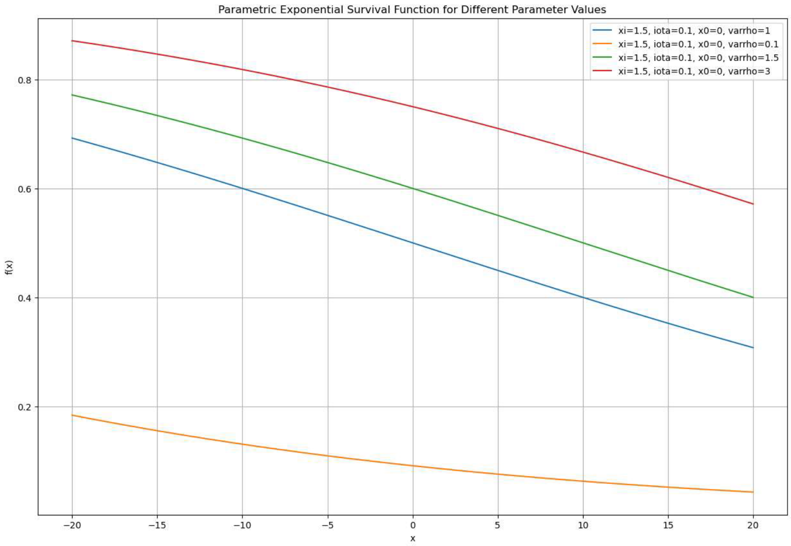

Figure 4: Behaviour of PES with respect to distinct parameter values of ρ.

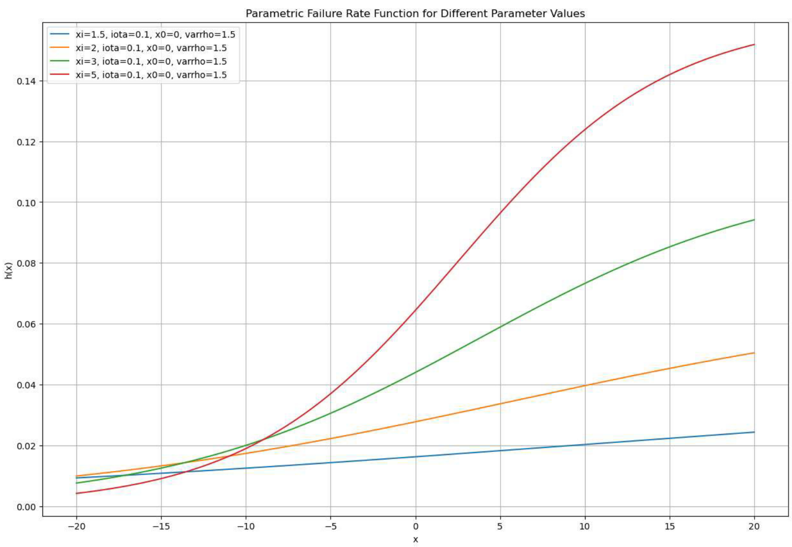

Figure 5: Behaviour of PFR with respect to arbitrary parameter values of ξ.

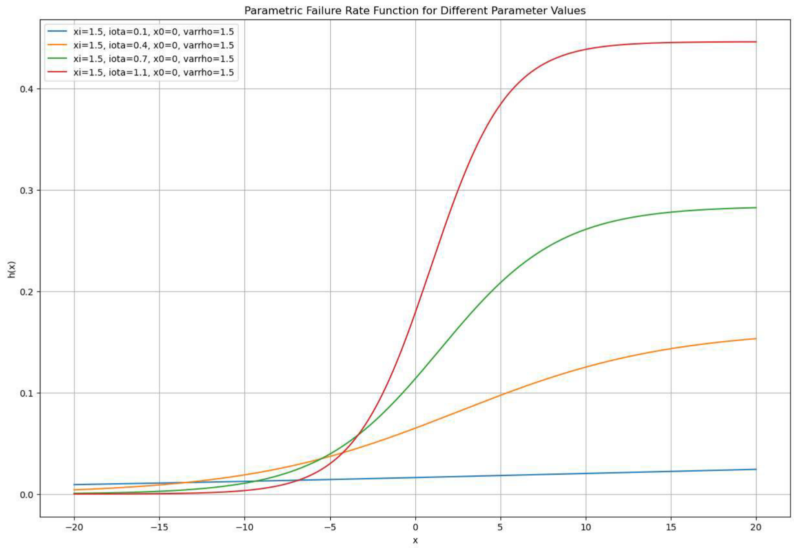

Figure 6: Behaviour of PFR with respect to arbitrary parameter values of ι.

