http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 74, Number 2, Pages 191–199 (2025)
https://doi.org/10.31801/cfsuasmas.1448617
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: March 7, 2024; Accepted: January 23, 2025

Summability of spliced double sequences

Emre TAŞ[1] and Sevcan DEMİRKALE[2]

[1,2] Kırşehir Ahi Evran University, Department of Mathematics, 40100, Kırşehir, TÜRKİYE

Abstract. In this paper, we introduce spliced double sequences and give the summability of this new notion by using four dimensional matrices. Note that there are some examples which show the effectiveness of spliced double sequences in summability theory.

2020 Mathematics Subject Classification. Primary 54A20; Secondary 40J05, 40A05, 40A35, 40G10, 40C15.
Keywords. Pringsheim convergence, double sequences, A-statistical convergence, spliced sequences.

1. Preliminaries

The concept of spliced sequences has been introduced by Osikiewicz [6]. The motivation behind the introduction of these sequences is to give alternative proof that there exist at least bounded sequences that a regular matrix cannot sum. Spliced sequences have been studied by many authors in different manners [1, 2, 11–14]. A spliced sequence is the combination of convergent sequences by using partitions. Yurdakadim and Ünver [14] have generalized this notion by using bounded sequences instead of convergent sequences. On the other hand, Bartoszewicz et al. [1] have generalized this concept by introducing the density of points.

In this paper, we describe spliced double sequences and give the summability of this new notion by using four dimensional matrices. Note that there are some examples which show the effectiveness of spliced double sequences in summability theory.

Let us firstly give a number of elemantary notions and definitions that will be required along the paper.

Definition 1. *[9] A double sequence $[x] = (x_{jk})$ is said to be Pringsheim convergent to L (denoted by $P - \lim x = L$) if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - L| < \epsilon$ for every $j, k > N$. In this case, L is called the Pringsheim limit of $[x]$ and the space of such sequences is denoted by $c^{(2)}$.*

Definition 2. *[8] A double sequence (x_{jk}) is said to be bounded in the sense of Pringsheim if there exists a positive number H such that $|x_{jk}| < H$ for each $j, k \in \mathbb{N}$. The norm of $[x]$ is $\|x\|_{\infty,(2)} = \sup_{j,k} |x_{jk}|$.*

Definition 3. *[4] Let $A = \left(a_{jk}^{mn}\right)$ be a four dimensional summability matrix and $[x] = (x_{jk})$ be a double sequence. If $[Ax] := \{(Ax)_{mn}\}$ is Pringsheim convergent to L then $[x]$ is said to be A-summable to L where*

$$(Ax)_{mn} := \sum_{j,k} a_{jk}^{mn} x_{jk}, \text{ for any } m, n \in \mathbb{N}.$$

A four dimensional matrix A is said to be RH-regular if it transforms every bounded Pringsheim convergent sequence into a Pringsheim convergent sequence with the same Pringsheim limit [4].

[1] emretas86@hotmail.com -Corresponding author; 0000-0002-6569-626X.
[2] sevcandemirkale@gmail.com; 0000-0003-0739-5044.

Let E be a subset of $\mathbb{N}\times\mathbb{N}$ and A be an RH-regular matrix. If the limit

$$P-\lim_{m,n}\sum_{(j,k)\in E}a_{jk}^{mn}$$

exists, we say that E has A-density and this is denoted by $\delta_A^{(2)}(E)$.

On the other hand, four dimensional Cesàro matrix $(C,1,1)=\left(c_{jk}^{mn}\right)$ is defined by

$$c_{jk}^{mn}=\begin{cases}\frac{1}{mn} & , \quad j\le m \text{ and } k\le n\\ 0 & , \quad \text{otherwise.}\end{cases}$$

In the definition of A-density, if we take $A=(C,1,1)$, the natural density of E is obtained. This is denoted by $\delta^{(2)}(E)$ [5].

2. Finite Spliced Sequences

In this section, we examine summability properties of M-spliced double sequences by using four dimensional matrices.

