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Research Article

Orlicz amalgam spaces on the affine group

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Abstract

Let $\mathbb{A}$ be the affine group, Φ_1, Φ_2 be Young functions. We study the Orlicz amalgam spaces $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ defined on $\mathbb{A}$, where the local and global component spaces are the Orlicz spaces $L^{\Phi_1}(\mathbb{A})$ and $L^{\Phi_2}(\mathbb{A})$, respectively. We obtain an equivalent discrete norm on the amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ using the constructions related to the affine group. Using the discrete norm we compute the dual space of $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$. We also prove that the Orlicz amalgam space is a left $L^1(\mathbb{A})$-module with respect to convolution under certain conditions. Finally, we investigate some inclusion relations between the Orlicz amalgam spaces.

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1. Introduction

An amalgam space consists of functions whose norm distinguishes between local and global properties. The first appearance of amalgam spaces was due to Wiener in his studies of generalized harmonic analysis [21–23]. Amalgam spaces of Lebesgue spaces are investigated by many authors [2–4, 14]. The most general definition of Wiener amalgam spaces was introduced by Feichtinger in 1980s [7–10].

Amalgam spaces have turned out to be very fruitful within pure and applied mathematics. In fact, these spaces are nowadays present in investigations that concern problems on pseudo differential operators, Strichartz estimates [6, 20] and mostly considered for the Lebesgue spaces on the real line. On the other hand, for $1 \leq p < \infty$, Heil and Kutyniok studied amalgam spaces $W(L^\infty(\mathbb{A}), L^p(\mathbb{A}))$ on the affine group $\mathbb{A}$ [12, 13], which is not abelian unlike the real line. They proved a useful convolution relation on the amalgam space $W(L^\infty(\mathbb{A}), L^1(\mathbb{A}))$.

Convolution relations have been intensively studied on IN groups, i.e., locally compact groups with a compact and invariant neighbourhood of identity. IN groups include all abelian groups as well as some non-abelian groups such as the reduced Heisenberg group which is important for time-frequency analysis. Unfortunately, the affine group which is important for wavelet theory is not an IN group. However, even for the affine group there are interesting, but more complicated, convolution relations.

An Orlicz space is a type of function space which generalizes the Lebesgue spaces L^p significantly. Besides the L^p spaces, a variety of function spaces arises naturally in analysis

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in this way such as $L\log^+ L$ which is a Banach space related to Hardy-Littelewood maximal functions. Orlicz spaces contain certain Sobolev spaces as subspaces.

In [1], the spaces $W(L^{\Phi}(\mathbb{A}), L^1(\mathbb{A}))$ and $W(L^{\infty}(\mathbb{A}), L^{\Phi}(\mathbb{A}))$ are defined on the affine group $\mathbb{A}$ and studied some properties such as translation invariance, inclusions and convolution.

The aim of this paper is to extend the results in [1] to a more general Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$. In order to do this we are motivated to study discrete norms on $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ using a specific partition of unity of the affine group. Using the discrete norm we prove duality and convolution theorems for amalgams, as well as inclusion relations.

This paper is organized as follows. In Section 2, we present some background and notation on weighted Orlicz spaces on locally compact groups. We also define the Orlicz amalgam spaces on the affine group which we denote $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$. In Section 3, we construct an equivalent discrete norm on the Orlicz amalgam spaces (Proposition 3.2). Using the equivalent discrete norm, we prove a duality theorem (Theorem 4.1) for the Orlicz amalgam space in Section 4. In Section 5, we give certain conditions under which the corresponding space over a non-IN group becomes a left $L^1(\mathbb{A})$-module with respect to convolution (Theorem 5.2). Finally, in Section 6, we investigate inclusion relations among the Orlicz amalgam spaces (Theorem 6.1, Theorem 6.5). Some results are also new for the Lebesgue spaces and the Orlicz spaces.

2. Prelimaniries

Throughout the paper, we consider the affine group $\mathbb{A} = \mathbb{R}^+ \times \mathbb{R}$ with the multiplication

$$(a,b)(x,y) = \Big(ax, \frac{b}{x} + y\Big),$$

where $\mathbb{R}^+$ denotes the multiplicative group of positive real numbers. The identity element and inverses of $\mathbb{A}$ are given by

$$e = (1,0), \quad (a,b)^{-1} = \Big(\frac{1}{a}, -ab\Big)$$

for $(a,b) \in \mathbb{A}$, respectively. It is easy to see that $\mathbb{A}$ is a non-abelian group under its multiplication.

One can see that the left Haar measure on $\mathbb{A}$ is $d\mu = \frac{dx}{x}dy$. The affine group $\mathbb{A}$ is not unimodular.

Let f, g be measurable functions on $\mathbb{A}$. The convolution product of f and g is defined by

$$(f * g)(x,y) = \int_{\mathbb{A}} f(a,b)g((a,b)^{-1}(x,y))\frac{da}{a}db, \quad (x,y) \in \mathbb{A},$$

whenever the integral exists.

We consider Orlicz spaces on the affine group $\mathbb{A}$. An Orlicz space is determined by a Young function. A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. For a Young function Φ, the complementary function Ψ of Φ is given by

$$\Psi(y) = \sup\{xy - \Phi(x) : x > 0\}, \quad y \geq 0,$$

and Ψ is also a Young function. So (Φ, Ψ) is called a complementary Young pair. We have the Young inequality

$$xy \leq \Phi(x) + \Psi(y), \quad x, y \geq 0$$

for complementary functions Φ and Ψ.

By our definition, a Young function can have the value ∞ at a point, and hence be discontinuous at such a point. However, we always consider the pair of complementary

Young functions (Φ, Ψ) with Φ being real valued and continuous on $[0, \infty)$ and increasing on $(0, \infty)$. Note that even though Φ is continuous, it may happen that Ψ is not continuous.

Let Φ_1, Φ_2 be two Young functions. If there exist a $c > 0$ and $x_0 \geq 0$ (depending on c) such that $\Phi_1(x) \leq \Phi_2(cx)$ for all $x \geq x_0$, then we say that Φ_2 is stronger than Φ_1 and denote this by $\Phi_1 \prec \Phi_2$. If $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$, then we write $\Phi_1 \asymp \Phi_2$.

A Young function Φ satisfies the Δ_2 condition if there exist a constant $K > 0$ and an $x_0 \geq 0$ such that $\Phi(2x) \leq K\Phi(x)$ for all $x \geq x_0$. In this case, we write $\Phi \in \Delta_2$.