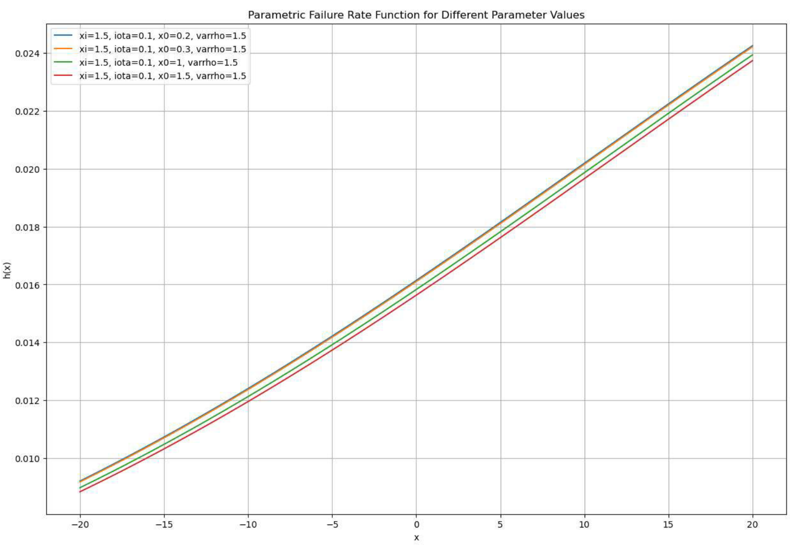

Figure 7: Behaviour of PFR with respect to arbitrary parameter values of x_0.