Definition 4. *$\{E_1,E_2,...,E_M\}$ is said to be an M-partition of $\mathbb{N}\times\mathbb{N}$ if $E_1\cup E_2\cup...\cup E_M=\mathbb{N}\times\mathbb{N}$ and $E_i\cap E_s=\varnothing$ for $i\neq s$ and $E_l=\{(v_l(j),\mu_l(k))\}_{j,k=1,1}^{\infty,\infty}$ is a subset of $\mathbb{N}\times\mathbb{N}$ and v_l,μ_l are injective on $\mathbb{N}$ for $l=1,2,...,M$.*

Definition 5. *Let $\gamma^{(l)}$ be a Pringsheim convergent double sequence such that $\gamma^{(l)}=(\gamma_{jk}^{(l)})_{j,k=1,1}^{\infty,\infty}$ and $P-\lim_{j,k}\gamma_{jk}^{(l)}=\Gamma^{(l)}$ for $l=1,2,...,M$. By the M-partition $\{E_1,E_2,...,E_M\}$, if $(r,s)\in E_l$ such that $(r,s)=(v_l(j),\mu_l(k))$ for some j,k, then $[x]:=(x_{rs})$ with $x_{rs}=x_{v_l(j),\mu_l(k)}=\gamma_{jk}^{(l)}$ is called M-spliced double sequence of the sequences $\gamma^{(1)},...,\gamma^{(M)}$.*

Theorem 1. *Let A be a nonnegative RH-regular matrix and $\{E_1,E_2,...,E_M\}$ is a fixed M-partition. If $\delta_A^{(2)}(E_l)$ exists for every $l=1,2,...,M$, then the matrix A has the splicing property over $\{E_1,E_2,...,E_M\}$, i.e.,*

$$P-\lim_{m,n}(Ax)_{mn}=\sum_{l=1}^{M}\delta_A^{(2)}(E_l)\,\Gamma^{(l)}$$

holds for every M-splice over $\{E_1,E_2,...,E_M\}$.

Proof. Let $\delta_A^{(2)}(E_l)$ exist for any $l=1,2,...,M$ and $[x]$ be an M-splice. Then one can write the following:

$$\begin{aligned}(Ax)_{mn}&=\sum_{j,k=1,1}^{\infty,\infty}a_{jk}^{mn}x_{jk}=\sum_{l=1}^{M}\left(\sum_{(j,k)\in E_l}a_{jk}^{mn}x_{jk}\right)\\&=\sum_{l=1}^{M}\left(\sum_{j,k=1,1}^{\infty,\infty}a_{v_l(j),\mu_l(k)}^{mn}x_{v_l(j),\mu_l(k)}\right)\\&=\sum_{l=1}^{M}\left(\sum_{j,k=1,1}^{\infty,\infty}a_{v_l(j),\mu_l(k)}^{mn}\gamma_{jk}^{(l)}\right)\\&=\sum_{l=1}^{M}\left(\sum_{j,k=1,1}^{\infty,\infty}b_{jk}^{mn}\gamma_{jk}^{(l)}\right).\end{aligned}$$

It is easy to see that $B(l)=(b_{jk}^{mn}(l))=\left(a_{v_l(j),\mu_l(k)}^{mn}\right)$ is multiplicative with multiplier $\delta_A^{(2)}(E_l)$ according to [8] for each $l=1,2,...,M$. That is, for every sequence $[x]\in c^{(2)}$, $P-\lim_{m,n}(Bx)_{mn}=\delta_B^{(2)}(E_l).P-\lim_{m,n}(x)_{mn}$ holds. Therefore

$$P-\lim_{m,n}(Ax)_{mn}=P-\lim_{m,n}\sum_{l=1}^{M}\left(\sum_{j,k=1,1}^{\infty,\infty}a_{v_l(j),\mu_l(k)}^{mn}\gamma_{jk}^{(l)}\right)$$

$$= \sum_{l=1}^{M} \left(P - \lim_{m,n} \sum_{j,k=1,1}^{\infty,\infty} a^{mn}_{v_l(j),\mu_l(k)} \gamma^{(l)}_{jk} \right)$$
$$= \sum_{l=1}^{M} \delta^{(2)}_{A} (E_l) \Gamma^{(l)}.$$

That is, $[x]$ is A-summable to $\sum_{l=1}^{M} \delta^{(2)}_{A} (E_l) \Gamma^{(l)}$. Hence A has splicing property over $\{E_1, E_2, ..., E_M\}$. This completes the proof. □

Theorem 2. *Let A be an $RH-regular$ matrix. For any $M \geq 2$, there exists an M-partition $\{E_1, E_2, ..., E_M\}$ such that A does not have the splicing property over $\{E_1, E_2, ..., E_M\}$.*

Proof. According to Theorem 3.1 in [7], we know that there exists at least bounded double sequence $[x]$ consisting of only $0's$ and $1's$ which is not A-summable. For $M \geq 2$, consider the partition $\{E_1, E_2, ..., E_M\}$ where $E_1 = \{(j,k) : x_{jk} = 1\}$ and $E_2, E_3, ..., E_M$ can be constructed such that they are disjoint, infinite, and $\bigcup_{l=2}^{M} E_l = (\mathbb{N} \times \mathbb{N}) \backslash E_1$. Then $[x]$ is an $M-spliced$ double sequence over the partition $\{E_1, E_2, ..., E_M\}$ of sequences $\gamma^{(1)}_{jk} = 1$ and $\gamma^{(l)}_{jk} = 0$ for $l = 2, ..., M$. Therefore since $[x]$ is not $A-summable$, A has not the splicing property over $\{E_1, E_2, ..., E_M\}$. □

A four dimensional matrix A is said to be RH-preserving zero limit matrix if it transforms any bounded Pringsheim convergent sequence to zero into a bounded Pringsheim convergent sequence to zero.