Let $\mathbb{A}$ be given with the left Haar measure $d\mu = \frac{dx}{x}dy$. Given a Young function Φ, the Orlicz space on $\mathbb{A}$ is defined by

$$L^\Phi(\mathbb{A}) = \Big\{ f : \mathbb{A} \to \mathbb{C} \text{ measurable} : \int_{\mathbb{A}} \Phi(\alpha|f(x,y)|)\frac{dx}{x}dy < \infty \text{ for some } \alpha > 0 \Big\}. \tag{2.1}$$

Then the Orlicz space is a Banach space under the Orlicz norm $\|\cdot\|_{L^\Phi(\mathbb{A})}$ defined for $f \in L^\Phi(\mathbb{A})$ by

$$\|f\|_{L^\Phi(\mathbb{A})} = \sup\Big\{ \int_{\mathbb{A}} |f(x,y)g(x,y)|\frac{dx}{x}dy \ : \ \int_{\mathbb{A}} \Psi(|g(x,y)|)\frac{dx}{x}dy \leq 1 \Big\},$$

where Ψ is the complementary Young function of Φ.

Letting

$$B_\Psi[0,1] = \Big\{ g \in L^\Psi(\mathbb{A}) \ : \ \int_{\mathbb{A}} \Psi(|g(x,y)|)\frac{dx}{x}dy \leq 1 \Big\},$$

we have

$$\|f\|_{L^\Phi(\mathbb{A})} = \sup\Big\{ \int_{\mathbb{A}} |f(x,y)g(x,y)|\frac{dx}{x}dy \ : \ g \in B_\Psi[0,1] \Big\}.$$

One can also define the Luxemburg norm $\|\cdot\|^o_{L^\Phi(\mathbb{A})}$ on $L^\Phi(\mathbb{A})$ by

$$\|f\|^o_{L^\Phi(\mathbb{A})} = \inf\Big\{ k > 0 : \int_{\mathbb{A}} \Phi\Big(\frac{|f(x,y)|}{k}\Big)\frac{dx}{x}dy \leq 1 \Big\}.$$

It is known that these norms are equivalent, that is,

$$\|\cdot\|^o_{L^\Phi(\mathbb{A})} \leq \|\cdot\|_{L^\Phi(\mathbb{A})} \leq 2\|\cdot\|^o_{L^\Phi(\mathbb{A})}$$

and

$$\|f\|^o_{L^\Phi(\mathbb{A})} \leq 1 \text{ if and only if } \int_{\mathbb{A}} \Phi(|f(x,y)|)\frac{dx}{x}dy \leq 1.$$

If (Φ, Ψ) is a complementary Young pair and $\Phi \in \Delta_2$, the dual space $L^\Phi(\mathbb{A})^*$ is $L^\Psi(\mathbb{A})$. If, in addition, $\Psi \in \Delta_2$, then the Orlicz space $L^\Phi(\mathbb{A})$ is a reflexive Banach space [15, 16].

Let $C_c(\mathbb{A})$ denote the space of all continuous complex valued functions on $\mathbb{A}$ with compact support. If $\Phi \in \Delta_2$, then $C_c(\mathbb{A})$ is dense in $L^\Phi(\mathbb{A})$ [15, 16].

A normed space $(Y, \|\cdot\|_Y)$ consisting of measurable of complex valued functions on a measurable space X is called solid if for each measurable $f : X \to \mathbb{C}$ satisfying $|f| \leq |g|$ almost everywhere for some $g \in Y$, then $f \in Y$ and $\|f\|_Y \leq \|g\|_Y$. Since the Young function Φ is increasing, the Orlicz space $L^\Phi(\mathbb{A})$ is a solid space. That is if any measurable function f for which there exists $g \in L^\Phi(\mathbb{A})$ such that $|f| \leq |g|$ locally almost everywhere belongs to $L^\Phi(\mathbb{A})$, with $\|f\|_{L^\Phi(\mathbb{A})} \leq \|g\|_{L^\Phi(\mathbb{A})}$ [18]. Also, if the right derivative of a Young function Φ at zero is positive, i.e., $\Phi'_+(0) > 0$, then the inclusion $L^\Phi(\mathbb{A}) \subseteq L^1(\mathbb{A})$ is valid. This implies that there exists a constant $c > 0$ such that

$$\|f\|_{L^1(\mathbb{A})} \leq c\|f\|_{L^\Phi(\mathbb{A})} \tag{2.2}$$

holds for every $f \in L^\Phi(\mathbb{A})$ [19, Theorem 3.1.2].

Let us remind some basic properties of $L^\Phi(\mathbb{A})$ which are given by [1].

Let $f \in L^\Phi(\mathbb{A})$ and $(a,b) \in \mathbb{A}$. The left translation, right translation and re-normalized right translation on $\mathbb{A}$ are defined by

$$L_{(a,b)}f(x,y) = f((a,b)^{-1}(x,y)),$$
$$R_{(a,b)}f(x,y) = f((x,y)(a,b)^{-1}),$$
$$A_{(a,b)}f(x,y) = aR_{(a,b)}f(x,y) = af((x,y)(a,b)^{-1})$$

for all $(x,y) \in \mathbb{A}$.

Lemma 2.1. *For $f \in L^\Phi(\mathbb{A})$, the following hold.*

(a) $\|L_{(a,b)}f\|_{L^\Phi(\mathbb{A})} = \|f\|_{L^\Phi(\mathbb{A})}$,
(b) $\|R_{(a,b)}f\|_{L^\Phi(\mathbb{A})} = \frac{1}{a}\|f\|_{L^\Phi(\mathbb{A})}$,
(c) $\|A_{(a,b)}f\|_{L^\Phi(\mathbb{A})} = \|f\|_{L^\Phi(\mathbb{A})}$.

Lemma 2.2. *Let Φ be a Young function with ${\Phi'}_+(0) > 0$. Then $L^\Phi(\mathbb{A})$ is a left Banach algebra with respect to convolution, that is,*

$$\|f * g\|_{L^\Phi(\mathbb{A})} \leq \|f\|_{L^\Phi(\mathbb{A})}\|g\|_{L^\Phi(\mathbb{A})}$$

holds for all $f, g \in L^\Phi(\mathbb{A})$.

Let us note that, without any condition on the Young function Φ, $L^\Phi(\mathbb{A})$ is a left $L^1(\mathbb{A})$-module with respect to convolution, i.e.,

$$\|f * g\|_{L^\Phi(\mathbb{A})} \leq \|f\|_{L^1(\mathbb{A})}\|g\|_{L^\Phi(\mathbb{A})}, \quad f \in L^1(\mathbb{A}),\ g \in L^\Phi(\mathbb{A}).$$

Lemma 2.3. *Let Φ be a Young function with ${\Phi'}_+(0) > 0$. Then, the equality*

$$\|A_{(a,b)}f * g\|_{L^\Phi(\mathbb{A})} = \|f * L_{(a,b)}g\|_{L^\Phi(\mathbb{A})}$$

holds for all $f, g \in L^\Phi(\mathbb{A})$.