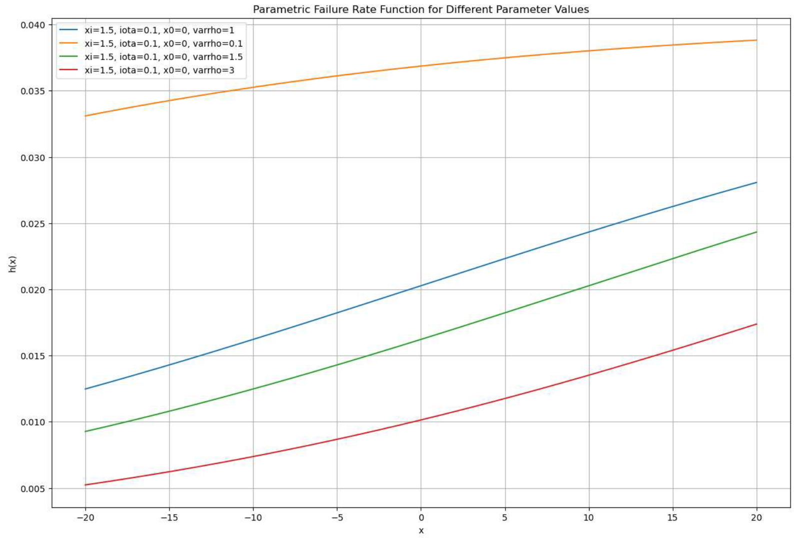

Figure 8: Behaviour of PFR with respect to arbitrary parameter values of ρ.

```
#Parametric Exponential Survival Function for Different Parameter Values
# "xi">1, "varrho"> 0, "iota"> 0
#Investigation and Comparison of Parameters
# - "xi" = Maxinum value of the curve
# - x0 = Horizontal shifting parameter. It determines where the function starts.
# - "iota" = Slope parameter.It controls how quickly the function changes.(growth rate)

import numpy as np
import matplotlib.pyplot as plt

def survival(x,varrho,xi,iota,x0):
    return varrho / (varrho + pow(xi,iota*(x-x0)))

def plot_survival(xi_values,iota_values,x0_values,varrho_values):
    x_values = np.linspace(-20, 20, 1000)
    plt.figure(figsize=(15, 10))

    for xi in xi_values:
        for iota in iota_values:
            for x0 in x0_values:
                for varrho in varrho_values:
                    y_values = survival(x_values, varrho, xi,iota, x0)
                    label = f'xi={xi}, iota={iota}, x0={x0}, varrho={varrho}'
                    plt.plot(x_values, y_values, label=label)

    plt.title('Parametric Exponential Survival Function for Different Parameter Values')
    plt.xlabel('x')
    plt.ylabel('f(x)')
    plt.legend()
    plt.grid(True)
    plt.show()


# Different Parameter Values
xi_values = [1.5, 2, 3, 5]
iota_values =[ 0.1]
x0_values = [0]
varrho_values = [1.5]
plot_survival(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1,0.4,0.7,1.1]
x0_values = [0]
varrho_values = [1.5]
plot_survival(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1]
x0_values = [0.2,0.3,1,1.5]
varrho_values = [1.5]
plot_survival(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1]
x0_values = [0]
varrho_values = [1,0.1,1.5,3]
plot_survival(xi_values,iota_values,x0_values,varrho_values)
```

Figure 9: Algorithm 1.

```
#Parametric Failure Rate Function for Different Parameter Values
# "xi">1, "varrho" > 0, "iota" > 0
#Investigation and Comparison of Parameters
# - "xi" = Maxinum value of the curve
# - x0 = Horizontal shifting parameter. It determines where the function starts.
# - "iota" = Slope parameter.It controls how varrhouickly the function changes.(growth rate)

import numpy as np
import math
import matplotlib.pyplot as plt

def failure_rate(x,varrho,xi,iota,x0):
    return (iota*(math.log(xi)/math.log(math.e))*pow(xi,iota*(x-x0)))/(varrho+pow(xi,iota*(x-x0)))

def plot_failure_rate(xi_values,iota_values,x0_values,varrho_values):
    x_values = np.linspace(-20, 20, 1000)
    plt.figure(figsize=(15, 10))

    for xi in xi_values:
        for iota in iota_values:
            for x0 in x0_values:
                for varrho in varrho_values:
                    y_values = failure_rate(x_values, varrho, xi,iota, x0)
                    label = f'xi={xi}, iota={iota}, x0={x0}, varrho={varrho}'
                    plt.plot(x_values, y_values, label=label)

    plt.title('Parametric Failure Rate Function for Different Parameter Values')
    plt.xlabel('x')
    plt.ylabel('h(x)')
    plt.legend()
    plt.grid(True)
    plt.show()


# Different Parameter Values
xi_values = [1.5, 2, 3, 5]
iota_values =[ 0.1]
x0_values = [0]
varrho_values = [1.5]
plot_failure_rate(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1,0.4,0.7,1.1]
x0_values = [0]
varrho_values = [1.5]
plot_failure_rate(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1]
x0_values = [0.2,0.3,1,1.5]
varrho_values = [1.5]
plot_failure_rate(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1]
x0_values = [0]
varrho_values = [1,0.1,1.5,3]
plot_failure_rate(xi_values,iota_values,x0_values,varrho_values)
```

Figure 10: Algorithm 2.

Moreover, graphs of the PES functions are visualized in Fig. 1 - Fig. 4 using computer programming language Python 3.9, as you see [16].

In the second place, we focus on the PFR function of the proposed distribution with arbitrary parameter values obtained as in Fig. 5 - Fig. 8. This is the function that gives the steadily revised immediate probability of a critical event. With this feature, it finds applications in many disciplines such as health sciences, and mathematical psychology [17]. Also, while the PES function serves for surviving, the PFR function deals with the failing [18].