We express a fact on four dimensional matrix which is RH-preserving zero limit in a lemma. The concept has been studied in [3] but the following Lemma can not be deduced from Theorem 2.3 in [3].

Lemma 1. *Four dimensional matrix $A = \left(a^{mn}_{jk}\right)$ preserves zero limit if and only if*

(a) $P - \lim_{m,n} a^{mn}_{jk} = 0$ *for any j and k,*

(b) $\sum_{j,k} \left|a^{mn}_{jk}\right|$ *has Pringsheim limit for any m and n,*

(c) $P - \lim_{m,n} \sum_{j} \left|a^{mn}_{jk}\right| = 0$ *for any k,*

(d) $P - \lim_{m,n} \sum_{k} \left|a^{mn}_{jk}\right| = 0$ *for any j,*

(e) $\sum_{j,k} \left|a^{mn}_{j,k}\right| \leq H$ *for any m and n.*

Proof. Sufficiency: Let $[x] = (x_{jk})$ be a bounded Pringsheim convergent sequence to 0. We show that $[Ax]$ is also a bounded Pringsheim convergent to zero by using $(a) - (e)$. We can write

$$|y_{mn}| = \left| \sum_{j,k}^{\infty,\infty} a^{mn}_{jk} x_{jk} \right|$$
$$\leq \left| \sum_{j=1,k=1}^{p,q} a^{mn}_{jk} x_{jk} \right| + \left| \sum_{j=1,k=q+1}^{p,\infty} a^{mn}_{jk} x_{jk} \right| + \left| \sum_{j=p+1,k=1}^{\infty,q} a^{mn}_{jk} x_{jk} \right| + \left| \sum_{j=p+1,k=q+1}^{\infty,\infty} a^{mn}_{jk} x_{jk} \right|.$$

For $\epsilon > 0$, we can find p and q so large that $|x_{jk}| \leq \frac{\epsilon}{4H}$ for $j > p$, $k > q$. Let L be the greatest of the numbers $|x_{jk}|$ for each j and k. Using conditions (a), (c), (d) we can find two integers T and N such that whenever $m \geq T$, $n \geq N$,

$$\sum_{j=1,k=1}^{p,q} \left|a^{mn}_{jk}\right| < \frac{\epsilon}{4pqL}$$
$$\sum_{k=1}^{\infty} \left|a^{mn}_{jk}\right| < \frac{\epsilon}{4pL}, \quad (j = 1, 2, 3, ..., p)$$

$$\sum_{j=1}^{\infty}\left|a_{jk}^{mn}\right| < \frac{\epsilon}{4qL}, \ \ (k=1,2,3,...,q).$$

Therefore, for $m \geq T$, $n \geq N$ we have,

$$|y_{mn}| \leq \frac{\epsilon}{4Lpq}Lpq + \frac{\epsilon}{4Lp}Lp + \frac{\epsilon}{4Lq}Lq + \frac{\epsilon}{4H}H$$

which means that

$$P - \lim_{m,n} y_{mn} = 0.$$

Necessity: To see (a), define a sequence (x_{mn}) as follows:

$$x_{mn} = \begin{cases} 1 & , \quad m = p \text{ and } n = q \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Here $y_{mn} = a_{pq}^{mn}$. Hence condition (a) is necessary.

(b) It is well known that if $\sum_{m=1,n=1}^{\infty,\infty} u_{mn}$ is not absolutely convergent, then one can choose a bounded double sequence s_{mn} such that $P - \lim_{m,n} s_{mn} = 0$ and $\sum_{m,n=1,1}^{\infty,\infty} u_{mn}s_{mn}$ diverges to $+\infty$ [10]. By using this corollary, one can choose any fixed m and n and suppose the series $\sum_{j,k}\left|a_{jk}^{mn}\right|$ diverges. In this case there is a bounded sequence x_{jk} such that $[x]$ has zero limit, and $\sum_{j,k} a_{jk}^{mn}x_{jk}$ diverges. This contradicts with our hypothesis. Therefore (b) holds. The proofs of (c), (d), (e) has same way in page 61, Theorem 2 in [10] $(d) - (f)$, respectively. □