Orlicz spaces are a kind of generalization of Lebesgue spaces. If the Young function Φ is $\frac{x^p}{p}$ or x^p for $1 < p < \infty$, then the space $L^\Phi(\mathbb{A})$ becomes the classical Lebesgue space $L^p(\mathbb{A})$ and the norm $\|\cdot\|_{L^\Phi}$ is equivalent to the classical norm $\|\cdot\|_{L^p}$.

If $p = 1$, then we obtain the space $L^1(\mathbb{A})$. In this case, the complementary Young function of $\Phi(x) = x$ is

$$\Psi(x) = \begin{cases} 0, & 0 \leq x \leq 1, \\ \infty, & x > 1, \end{cases} \tag{2.3}$$

and $\|f\|_{L^\Phi} = \|f\|_{L^1}$ for all $f \in L^1(\mathbb{A})$. If $p = \infty$, then for the Young function Φ given in (2.3), the space $L^\Phi(\mathbb{A})$ is equal to the space $L^\infty(\mathbb{A})$ and we have $\|f\|_{L^\Phi} = \|f\|_{L^\infty}$ for all $f \in L^\infty(\mathbb{A})$.

There are other examples of complementary Young pairs.

(i) If $\Phi(x) = e^{ax} - 1$ with $a > 0$, then $\Psi(x) = \frac{1}{a}\ln(\frac{y}{a})y - \frac{y}{a} + 1$.
(ii) If $\Phi(x) = e^x - x - 1$, then $\Psi(x) = (1+x)\ln(1+x) - x$.
(iii) If

$$\Phi(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ \infty, & x > 1, \end{cases}$$

then

$$\Psi(x) = \begin{cases} 0, & 0 \leq x \leq 1, \\ x-1, & x > 1. \end{cases}$$

Note that if we take $\Phi(x) = e^{x^\beta} - 1$ with $\beta > 0$, then the Orlicz space L^Φ becomes the Zygmund space $\exp L^\beta$.

For further information on Orlicz spaces, the reader is referred to [18] and [19].

Let us now define the Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ consisting of functions that are locally in $L^{\Phi_1}(\mathbb{A})$ and globally in $L^{\Phi_2}(\mathbb{A})$. In our results, the translation invariance and solidity of $L^{\Phi_1}(\mathbb{A})$ and $L^{\Phi_2}(\mathbb{A})$ will play important roles. In light of this, we start with the following.

Definition 2.4. Let Q be a fixed compact subset of $\mathbb{A}$ with nonempty interior. The Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})$ consists of all measurable functions $f : \mathbb{A} \to \mathbb{C}$ such that $f\chi_{(x,y)Q} \in L^{\Phi_1}(\mathbb{A})$ for each $(x, y) \in \mathbb{A}$ and the control function

$$F_f(x, y) = F_f^Q(x, y) = \|f\chi_{(x,y)Q}\|_{L^{\Phi_1}(\mathbb{A})}$$

is in $L^{\Phi_2}(\mathbb{A})$. The Orlicz amalgam norm on $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ is defined by

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} := \|F_f\|_{L^{\Phi_2}(\mathbb{A})} = \|\|f\chi_{(x,y)Q}\|_{L^{\Phi_1}(\mathbb{A})}\|_{L^{\Phi_2}(\mathbb{A})}. \tag{2.4}$$

Similar to that in Orlicz spaces, we define the Luxemburg norm $\|\cdot\|^o_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))}$ on $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ by

$$\|f\|^o_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} = \|\|f\chi_{(x,y)Q}\|^o_{L^{\Phi_1}(\mathbb{A})}\|^o_{L^{\Phi_2}(\mathbb{A})}.$$

By the equivalence of the Orlicz norm and the Luxemburg norm in Orlicz spaces [18], we have

$$\|f\|^o_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} \le \|f\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} \le 4\|f\|^o_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))}. \tag{2.5}$$

Throughout the paper, we consider the Orlicz norm on $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$.

Note that $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ is a Banach space and its definition is independent of the choice of the compact subset $Q \subset \mathbb{A}$, in the sense that different compact subsets yield equivalent Orlicz amalgam space norms. By using the completeness and the translation invarinace of the Orlicz spaces $L^{\Phi_1}(\mathbb{A})$, $L^{\Phi_2}(\mathbb{A})$, the proofs can be done to that of [1, Theorem 4.2, Theorem 4.3].

Our reason for studying Orlicz amalgam spaces comes from the fact that they generalize the Orlicz spaces. In particular, setting $\Phi_1 \asymp \Phi_2$ will actually result in an Orlicz space. Hence, we can give the following proposition in [1].

Proposition 2.5. *Let (Φ, Ψ) be a complementary Young pair with $\Phi'_+(0) > 0$ and $\Psi'_+(0) > 0$. Then $W(L^{\Phi}(\mathbb{A}), L^{\Phi}(\mathbb{A})) = L^{\Phi}(\mathbb{A})$.*

There is also a Hölder inequality for Orlicz amalgam spaces similar to that in Orlicz spaces. It can be extended to a duality theorem. However, the duality of Orlicz amalgam spaces will be proved after having defined discrete norms in the next section. For now we state Hölder inequality for Orlicz amalgam spaces.

Proposition 2.6. *Let $(\Phi_1, \Psi_1), (\Phi_2, \Psi_2)$ be a complementary Young pairs. Then, we have*

$$\|fg\|_{L^1(\mathbb{A})} \le \|f\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))}\|g\|^o_{W(L^{\Psi_1}(\mathbb{A}),L^{\Psi_2}(\mathbb{A}))} \tag{2.6}$$

for all $f \in W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ and $g \in W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))$.

Proof. Let $f \in W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})$ and $g \in W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))$. By using the Hölder inequality in the Orlicz spaces [18], we obtain

$$\|fg\|_{W(L^1,L^1)} = \|\|(f\chi_{(x,y)Q})(g\chi_{(x,y)Q})\|_{L^1}\|_{L^1} \le \|f\|_{W(L^{\Phi_1},L^{\Phi_2})}\ \|g\|^o_{W(L^{\Psi_1},L^{\Psi_2})}.$$

The result then follows from the fact that $W(L^1(\mathbb{A}), L^1(\mathbb{A})) = L^1(\mathbb{A})$ [11]. □

Let us remark that by (2.5), we have

$$\|fg\|_{L^1(\mathbb{A})} \le 4\|f\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))}\|g\|_{W(L^{\Psi_1}(\mathbb{A}),L^{\Psi_2}(\mathbb{A}))} \tag{2.7}$$

for all $f \in W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ and $g \in W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))$.

3. Discrete norms

In this section, we construct an equivalent discrete norm on the Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ using partitions of unity of $\mathbb{A}$. This allows to prove some basic properties of the Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ such as duality and inclusion relations for the Orlicz amalgam spaces based on the global components. In particular, by using the equivalent discrete norm, we want to prove the Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ is a left $L^1(\mathbb{A})$-module in Section 5.