4. Conclusion

In this paper, some important inequalities like concavity, super multiplicativity, and sub-additivity of the (ι, x_0)-generalized logistic-type function have been proved. ”Ceteris Paribus” plotting for parametric exponential survival (PES) and also parametric failure (hazard) rate (PFR) functions with four variables have been performed. Thus, when the survival function we parameterized is compared to a function that is not parametrized; we can say that the parametric one may provide more detailed and sophisticated modeling in survival analysis. In short, we may obtain higher accuracy values in the validation data of the models with the help of functions containing four parameters, that is, to make the models more robust.

Declarations

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Availability of Data and Materials: Data sharing not applicable.

ORCID
Seda Karateke https://orcid.org/0000-0003-1219-0115

References

[1] J. Zhang, L. Yin and W. Cui, *The monotonic properties of (p,a)-generalized sigmoid function with application*, Pak. J. Statist., **35**(2) (2019), 171-185. [Scopus]
[2] M.I. Jordan, *Why the logistic function? A tutorial discussion on probabilities and neural networks*, (1995).
[3] R.M. Neal, *Connectionist learning of belief networks*, Artif. Intell., **56**(1), 1992, 71-113. [CrossRef] [Scopus] [Web of Science]
[4] P. McCullagh and J. Nelder, *Generalized Linear Models*, Second Edition. Chapman & Hall., (1989), ISBN: 9780412317606. [CrossRef]
[5] D. Yu, *Softmax function based intuitionistic fuzzy multi-criteria decision making and applications*, Oper. Res. Int. J., **16** (2016), 327-348. [CrossRef] [Scopus] [Web of Science]
[6] R. Torres, R. Salas and H. Astudillo, *Time-based hesitant fuzzy information aggregation approach for decision making problems*, Int. J. Intell. Syst., **29**(6) (2014), 579-595. [CrossRef] [Scopus] [Web of Science]
[7] I. Goodfellow, Y. Bengio and A. Courville, *6.2.2.3 Softmax Units for Multinoulli Output Distributions*, Deep Learning. MIT Press., (2016), 180-184. ISBN 978-0-26203561-3.
[8] G.A. Anastassiou, *Banach Space Valued Multivariate Multi Layer Neural Network Approximation Based on q-Deformed and λ-Parametrized A-Generalized Logistic Function*. In: Parametrized, Deformed and General Neural Networks, Studies in Computational Intelligence, **1116**. Springer, Cham., (2023), 365-394. [CrossRef] [Scopus]
[9] A. Arai, *Exactly solvable supersymmetric quantum mechanics*, J. Math. Anal. Appl., **158**(1) (1991), 63-79. [CrossRef] [Scopus] [Web of Science]
[10] G.D. Anderson, M. Vamanamurthy, and M.Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl., **335**(2) (2007), 1294-1308. [CrossRef] [Scopus] [Web of Science]
[11] P.S. Bullen, *Handbook of Means and Their Inequalities*, Springer Science & Business Media (**560**), 2013. [CrossRef]
[12] P.S. Bullen, D.S. Mitrinovic and M. Vasic, *Means and Their Inequalities*, Springer Science & Business Media **31**, 2013. [CrossRef]
[13] I. Pinelis, *L'Hospital type rules for monotonicity, with applications*, J. Ineq. Pure & Appl. Math.**3**(1) (2002), 1-5.
[14] R.G. Miller, *Survival Analysis*, John Wiley & Sons, (1997), ISBN 0-471-25218-2.
[15] B. George, S. Seals and I. Aban, *Survival analysis and regression models*, J. Nucl. Cardiol., **21** (2014), 686-694. [CrossRef] [Scopus] [Web of Science]
[16] S. Karateke, M. Zontul, V.N. Mishra and A.R. Gairola, *On the Approximation by Stancu-Type Bivariate Jakimovski–Leviatan–Durrmeyer Operators*, La Matematica, **3** (2024), 211–233. [CrossRef]
[17] R.A. Chechile, *Mathematical tools for hazard function analysis*, J. Math. Psychol., **47**(5-6) (2003), 478-494. [CrossRef] [Scopus] [Web of Science]
[18] F. Emmert-Streib and M. Dehmer, *Introduction to survival analysis in practice*, Mach. Learn. Knowl. Extr., **1**(3) (2019), 1013-1038. [CrossRef] [Scopus] [Web of Science]

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