Theorem 3. *Let a four dimensional matrix A be an RH-preserving zero limit matrix. If $\gamma \in c^{(2)} \backslash c_0^{(2)}$ is an A-summable bounded double sequence to L, then A is multiplicative with multiplier L/Γ with $P - \lim_{j,k} \gamma_{jk} = \Gamma \neq 0$.*

Proof. From Lemma 1 and Theorem 2.3. in [8], since A is four dimensional matrix which is RH-preserving zero limit in order to show that L/Γ-multiplicative, it is enough to show that $P - \lim_{m,n} \sum_{j,k} a_{jk}^{mn} = L/\Gamma$. Since $P - \lim_{j,k} \gamma_{jk} = \Gamma \neq 0$, it can be expressed as $\gamma_{jk} = \Gamma e_{jk} + \epsilon_{jk}$ where all terms of the sequence (e_{jk}) is 1 and $P - \lim_{j,k} \epsilon_{jk} = 0$. Then, for a given m and n, we obtain

$$\begin{aligned}(A\gamma)_{mn} &= \sum_{j,k} a_{jk}^{mn}\gamma_{jk} = \sum_{j,k} a_{jk}^{mn}\left(\Gamma e_{jk} + \epsilon_{jk}\right) \\ &= \Gamma \sum_{j,k} a_{jk}^{mn} e_{jk} + \sum_{j,k} a_{jk}^{mn}\epsilon_{jk}.\end{aligned}$$

Since γ is A-summable to L and A preserves zero limits, we also have

$$\begin{aligned}L = P - \lim_{m,n}(A\gamma)_{mn} &= \Gamma P - \lim_{m,n}\sum_{j,k} a_{jk}^{mn} + P - \lim_{m,n}\sum_{j,k} a_{jk}^{mn}\epsilon_{jk} \\ &= \Gamma P - \lim_{m,n}\sum_{j,k} a_{jk}^{mn} + 0,\end{aligned}$$

since A preserves zero limits and $P - \lim_{m,n}(A\gamma)_{mn} = L$. That is, $P - \lim_{m,n}\sum_{j,k} a_{jk}^{mn} = \frac{L}{\Gamma}$. Hence A is multiplicative with multiplier $\frac{L}{\Gamma}$. □

Now we use the following notation to give a new result.

Consider a four dimensional matrix $A = \left(a_{jk}^{mn}\right)$ and a subset $E = \{(v(r), \mu(s))\}_{r,s=1,1}^{\infty,\infty}$ of $\mathbb{N} \times \mathbb{N}$. Then define $A^{[E]} = \left(b_{jk}^{mn}\right)$ by $b_{jk}^{mn} = a_{v(j),\mu(k)}^{mn}$. For a double sequence $[x]$, we write

$$\left(A^{[E]}x\right)_{mn} = \sum_{j,k} b_{jk}^{mn}x_{jk} = \sum_{j,k} a_{v(j),\mu(k)}^{mn}x_{jk}.$$

Theorem 4. *Let A be a nonnegative RH-regular matrix and $E \subset \mathbb{N} \times \mathbb{N}$ be an infinite set. If $\delta_A^{(2)}(E)$ exists, then $A^{[E]}$ is RH-multiplicative with multiplier $\delta_A^{(2)}(E)$ and conversely, if $A^{[E]}$ is multiplicative with the constant t, then $\delta_A^{(2)}(E)$ exists and equals to t.*

Proof. For every m and n, we have

$$\left(A^{[E]}e\right)_{mn} = \sum_{j,k} a^{mn}_{v(j),\mu(k)} = \sum_{(j,k)\in E} a^{mn}_{jk}.$$

If $\delta_A^{(2)}(E)$ exists, by Theorem 3, $A^{[E]}$ is RH-multiplicative with multiplier $\delta_A^{(2)}(E)$. Conversely, if $A^{[E]}$ is RH-multiplicative with multiplier t, then we have

$$t = P - \lim_{m,n}\left(A^{[E]}e\right) = P - \lim_{m,n} \sum_{(j,k)\in E} a^{mn}_{jk} = \delta_A^{(2)}(E),$$

where all terms of the sequence (e_{jk}) is 1. □

Theorem 5. *Let $[x]$ be a 2-splice of bounded double sequences $\gamma^{(1)}$ and $\gamma^{(2)}$ over $\{E_1, E_2\}$ with $\Gamma^{(1)} \neq \Gamma^{(2)}$. If A is a nonnegative RH-regular matrix which sums $[x]$ to the value L, then $\delta_A^{(2)}(E_1)$ and $\delta_A^{(2)}(E_2)$ exist with*

$$\delta_A^{(2)}(E_1) = \frac{\Gamma^{(2)} - L}{\Gamma^{(2)} - \Gamma^{(1)}} \text{ and } \delta_A^{(2)}(E_2) = \frac{L - \Gamma^{(1)}}{\Gamma^{(2)} - \Gamma^{(1)}}.$$