For our goal, we need the following notation and lemma which we use in [1].

Let $\{Q_h\}_{h>0}$ denote a fixed family of increasing, exhaustive neighborhoods of identity in $\mathbb{A}$ and we will take $Q_h = [e^{-h}, e^h) \times [-h, h)$. As above, $(x, y)Q_h$ is the set Q_h left translated by $(x, y) \in \mathbb{A}$, i.e.,

$$(x, y)Q_h = \Big\{\Big(xa, \frac{y}{a} + b\Big) \ : \ a \in [e^{-h}, e^h),\ b \in [-h, h)\Big\}.$$

The Haar measure of the translated set $(x, y)Q_h$ is

$$\mu((x, y)Q_h) = \mu(Q_h) = \int_{-h}^{h}\int_{e^{-h}}^{e^h} \frac{dx}{x}dy = 4h^2. \tag{3.1}$$

Given $h > 0$, for $k, j \in \mathbb{Z}$, we define particular translates of Q_h and Q_{2h}, by

$$B_{jk} = (e^{2jh}, 2khe^{-h})Q_h,$$
$$B'_{jk} = (e^{2jh}, 2khe^{-h})Q_{2h}.$$

Let us note that $B_{jk} \subseteq B'_{jk}$.

To obtain an equivalent discrete norm on these spaces the following lemma is a key observation.

Lemma 3.1 ([12]). *If $h > 0$, then*

(a) $\bigcup_{j,k\in\mathbb{Z}} B_{jk} = \mathbb{A}$,

(b) *given $m, n \in \mathbb{Z}$, the box B'_{mn} can intersect at most $N = 5(2e^{3h} + 1)$ boxes B'_{jk} for $j, k \in \mathbb{Z}$.*

Hence the set $X = \{(e^{2jh}, 2khe^{-h}) \ : \ j, k \in \mathbb{Z}\}$ for $h > 0$ becomes a well-spread family [9, 10].

By Urysohn's lemma, there exist continuous functions $\phi_{jk} : \mathbb{A} \to \mathbb{R}$ such that $0 \le \phi_{jk}(x, y) \le 1$, $\mathrm{supp}(\phi_{jk}) \subseteq B'_{jk}$ and $\phi_{jk}(x, y) = 1$ for $(x, y) \in B_{jk}$. Define

$$\psi_{jk} = \frac{\phi_{jk}}{\sum\limits_{m,n\in\mathbb{Z}} \phi_{mn}}.$$

Thus $\{\psi_{jk}\}_{j,k\in\mathbb{Z}}$ is a bounded uniform partition of unity (BUPU). Then, by [7, Theorem 2] we have the following equivalence on $W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi_2}(\mathbb{A}))$

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi_2}(\mathbb{A}))} \approx \Big\| \sum_{j,k\in\mathbb{Z}} \|f\psi_{jk}\|_{L^{\Phi_1}(\mathbb{A})}\chi_{B'_{jk}} \Big\|_{L^{\Phi_2}(\mathbb{A})}. \tag{3.2}$$

On the other hand, $\{B_{jk}\}_{j,k\in\mathbb{Z}}$ is a partition of $\mathbb{A}$ and $\{\chi_{B_{jk}}\}_{j,k\in\mathbb{Z}}$ becomes a BUPU [12]. Therefore, if we take compact sets B_{jk} instead of the sets B'_{jk} in (3.2), then we obtain the following proposition which gives us an equivalent discrete norm.

Proposition 3.2. *Let Φ_1, Φ_2 be Young functions. Then, we have*

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} \approx \Big\|\big(\|f\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})}\big)_{j,k\in\mathbb{Z}}\Big\|_{\ell^{\Phi_2}}.$$

In other words, $W(L^{\Phi_1}, L^{\Phi_2}) = W(L^{\Phi_1}, \ell^{\Phi_2})$.

Proof. Let us take the well-spread family $X = \{(e^{2jh}, 2khe^{-h}) \; : \; j,k \in \mathbb{Z}\}$ for $h > 0$. Then we have a BUPU $\{\psi_{jk}\}_{j,k\in\mathbb{Z}}$. Since $\chi_{B_{jk}} \le N\psi_{jk} \le N\chi_{B'_{jk}}$ by Lemma 3.1, by using a property of a well-spread family, we have

$$\begin{aligned}\frac{1}{N}\sum_{j,k\in\mathbb{Z}} \|f\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})} &\le \sum_{j,k\in\mathbb{Z}} \|f\psi_{jk}\|_{L^{\Phi_1}(\mathbb{A})} \\ &\le \sum_{j,k\in\mathbb{Z}} \|f\chi_{B'_{jk}}\|_{L^{\Phi_1}(\mathbb{A})} \le N\sum_{j,k\in\mathbb{Z}} \|f\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})}.\end{aligned} \tag{3.3}$$

By using (3.2), (3.3) and a property of the Bochner integral [5, Appendix E.11], we obtain

$$\begin{aligned}&\Big\|\sum_{j,k\in\mathbb{Z}} \|f\psi_{jk}\|_{L^{\Phi_1}(\mathbb{A})}\chi_{B_{jk}}\Big\|_{L^{\Phi_2}(\mathbb{A})} \\ &= \sup\Big\{\int_{B_{jk}}\sum_{j,k\in\mathbb{Z}} \|f\psi_{jk}\|_{L^{\Phi_1}(\mathbb{A})}|g(z)|d\mu(z) \; : \; \sum_{j,k\in\mathbb{Z}}\int_{B_{jk}}\Psi_2(|g(z)|)d\mu(z) \le 1\Big\} \\ &= \sup\Big\{\sum_{j,k\in\mathbb{Z}} \|f\psi_{jk}\|_{L^{\Phi_1}(\mathbb{A})}\int_{B_{jk}}|g(z)|d\mu(z) \; : \; \sum_{j,k\in\mathbb{Z}}\Psi_2(\int_{B_{jk}}|g(z)|d\mu(z)) \le 1\Big\} \\ &\le N\Big\|(\|f\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})})_{j,k\in\mathbb{Z}}\Big\|_{\ell^{\Phi_2}}.\end{aligned}$$

On the other hand, by using the left side of (3.3), we obtain

$$\Big\|\sum_{j,k\in\mathbb{Z}} \|f\psi_{jk}\|_{L^{\Phi_1}(\mathbb{A})}\chi_{B_{jk}}\Big\|_{L^{\Phi_2}(\mathbb{A})} \ge \frac{1}{N}\Big\|(\|f\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})})_{j,k\in\mathbb{Z}}\Big\|_{\ell^{\Phi_2}}.$$

Hence, by the equivalence (3.2), we find

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}),\; L^{\Phi_2}(\mathbb{A}))} \approx \Big\|(\|f\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})})_{j,k\in\mathbb{Z}}\Big\|_{\ell^{\Phi_2}}.$$

□

In particular, if we consider the Young functions

$$\Phi_1(x) = \begin{cases} 0, & 0 \le x \le 1, \\ \infty, & x > 1, \end{cases}$$

and $\Phi_2(x) = x^p$ for $1 \le p < \infty$ in Proposition 3.2, then we obtain the following equivalent norm on $W(L^\infty(\mathbb{A}), L^p(\mathbb{A}))$

$$\|f\|_{W(L^\infty(\mathbb{A}), L^p(\mathbb{A}))} \approx \Big\|(\|f\chi_{B_{jk}}\|_{L^\infty(\mathbb{A})})_{j,k\in\mathbb{Z}}\Big\|_{\ell^p}$$

which is given in [12, Proposition 3.3].