Proof. Let $[x]$ be a $2-$*splice* of the bounded double sequences $\gamma^{(1)}$ and $\gamma^{(2)}$ over $\{E_1, E_2\}$ with $\Gamma^{(1)} \neq \Gamma^{(2)}$, such that $[x]$ is A-summable to L. $A^{[E_l]} = \left(a^{mn}_{v_l(j),\mu_l(k)}\right)$ matrices with $E_l = \{(v_l(j), \mu_l(k)) : (j,k) \in \mathbb{N}\times\mathbb{N}\}$, $(l = 1, 2)$ fulfill the conditions of Lemma 1 that is, they preserve zero limits. For any given m and n, we have

$$\begin{aligned}\left(A\left(x - \Gamma^{(1)}\right)\right)_{mn} &= \sum_{j,k} a^{mn}_{jk}\left(x_{jk} - \Gamma^{(1)}\right)\\ &= \sum_{j,k} a^{mn}_{v_1(j),\mu_1(k)}\left(\gamma^{(1)}_{jk} - \Gamma^{(1)}\right) + \sum_{j,k} a^{mn}_{v_2(j),\mu_2(k)}\left(\gamma^{(2)}_{jk} - \Gamma^{(1)}\right)\\ &= \left(A^{[E_1]}\left(\gamma^{(1)} - \Gamma^{(1)}\right)\right)_{mn} + \left(A^{[E_2]}\left(\gamma^{(2)} - \Gamma^{(1)}\right)\right)_{mn}.\end{aligned}$$

Since $\gamma^{(1)} - \Gamma^{(1)} \in c_0^{(2)}$, we have

$$\begin{aligned}\left(A^{[E_2]}\left(\gamma^{(2)} - \Gamma^{(1)}\right)\right) &= \left(A\left(x - \Gamma^{(1)}\right)\right)_{mn} - \left(A^{[E_1]}\left(\gamma^{(1)} - \Gamma^{(1)}\right)\right)_{mn}\\ &= L - \Gamma^{(1)} + o(1).\end{aligned}$$

Thus, $\gamma^{(2)} - \Gamma^{(1)}$ is $A^{[E_2]}$-summable to $L - \Gamma^{(1)}$. Since $\gamma^{(2)} - \Gamma^{(1)} \in c^{(2)} \backslash c_0^{(2)}$, Lemma 1 implies that $A^{[E_2]}$ is multiplicative with multiplier $\left(L - \Gamma^{(1)}\right) / \left(\Gamma^{(2)} - \Gamma^{(1)}\right)$. Then, Theorem 3 implies that

$$\delta_A^{(2)}(E_2) = \frac{L - \Gamma^{(1)}}{\Gamma^{(2)} - \Gamma^{(1)}}$$

and

$$\delta_A^{(2)}(E_1) = 1 - \delta_A^{(2)}(E_2) = 1 - \frac{L - \Gamma^{(1)}}{\Gamma^{(2)} - \Gamma^{(1)}} = \frac{\Gamma^{(2)} - L}{\Gamma^{(2)} - \Gamma^{(1)}}.$$

□

3. Infinite Spliced Sequences

In this section, we examine summability properties of ∞-spliced double sequences by using four dimensional matrices. ∞-partition of $\mathbb{N}\times\mathbb{N}$ is similarly defined as in Definition 4.

Definition 6. *Let $\{E_l\}$ be a fixed $\infty -$ partition of $\mathbb{N}\times\mathbb{N}$. For $l\in\mathbb{N}$, let $\gamma^{(l)}=\left(\gamma_{j,k}^{(l)}\right)_{j,k=1,1}^{\infty,\infty}$ be convergent double sequences with $P-\lim_{j,k}\gamma_{j,k}^{(l)}=\Gamma^{(l)}$. Then the $\infty -$ splice of the sequences $\gamma^{(l)}$, $l\in\mathbb{N}$ over the $\infty-$partition $\{E_l\}$ is the sequence $[x]$ defined as follows: if $(r,s)\in E_l$, then $(r,s)=(v_l(j),\mu_l(k))$ for some $j,k\in\mathbb{N}$ and $x_{r,s}=x_{v_l(j),\mu_l(k)}:=\gamma_{j,k}^{(l)}$.*

Definition 7. *Let A be an $RH-$ regular matrix and consider a fixed $\infty -$ partition $\{E_l\}$. If A sums every bounded double $\infty -$ spliced sequences over the $\infty -$ partition $\{E_l\}$, then A is said to have the splicing property over $\{E_l\}$.*

Theorem 6. *Let A be a nonnegative $RH -$ regular matrix and $\{E_l\}$ be a $\infty -$ partition of $\mathbb{N}\times\mathbb{N}$. If for every $l\in\mathbb{N}$, $\delta_A^{(2)}(E_l)$ exists and $\sum_l\delta_A^{(2)}(E_l)=1$, for bounded $\infty -$ spliced sequences over $\{E_l\}$, then A has the splicing property over $\{E_l\}$ with*

$$P-\lim_{m,n}(Ax)_{mn}=\sum_{l=1}^{\infty}\delta_A^{(2)}(E_l)\,\Gamma^{(l)}.$$

Proof. Let $\delta_A^{(2)}(E_l)$ exists for every $l\in\mathbb{N}$, $\sum_{l=1}^{\infty}\delta_A^{(2)}(E_l)=1$ and let $[x]$ be a bounded $\infty -$ *spliced* sequence over $\{E_l\}$. Then for a given m and n, we have

$$\begin{aligned}(Ax)_{mn}&=\sum_{j,k=1,1}^{\infty,\infty}a_{jk}^{mn}x_{jk}=\sum_{l=1}^{\infty}\left(\sum_{(j,k)\in E_l}a_{jk}^{mn}x_{jk}\right)\\&=\sum_{l=1}^{\infty}\left(\sum_{j,k=1,1}^{\infty,\infty}a_{v_l(j),\mu_l(k)}^{mn}x_{v_l(j),\,\mu_l(k)}\right)\\&=\sum_{l=1}^{\infty}\left(\sum_{j,k=1,1}^{\infty,\infty}a_{v_l(j),\mu_l(k)}^{mn}\gamma_{j,k}^{(l)}\right)\\&=\sum_{l=1}^{\infty}\left(B\gamma^{(l)}\right)_{mn}\end{aligned}$$

where $B=\left(b_{jk}^{mn}\right)=\left(a_{v_l(j),\,\mu_l(k)}^{mn}\right)$. Define f_{mn} and g_{mn} as follows:

$f_{mn}:\mathbb{N}\longrightarrow\mathbb{C}$ by $f_{mn}(l):=\left(B\gamma^{(l)}\right)_{mn}$ and $g_{mn}:\mathbb{N}\longrightarrow\mathbb{C}$ by $g_{mn}(l):=M.(Be)_{mn}$, where $M=\|x\|_{\infty,(2)}$ and all terms of the sequence $[e]=(e_{jk})$ is 1.

Since $\delta_A^{(2)}(E_l)$ exists for every l, the matrix $B=\left(b_{jk}^{mn}\right)$ is multiplicative with multiplier $\delta_A^{(2)}(E_l)$ and

$$f(l):=P-\lim_{m,n}f_{mn}(l)=P-\lim_{m,n}\left(B\gamma^{(l)}\right)_{mn}=\delta_A^{(2)}(E_l)\,\Gamma^{(l)}$$

and

$$g(l):=P-\lim_{m,n}g_{mn}(l)=P-\lim_{m,n}M\,(Be)_{mn}=M\delta_A^{(2)}(E_l)\,.$$

If μ represents the counting measure, then we get

$$\begin{aligned}P-\lim_{m,n}\int_{\mathbb{N}}g_{mn}(l)\,d\mu&=P-\lim_{m,n}\sum_{l=1}^{\infty}M\,(Be)_{mn}\\&=M.P-\lim_{m,n}\sum_{l=1}^{\infty}\left(\sum_{(j,k)\in E_l}a_{jk}^{mn}\right)\\&=M.P-\lim_{m,n}\sum_{j,k=1,1}^{\infty,\infty}a_{jk}^{mn}\\&=M.1\end{aligned}$$

$$= M \sum_{l=1}^{\infty} \delta_A^{(2)} (E_l)$$

$$= \int_{\mathbb{N}} g(l)\, d\mu.$$

In other words,

$$P - \lim_{m,n} \int_{\mathbb{N}} g_{mn}(l)\, d\mu = \int_{\mathbb{N}} P - \lim_{m,n} g_{mn}(l)\, d\mu.$$