4. Duality

To prove the following duality theorem for the space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$, the equivalent discrete norm in Proposition 3.2 will play a key role.

Theorem 4.1. *Let $(\Phi_1, \Psi_1), (\Phi_2, \Psi_2)$ be complementary Young pairs with $\Phi_1, \Phi_2 \in \Delta_2$. Then, the dual space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))^*$ is $W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))$.*

Proof. Define the following map for $f \in W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ as

$$\begin{aligned}&\phi_g : W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A})) \to W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))^* \\ &\qquad\qquad g \to \phi_g(f) = \langle f, g\rangle = \int_{\mathbb{A}} |f(x,y)g(x,y)|\frac{dx}{x}dy.\end{aligned}$$

By the Hölder inequality (2.6), we have

$$\int_{\mathbb{A}} |f(x,y)g(x,y)|\frac{dx}{x}dy \le \|f\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))}\ \|g\|^{o}_{W(L^{\Psi_1}(\mathbb{A}),L^{\Psi_2}(\mathbb{A}))},$$

which implies that $\langle f,g\rangle$ is well-defined and so g determines a continuous linear functional ϕ_g on $W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))$.

Let $\varphi \in W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))^*$ be given. For simplicity, let us consider $\{\chi_{B_{jk}}\}_{j,k\in\mathbb{Z}}$ and fix $j,k\in\mathbb{Z}$. Then, $L^{\Phi_1}(B_{jk})$ which is the space of $L^{\Phi_1}(\mathbb{A})$ functions supported in B_{jk}, is contained in $W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))$, i.e.,

$$L^{\Phi_1}(B_{jk}) = \{h \in L^{\Phi_1}(\mathbb{A}) : \operatorname{supp}(h) \subseteq B_{jk}\} \subseteq W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A})).$$

Thus, φ restricts to a bounded linear functional $\varphi_{jk} := \varphi_{B_{jk}}$ on $L^{\Phi_1}(B_{jk})$ for each $j,k\in\mathbb{Z}$. Therefore, there exists a $g_{jk} \in L^{\Phi_1}(B_{jk})^* = L^{\Psi_1}(B_{jk})$ such that

$$\|g_{jk}\|^{\circ}_{L^{\Psi_1}(\mathbb{A})} = \|\varphi_{jk}\| \ \text{ and } \ \langle h,\varphi\rangle = \langle h, g_{jk}\rangle,$$

for all $h \in L^{\Phi_1}(B_{jk})$, where $g_{jk} = g\chi_{B_{jk}}$.

Since $\operatorname{supp}(g_{jk}) \subseteq B_{jk}$ and the family $\{B_{jk}\}_{j,k\in\mathbb{Z}}$ is a partition of $\mathbb{A}$, we can define $g = \sum_{j,k\in\mathbb{Z}} g_{jk}$.

It is easy to see that $\|\varphi\| \le \|g\|^{o}_{W(L^{\Psi_1}(\mathbb{A}),L^{\Psi_2}(\mathbb{A}))}$ by the first part of the proof.

Given $\varepsilon > 0$, for $j,k\in\mathbb{Z}$, choose $h_\varepsilon\chi_{B_{jk}} \in L^{\Phi_1}(B_{jk})$ such that $\|h_\varepsilon\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})} = 1$ and

$$\varphi_{jk}(h_\varepsilon\chi_{B_{jk}}) = |\varphi_{B_{jk}}(h_\varepsilon\chi_{B_{jk}})| > (1-\varepsilon)\|\varphi_{jk}\| = (1-\varepsilon)\|g_{jk}\|^{\circ}_{L^{\Psi_1}(\mathbb{A})}. \tag{4.1}$$

For an arbitrary $(\alpha_{jk})_{j,k\in\mathbb{Z}} \in \ell^{\Phi_2}$, define $f = |\alpha_{jk}|h_\varepsilon\chi_{B_{jk}}$. By using Proposition 3.2, we have

$$\begin{aligned}\|f\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} &\approx \|(\|f\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})})_{j,k\in\mathbb{Z}}\|_{\ell^{\Phi_2}}\\ &= \|(|\alpha_{jk}|\|h_\varepsilon\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})})_{j,k\in\mathbb{Z}}\|_{\ell^{\Phi_2}} = \|\alpha_{jk}\|_{\ell^{\Phi_2}}.\end{aligned}$$

This implies that $f \in W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))$.

On the other hand, by inequality (4.1), we obtain

$$\begin{aligned}\|\varphi\|\|\alpha_{jk}\|_{\ell^{\Phi_2}} &= \|\varphi\|\|f\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} \ge |\varphi(f)|\\ &= \Big|\sum_{j,k\in\mathbb{Z}}\int_{B_{jk}} |\alpha_{jk}|h_\varepsilon\chi_{B_{jk}}g\,d\mu\Big|\\ &= \sum_{j,k\in\mathbb{Z}}|\alpha_{jk}|\varphi_{B_{jk}}(h_\varepsilon\chi_{B_{jk}}) \ge (1-\varepsilon)\sum_{j,k\in\mathbb{Z}}|\alpha_{jk}|\|g\chi_{B_{jk}}\|^{\circ}_{L^{\Psi_1}(\mathbb{A})}.\end{aligned}$$

Since $(\alpha_{jk})_{j,k\in\mathbb{Z}}$ is arbitrary in ℓ^{Φ_2}, it follows that $(1-\varepsilon)\|g\chi_{B_{jk}}\|^{\circ}_{L^{\Psi_1}(\mathbb{A})}$ is in ℓ^{Ψ_2} and has norm not bigger than $\|\varphi\|$. Since $\varepsilon > 0$ is arbitrarily small, we obtain $\|\varphi\| \ge \|g\|^{o}_{W(L^{\Psi_1}(\mathbb{A}),L^{\Psi_2}(\mathbb{A}))}$. □

Corollary 4.2. *Let $(\Phi_1,\Psi_1),(\Phi_2,\Psi_2)$ be complementary Young pairs. If $\Phi_i,\Psi_i, i=1,2$ satisfy the Δ_2 condition, then $W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))$ is reflexive.*

Note that if we take the Young functions $\Phi_1(x) = x^p$ and $\Phi_2(x) = x^q$ for $1 \le p,q < \infty$ in Theorem 4.1, then we obtain the following result [11].