Also for every m and n,

$$|f_{mn}(l)| = \left| \sum_{j,k=1,1}^{\infty,\infty} a^{mn}_{v_l(j),\mu_l(k)} \gamma^{(l)}_{j,k} \right|$$

$$\leq M \sum_{j,k=1,1}^{\infty,\infty} a^{mn}_{v_l(j),\mu_l(k)}$$

$$= M (Be)_{mn} = g_{mn}(l).$$

Therefore according to dominated convergence theorem with two parameters,

$$P - \lim_{m,n} (Ax)_{mn} = P - \lim_{m,n} \sum_{l=1}^{\infty} \left(B\gamma^{(l)} \right)_{mn}$$

$$= \sum_{l=1}^{\infty} P - \lim_{m,n} \left(B\gamma^{(l)} \right)_{mn}$$

$$= \sum_{l=1}^{\infty} \delta_A^{(2)} (E_l)\, \Gamma^{(l)}$$

is obtained. Thus x is A-summable to $\sum_l \delta_A^{(2)} (E_l)\, \Gamma^{(l)}$ and consequently A has the splicing property. □

Example 1. *There exists a 3-spliced sequence x over a 3-partition $\{E_1, E_2, E_3\}$ with $\Gamma^{(1)}$, $\Gamma^{(2)}$, $\Gamma^{(3)}$ all distinct, such that x is $(C,1,1)$-summable to 0, but $\delta^{(2)}(E_1)$, $\delta^{(2)}(E_2)$ and $\delta^{(2)}(E_3)$ do not exist. In order to see it, first recall that an RH-regular matrix can not sum every bounded sequence even if consists of 0's and 1's. Therefore there exists a subset $E_1 = \{v_l(j), \mu_l(k)\}_{j,k=1,1}^{\infty,\infty} \subset \mathbb{N} \times \mathbb{N}$ with $v_1(j) + 1 < v_1(j+1)$ and $\mu_1(k) + 1 < \mu_1(k+1)$ such that $\delta^{(2)}(E_1) = P - \lim \frac{1}{mn} |\{(i,j) : \{v_l(j), \mu_l(k)\} \in E_1\}|$ does not exist. Considering the set $E_3 = (\mathbb{N} \times \mathbb{N}) \setminus (E_1 \cup E_2)$, $\{E_1, E_2, E_3\}$ becomes a 3-partition of $\mathbb{N} \times \mathbb{N}$. If $\gamma^{(1)}_{j,k} := 1$, $\gamma^{(2)}_{j,k} := -1$ and $\gamma^{(3)}_{j,k} := 0$ over $\{E_1, E_2, E_3\}$, then the obtained 3-spliced sequence x is $(C,1,1)$-summable to 0. Actually*

$$((C,1,1)(x))_{mn} = \left| \frac{1}{mn} \sum_{\substack{(j,k)\in E_1 \\ 1\leq j\leq m \\ 1\leq k\leq n}} x_{jk} + \frac{1}{mn} \sum_{\substack{(j,k)\in E_2 \\ 1\leq j\leq m \\ 1\leq k\leq n}} x_{jk} + \frac{1}{mn} \sum_{\substack{(j,k)\in E_3 \\ 1\leq j\leq m \\ 1\leq k\leq n}} x_{jk} \right|$$

$$= \left| \frac{1}{mn} \sum_{\substack{j,k \\ v_1(j)\leq m \\ \mu_1(k)\leq n}} \gamma^{(1)}_{j,k} + \frac{1}{mn} \sum_{\substack{j,k \\ v_2(j)=v_1(j)+1\leq m \\ \mu_2(k)=\mu_1(k)+1\leq n}} \gamma^{(2)}_{j,k} + \frac{1}{mn} \sum_{\substack{j,k \\ v_3(j)\leq m \\ \mu_3(k)\leq n}} \gamma^{(3)}_{j,k} \right|$$

$$= \left| \frac{1}{mn} \sum_{\substack{j,k \\ v_1(j) \le m \\ \mu_1(k) \le n}} 1 - \frac{1}{mn} \sum_{\substack{j,k \\ v_1(j) \le m-1 \\ \mu_1(k) \le n-1}} 1 \right| \le \frac{1}{m} + \frac{1}{n} \longrightarrow 0.$$

Example 2. *Consider an ∞-partition of $\mathbb{N} \times \mathbb{N}$ as follows*

$$E_l = \left\{ \left(2^{l-1} (2j-1), k \right) \right\}_{j,k=1,1}^{\infty,\infty}, \quad l \in \mathbb{N}.$$

According to Mursaleen and Osama [5], for every l, we have

$$\delta^{(2)}(E_l) = P - \lim_{j,k} \frac{j.k}{2^{l-1}(2j-1).k} = \frac{1}{2^l}.$$

Hence, we obtain $\sum_{l=1}^{\infty} \delta^{(2)}(E_l) = \sum_{l=1}^{\infty} \frac{1}{2^l} = 1.$

Example 3. *Consider the following double sequence $[x]$:*

$$x_{jk} = \begin{cases} \sqrt{\frac{3r+1}{r}} & , \quad j = 3r-2, \ k \in \mathbb{N}, \\ \arctan r & , \quad j = 3r-1, \ k \in \mathbb{N}, \\ \frac{1}{2r} & , \quad j = 3r, \ k \in \mathbb{N}. \end{cases}$$