Corollary 4.3. *Let $1 \le p,q < \infty$ and let p',q' be the respective dual exponents. Then, the dual space of $W(L^p(\mathbb{A}),L^q(\mathbb{A}))$ is the space $W(L^{p'}(\mathbb{A}),L^{q'}(\mathbb{A}))$.*

5. Convolution

Feichtinger in [7, Theorem 3], gave an interesting convolution relation between Wiener amalgams over an IN group.

That is, if $B_1 * B_2 \subseteq B_3$ and $C_1 * C_2 \subseteq C_3$, then we have $W(B_1, C_1) * W(B_2, C_2) \subseteq W(B_3, C_3)$. However, there are other useful convolution relations that hold for amalgam spaces on non-IN groups; see, e.g., [13].

In this section, we give a condition on Young functions Φ_1 and Φ_2 for the Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ on the affine group $\mathbb{A}$ to be a left $L^1(\mathbb{A})$-module with respect to convolution.

Let us recall that the following proposition will plays an important role [1].

Proposition 5.1. *There exists a compact neighborhood $Q = Q^{-1}$ of identity in $\mathbb{A}$ and there exist points $(a_n, b_n) \in \mathbb{A}$, $n \in \mathbb{N}$, such that the following hold.*

(a) *If $g \in W(L^{\Phi}(\mathbb{A}), L^1(\mathbb{A}))$, then we find functions g_n belonging to $L^{\Phi}(\mathbb{A})$ with* $\operatorname{supp}(g_n) \subseteq Q$ *such that $g = \sum_{n\in\mathbb{N}} L_{(a_n,b_n)} g_n$.*

(b) *The following is equivalent norm on $W(L^{\Phi}(\mathbb{A}), L^1(\mathbb{A}))$:*

$$\|g\|_{W(L^{\Phi}(\mathbb{A}),\ L^1(\mathbb{A}))} \approx \|(\|g_n\|_{L^{\Phi}(\mathbb{A})})_{n\in\mathbb{N}}\|_{\ell^1}.$$

Let us now give the following convolution theorem on the affine group $\mathbb{A}$.

Theorem 5.2. *Let Φ_1, Φ_2 be Young functions with $(\Phi_1')_+(0) > 0$ and $(\Phi_2')_+(0) > 0$. Then, $W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi_2}(\mathbb{A}))$ is a left $L^1(\mathbb{A})$-module with respect to convolution.*

Proof. Let $f \in L^1(\mathbb{A})$ and $g \in W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi_2}(\mathbb{A}))$. Let Q, (a_n, b_n) and g_n be as given by Proposition 5.1.

For $(z, w) \in (x, y)Q$, we know that if $g_n((u, v)^{-1}(z, w)) \neq 0$, then $(u, v) \in (x, y)Q^2$ [1, Theorem 5.6]. Hence

$$g_n((u, v)^{-1}(z, w))\chi_{(x,y)Q}(z, w) = L_{(u,v)}g_n(z, w)\chi_{(x,y)Q^2}(u, v). \tag{5.1}$$

Moreover, by Lemma 2.1, we have $L_{(a_nb_n)}g_n \in L^{\Phi_1}(\mathbb{A})$. Since $L^{\Phi_1}(\mathbb{A})$ is a left $L^1(\mathbb{A})$-module, we obtain $f * L_{(a_nb_n)}g_n \in L^{\Phi_1}(\mathbb{A})$ for all $f \in L^1(\mathbb{A})$.

By using Lemma 2.3,

$$\begin{aligned} F^Q_{f*L_{(a_nb_n)}g_n}(x, y) &= \|(f * L_{(a_nb_n)}g_n)\chi_{(x,y)Q}\|_{L^{\Phi_1}(\mathbb{A})} \\ &= \|(A_{(a_nb_n)}f * g_n)\chi_{(x,y)Q}\|_{L^{\Phi_1}(\mathbb{A})}. \end{aligned} \tag{5.2}$$

Hence, for every $\varepsilon > 0$, there exists a function $h_\varepsilon \in B_{\Psi_1}[0, 1]$ such that

$$\begin{aligned} &\|(A_{(a_nb_n)}f * g_n)\chi_{(x,y)Q}\|_{L^{\Phi_1}(\mathbb{A})} - \varepsilon \\ &\qquad < \int_{\mathbb{A}} |(A_{(a_nb_n)}f * g_n)(z, w)\chi_{(x,y)Q}(z, w)h_\varepsilon(z, w)|\frac{dz}{z}dw. \end{aligned} \tag{5.3}$$

By using (5.1) and Fubini's Theorem, we obtain

$$\begin{aligned} &\int_{\mathbb{A}} |(A_{(a_nb_n)}f * g_n)(z, w)\chi_{(x,y)Q}(z, w)h_\varepsilon(z, w)|\frac{dz}{z}dw \\ &\leq \int_{\mathbb{A}} \left(\int_{\mathbb{A}} |A_{(a_nb_n)}f(u, v)g_n((u, v)^{-1}(z, w))|\frac{du}{u}dv\right) |\chi_{(x,y)Q}(z, w)h_\varepsilon(z, w)|\frac{dz}{z}dw \\ &= \int_{\mathbb{A}} |A_{(a_nb_n)}f(u, v)| \left(\int_{\mathbb{A}} |L_{(u,v)}g_n(z, w)\chi_{(x,y)Q^2}(u, v)h_\varepsilon(z, w)|\frac{dz}{z}dw\right) \frac{du}{u}dv \\ &\leq \|L_{(u,v)}g_n\|_{L^{\Phi_1}(\mathbb{A})}\ \|(A_{(a_nb_n)}f)\chi_{(x,y)Q^2}\|_{L^1(\mathbb{A})}. \end{aligned}$$

By (5.3) and Lemma 2.1 (a), for every $\varepsilon > 0$, we have

$$\|(A_{(a_nb_n)}f * g_n)\chi_{(x,y)Q}\|_{L^{\Phi_1}(\mathbb{A})} - \varepsilon < \|g_n\|_{L^{\Phi_1}(\mathbb{A})}\ \|(A_{(a_nb_n)}f)\chi_{(x,y)Q^2}\|_{L^1(\mathbb{A})}.$$

From (5.2), we find

$$F^{Q}_{f*L_{(a_nb_n)}g_n}(x,y) \le \|g_n\|_{L^{\Phi_1}(\mathbb{A})} F^{Q^2}_{A_{(a_nb_n)}f}(x,y).$$