This sequence can be expressed a 3-spliced of the sequences $\gamma^{(1)}$, $\gamma^{(2)}$, $\gamma^{(3)}$ over $\{E_1, E_2, E_3\}$, where

$$\gamma_{jk}^{(1)} = \sqrt{\frac{3j+1}{j}}, \quad \gamma_{jk}^{(2)} = \arctan j, \quad \gamma_{jk}^{(3)} = \frac{1}{2j}$$

and

$$E_1 = \{(3j-2,k)\}_{j,k=1,1}^{\infty,\infty}, \quad E_2 = \{(3j-1,k)\}_{j,k=1,1}^{\infty,\infty}, \quad E_3 = \{(3j,k)\}_{j,k=1,1}^{\infty,\infty}.$$

Observe that $\delta^{(2)}(E_l) = \frac{1}{3}$ for $l = 1,2,3$ and also $P - \lim_{j,k} \gamma_{jk}^{(1)} = \sqrt{3}$, $P - \lim_{j,k} \gamma_{jk}^{(2)} = \frac{\pi}{2}$, $P - \lim_{j,k} \gamma_{jk}^{(3)} = 0$. It is not easy to show directly that such the sequence x is $(C,1,1)$-summable. But by using Theorem 1, we easily have that

$$P - \lim_{m,n} \{(C,1,1)x\}_{mn} = \sum_{l=1}^{3} \delta^{(2)}(E_l) \Gamma^{(l)} = \frac{1}{3}\sqrt{3} + \frac{1}{3}\frac{\pi}{2} + \frac{1}{3}.0 = \frac{1}{3}\left(\sqrt{3} + \frac{\pi}{2}\right).$$

Author Contribution Statements The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

Declaration of Competing Interests It is declared that the authors have no competing interests.

References

[1] Bartoszewicz, A., Das, P., Głąb, S., On matrix summability of spliced sequences and A-density of points, *Linear Algebra Appl.*, 487 (2015), 22–42. https://doi.org/10.1016/j.laa.2015.08.031.

[2] Bose, K., Das, P., Sengupta, S., On spliced sequences and the density of points with respect to a matrix constructed by using a weight function, *Ukrainian Mathematical Journal*, 71 (2020), 1359–1374. https://doi.org/10.1007/s11253-020-01720-1.

[3] Gökhan, A., Çolak, R., Mursaleen, M., Some matrix transformation and generalized core of double sequences, *Mathematical and Computer Modelling*, 49 (2009), 1721–1731. https://doi.org/10.1016/j.mcm.2008.12.002.

[4] Hamilton, H. J., Transformations of multiple sequences, *Duke Mathematical Journal*, 2 (1936), 29–60.

[5] Mursaleen, E., Osama, H. H., Statistical convergence of double sequences, *J. Math. Anal. Appl.*, 288 (2003), 223–231.

[6] Osikiewicz, J. A., Summability of spliced sequences, *Rocky Mountain J. Math.*, 35 (2005), 977–996.

[7] Patterson, R. F., Analogues of some fundamental theorems of summability theory, *International Journal of Mathematics and Mathematical Sciences*, 23 (2000), 1–9.

[8] Patterson, R. F., Lemma, M., Four dimensional matrix characterization of double oscillation via RH-conservative and RH-multiplicative matrices, *Cent. Eur. J. Math.*, 6 (2008), 581–594. https://doi.org/10.2478/s11533-008-0043-7.

[9] Pringsheim, A., On the theory of doubly infinite sequences of numbers, *Math. Ann.*, 53 (1900), 289–321.

[10] Robinson, G. M., Divergent double sequences and series, *Transactions of the American Mathematical Society*, 28 (1926), 50–73.

[11] Ünver, M., Abel summability in topological spaces, *Monatshefte für Mathematik*, 178 (2015), 633–643. https://doi.org/10.1007/s00605-014-0717-0.

[12] Ünver, M., Khan, M. K., Orhan, C., *A*-distributional summability in topological spaces, *Positivity*, 18 (2014), 131–145. https://doi.org/10.1007/s11117-013-0235-7.

[13] Yardımcı, Ş., Gülfırat, M., Spliced sequences and summability with a rate, *Positivity* 27(17) (2023). https://doi.org/10.1007/s11117-023-00970-0.

[14] Yurdakadim, T., Ünver, M., Some results concerning the summability of spliced sequences, *Turk. J. Math.*, 40 (2016), 1134–1143. https://doi.org/10.3906/mat-1508-34.