By the solidity of $L^1(\mathbb{A})$,

$$\|F^{Q}_{f*L_{(a_nb_n)}g_n}\|_{L^1(\mathbb{A})} \le \|g_n\|_{L^{\Phi_1}(\mathbb{A})}\, \|F^{Q^2}_{A_{(a_nb_n)}f}\|_{L^1(\mathbb{A})}.$$

By the definition of the amalgam space, the fact that $W(L^1(\mathbb{A}),\ L^1(\mathbb{A})) = L^1(\mathbb{A})$ and [13, Lemma 2.1], we obtain

$$\|f * L_{(a_nb_n)}g_n\|_{W(L^{\Phi_1}(\mathbb{A}),\ L^1(\mathbb{A}))} \le \|g_n\|_{L^{\Phi_1}(\mathbb{A})}\|f\|_{L^1(\mathbb{A})}. \tag{5.4}$$

On the other hand, by hypothesis on Φ_2, we have $L^{\Phi_2}(\mathbb{A}) \subseteq L^1(\mathbb{A})$. This implies that $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) \subseteq W(L^{\Phi_1}(\mathbb{A}), L^1(\mathbb{A}))$, so there exists a constant $K_1 > 0$ such that

$$\|f * L_{(a_nb_n)}g_n\|_{W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi_2}(\mathbb{A}))} \le K_1\|f * L_{(a_nb_n)}g_n\|_{W(L^{\Phi_1}(\mathbb{A}),\ L^1(\mathbb{A}))}. \tag{5.5}$$

Combining (5.4) and (5.5),

$$\|f * L_{(a_nb_n)}g_n\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} \le K_1\|g_n\|_{L^{\Phi_1}(\mathbb{A})}\, \|f\|_{L^1(\mathbb{A})}. \tag{5.6}$$

Now, by using Proposition 5.1 and (5.6), we have

$$\begin{aligned} \|f * g\|_{W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi_2}(\mathbb{A}))} &= \Big\|f * \sum_{n\in\mathbb{N}} L_{(a_n,b_n)}g_n\Big\|_{W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi_2}(\mathbb{A}))} \\ &\le K_1 \sum_{n\in\mathbb{N}} \|g_n\|_{L^{\Phi_1}(\mathbb{A})}\, \|f\|_{L^1(\mathbb{A})} \\ &\le K_1K_2\|f\|_{L^1(\mathbb{A})}\, \|g\|_{W(L^{\Phi_1}(\mathbb{A}),L^1(\mathbb{A}))} \end{aligned} \tag{5.7}$$

for some $K_2 > 0$. Then, we have $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) \subseteq W(L^{\Phi_1}(\mathbb{A}), L^1(\mathbb{A}))$. Hence there exists a constant $K_3 > 0$ such that from (5.7)

$$\|f * g\|_{W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi_2}(\mathbb{A}))} \le K\|f\|_{L^1(\mathbb{A})}\, \|g\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))},$$

where $K = K_1K_2K_3$. Thus, the Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ is a left $L^1(\mathbb{A})$-module. □

6. Some inclusion relations

In this section, we investigate inclusion properties among the Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ with respect to the local and global components.

We first show that inclusion relations on the local components affect the inclusion relations for the Orlicz amalgam spaces.

Let us note that if K is a compact subset of $\mathbb{A}$, then $\Phi_1 \prec \Phi_2$ implies that $L^{\Phi_2}(K) \subseteq L^{\Phi_1}(K)$.

Theorem 6.1. *Let Φ, Φ_1, Φ_2 be Young functions. If $\Phi_1 \prec \Phi_2$, then we have $W(L^{\Phi_2}(\mathbb{A}), L^{\Phi}(\mathbb{A})) \subseteq W(L^{\Phi_1}(\mathbb{A}), L^{\Phi}(\mathbb{A}))$.*

Proof. Let $f \in W(L^{\Phi_2}(\mathbb{A}),\ L^1(\mathbb{A}))$ and Q be a compact subset of $\mathbb{A}$ with nonempty interior and $(x, y) \in \mathbb{A}$. Then

$$\|f\|_{W(L^{\Phi_2}(\mathbb{A}),L^{\Phi}(\mathbb{A}))} = \|\|f\chi_{(x,y)Q}\|_{L^{\Phi_2}(\mathbb{A})}\|_{L^{\Phi}(\mathbb{A})} < \infty.$$

By hypothesis, we have $\|f\chi_{(x,y)Q}\|_{L^{\Phi_1}(\mathbb{A})} \leq K\|f\chi_{(x,y)Q}\|_{L^{\Phi_2}(\mathbb{A})}$. By using the solidity of $L^{\Phi}(\mathbb{A})$, we obtain

$$\begin{aligned}\|f\|_{W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi}(\mathbb{A}))} &= \|\|f\chi_{(x,y)Q}\|_{L^{\Phi_1}(\mathbb{A})}\|_{L^{\Phi}(\mathbb{A})}\\ &\leq K\|\|f\chi_{(x,y)Q}\|_{L^{\Phi_2}(\mathbb{A})}\|_{L^{\Phi}(\mathbb{A})}\\ &= K\|f\|_{W(L^{\Phi_2}(\mathbb{A}),L^{\Phi}(\mathbb{A}))}.\end{aligned}$$

Thus we have $f \in W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi}(\mathbb{A}))$. □

Taking $\Phi_2 \prec \Phi_1$ in Theorem 6.1, we have $W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi}(\mathbb{A})) \subseteq W(L^{\Phi_2}(\mathbb{A}),\ L^{\Phi}(\mathbb{A}))$. Hence, we conclude the following result.

Corollary 6.2. *Let Φ, Φ_1, Φ_2 be Young functions. If $\Phi_1 \asymp \Phi_2$, then we have $W(L^{\Phi_1}(\mathbb{A}),\ L^{\Phi}(\mathbb{A})) = W(L^{\Phi_2}(\mathbb{A}),\ L^{\Phi}(\mathbb{A}))$.*

Remark 6.3. The converse of Theorem 6.1 is not true in general.

Example 6.4. Let us take $\Phi_1(x) = \frac{x^{p_1}}{p_1}$ and $\Phi_2(x) = \frac{x^{p_2}}{p_2}$ and $\Phi(x) = \frac{x^q}{q}$ for $1 \leq p_1 \leq p_2 < \infty$, $1 \leq q < \infty$. Then the Orlicz spaces $L^{\Phi_1}(\mathbb{A})$ and $L^{\Phi_2}(\mathbb{A})$ become the Lebesgue spaces $L^{p_1}(\mathbb{A})$ and $L^{p_2}(\mathbb{A})$, respectively. We know that if $p_1 < p_2$, we have $W(L^{p_2}(\mathbb{A}), L^q(\mathbb{A})) \subseteq W(L^{p_1}(\mathbb{A}),\ L^q(\mathbb{A}))$.

However, the relation $\Phi_1 \prec \Phi_2$ does not hold. Assuming that $\Phi_1 \prec \Phi_2$, there exists a constant $K > 0$ such that $\Phi_1(x) \leq \Phi_2(Kx)$ for every $x \geq 0$. Thus

$$\frac{x^{p_1}}{p_1} \leq \frac{(Kx)^{p_2}}{p_2} \quad \Rightarrow \quad \frac{1}{x^{p_2-p_1}} \leq \frac{p_1 K^{p_2}}{p_2}.$$

Taking limits as $x \to 0$, we obtain

$$+\infty = \lim_{x\to 0} \frac{1}{x^{p_2-p_1}} \leq \lim_{x\to\infty} \frac{p_1 K^{p_2}}{p_2} < +\infty,$$

and this is a contradiction.

An equivalent discrete norm gives us an easier way of understanding the inclusion relation between the Orlicz amalgam spaces $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ based on the global component.

Theorem 6.5. *Let Φ, Φ_1, Φ_2 be Young functions. If $\Phi_1 \prec \Phi_2$, then we have $W(L^{\Phi}(\mathbb{A}),\ L^{\Phi_1}(\mathbb{A})) \subseteq W(L^{\Phi}(\mathbb{A}),\ L^{\Phi_2}(\mathbb{A}))$.*

Proof. Let $f \in W(L^{\Phi}(\mathbb{A}), L^{\Phi_1}(\mathbb{A}))$ and $\{B_{jk}\}_{j,k\in\mathbb{Z}}$ be given as in Lemma 3.1. By Proposition 3.2, we have

$$\|f\|_{W(L^{\Phi}(\mathbb{A}),\ L^{\Phi_1}(\mathbb{A}))} \approx \left\|\left(\|f\chi_{B_{jk}}\|_{L^{\Phi}(\mathbb{A})}\right)_{j,k\in\mathbb{Z}}\right\|_{\ell^{\Phi_1}} < \infty. \tag{6.1}$$

Since $\Phi_1 \prec \Phi_2$, we have $\ell^{\Phi_1} \subseteq \ell^{\Phi_2}$ [17]. Hence there exists a constant $K > 0$ such that

$$\|(\|f\chi_{B_{jk}}\|_{L^{\Phi}(\mathbb{A})})_{j,k\in\mathbb{Z}}\|_{\ell^{\Phi_2}} \leq K\left\|\left(\|f\chi_{B_{jk}}\|_{L^{\Phi}(\mathbb{A})}\right)_{j,k\in\mathbb{Z}}\right\|_{\ell^{\Phi_1}} \tag{6.2}$$

for all $(\|f\chi_{B_{jk}}\|_{L^{\Phi}(\mathbb{A})})_{j,k\in\mathbb{Z}} \in \ell^{\Phi_1}$. By using (6.1) and (6.2), we obtain

$$\begin{aligned}\|f\|_{W(L^{\Phi}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} &\approx \left\|\left(\|f\chi_{B_{jk}}\|_{L^{\Phi}(\mathbb{A})}\right)_{j,k\in\mathbb{Z}}\right\|_{\ell^{\Phi_2}}\\ &\leq K\left\|\left(\|f\chi_{B_{jk}}\|_{L^{\Phi}(\mathbb{A})}\right)_{j,k\in\mathbb{Z}}\right\|_{\ell^{\Phi_1}}\\ &\approx K\|f\|_{W(L^{\Phi}(\mathbb{A}),L^{\Phi_1}(\mathbb{A}))},\end{aligned}$$

which implies that $f \in W(L^{\Phi}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$. □

Taking $\Phi_2 \prec \Phi_1$ in Theorem 6.5, we have $W(L^{\Phi}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) \subseteq W(L^{\Phi}(\mathbb{A}), L^{\Phi_1}(\mathbb{A}))$. Thus we deduce the following result.

Corollary 6.6. *Let Φ, Φ_1, Φ_2 be Young functions. If $\Phi_1 \asymp \Phi_2$, then we have $W(L^{\Phi}(\mathbb{A}), L^{\Phi_1}(\mathbb{A})) = W(L^{\Phi}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$.*

Let us now give inclusion relation between the Orlicz amalgam space and its global and local components.

Theorem 6.7. *Let Φ_1, Φ_2 be Young functions with $(\Phi_i')_+(0) > 0$, $(\Psi_i')_+(0) > 0$ for $i = 1, 2$.*

(i) *If $\Phi_1 \prec \Phi_2$, then $L^{\Phi_1}(\mathbb{A}) \cup L^{\Phi_2}(\mathbb{A}) \subseteq W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$.*

(ii) *If $\Phi_2 \prec \Phi_1$, then $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) \subseteq L^{\Phi_1}(\mathbb{A}) \cap L^{\Phi_2}(\mathbb{A})$.*

Proof. (i) Let $f \in L^{\Phi_2}(\mathbb{A})$. By Theorem 2.5 and Theorem 6.1, we have

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} \leq K_1 \|f\|_{W(L^{\Phi_2}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} \leq K_2 \|f\|_{L^{\Phi_2}(\mathbb{A})} < \infty,$$

which implies that $L^{\Phi_2}(\mathbb{A}) \subseteq W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$. Similarly the inclusion $L^{\Phi_1}(\mathbb{A}) \subseteq W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ can be easily proved by using Theorem 2.5 and Theorem 6.5.

(ii) By Theorem 2.5 and Theorem 6.1, we have

$$\|f\|_{L^{\Phi_2}(\mathbb{A})} \leq K_1 \|f\|_{W(L^{\Phi_2}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} \leq K_2 \|f\|_{W(L^{\Phi_1}(\mathbb{A}),L^{\Phi_2}(\mathbb{A}))} < \infty,$$

which implies that $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) \subseteq L^{\Phi_2}(\mathbb{A})$. Similarly $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) \subseteq L^{\Phi_1}(\mathbb{A})$ is obtained by using Theorem 2.5 and Theorem 6.5. $\square$

Note that if we consider the Young functions $\Phi_1(x) = x^p$ and $\Phi_2(x) = x^q$ for $1 < p, q < \infty$ in the case $L^p(\mathbb{A}) \subseteq L^1(\mathbb{A})$ and $L^q(\mathbb{A}) \subseteq L^1(\mathbb{A})$, then we obtain the following corollary.

Corollary 6.8. *Let $1 < p, q < \infty$. Following hold:*

(i) *If $p \leq q$, then $L^p(\mathbb{A}) \cup L^q(\mathbb{A}) \subseteq W(L^p(\mathbb{A}), L^q(\mathbb{A}))$.*

(ii) *If $q \leq p$, then $W(L^p(\mathbb{A}), L^q(\mathbb{A})) \subseteq L^p(\mathbb{A}) \cap L^q(\mathbb{A})$.*

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