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Research Paper

Level Polynomials of Rooted Trees

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Abstract— Distance is the most important graph invariant and its history goes back to 1940s. Today total distance or Wiener index is widely studied in mathematics, computer science, statistics and related fields. Level index is also a distance based graph invariant which was introduced in 2017 for rooted trees. Level index is the numerator of the Gini index which is a statistical tool but Balaji and Mahmoud defined the graph theoretical applications of this index for statistical analysis of graphs. In this paper we define a new graph polynomial which is called level polynomial and calculate the level polynomial of some classes of trees. We obtain some interesting relations between the level polynomials and some integer sequences. Finally, we give an open problem at the end of the paper.

Keywords. *Level index, Level polynomial, Triangular Numbers, Subdivision of Stars, Dendrimers*

1. Introduction

The Gini index was defined by Gini (1912). It shows the income inequality of social groups and is used by The World Bank for the economical investigations. The graph theoretical application of Gini index was introduced by Balaji and Mahmoud (2017) for rooted trees. They introduced two distance based topological indices, Gini index and level index. Moreover, degree based Gini index was defined by Domicolo and Mahmoud (2019).

The first distance based topological index was introduced by Wiener (1947). Wiener showed that there is a correlation between the physico-chemical properties of molecules and distances between the atoms. Hosoya defined a distance counting polynomial (Hosoya, 1988) which is called Hosoya polynomial in the literature. The first derivative of Hosoya polynomial gives Wiener index and second derive gives the Wiener polarity index. Derivatives of Hosoya polynomial were used as molecular descriptors by Konstantinova and Diudea (2000), Estrada et al. (1998). Moreover vertex-weighted Wiener polynomials were studied by Došlić (2008).

The level concept was used in the papers (Flajolet and Prodinger, 1987) and (Tangora, 1991) for rooted trees. Flajolet and Prodinger (1987) obtained a number sequence and investigated properties of this sequence. Statistical analysis of level index was studied by Balaji and Mahmoud (2017). They adapted the original Gini index to graph theory and calculated the Gini index caterpillar graphs. Because caterpillar graphs represent the structural formula of some chemically important molecules and represent the distribution of organisms at a spine.

Since the distance is the most important graph invariant and it is widely studied in different sciences, we decided to introduce a new graph polynomial. In this paper we define a new distance based graph polynomial which is called “Level Polynomial”. The first derivative of level polynomial gives the level index of graphs. Moreover, we compute the level polynomial and level index of triangular numbers, caterpillar graphs, subdivisions of stars and regular dendrimer graphs. We obtain some interesting relations between the coefficients of level polynomials of graphs and some integer sequences.

It is known that average distance is also computed by Hosoya (Wiener) polynomials of graphs. Similar to other average graph measures, average measure can be defined as a graph invariant. At the end of the paper, we give the formula of the average level in terms of level polynomials of graphs. With respect to our observations average distance and average level are incomparable for rooted trees but equality is attained for only paths. It means that the relations between these average measures can be investigated.

2. Preliminaries

We use only simple, connected and undirected graphs. The degree of a vertex u is denoted by $\deg(u)$. A vertex with degree one is named a leaf. The notation $d(u,v)$ is used to show the distance between two any vertices u and v in a graph.

In a graph G, the number of vertices n is called order. The path and star graphs with n vertices are denoted by P_n and S_n, respectively.

Definition 2.1. The total distance from a vertex $u \in V(G)$ to other vertices is presented by the following phrase

$$D(u) = \sum_{v \in V(G)} d(u,v).$$

Definition 2.2. The Wiener index for a graph G is defined by the following equation (Wiener, 1947)

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} D(u).$$

Definition 2.3. The Hosoya (Wiener) polynomial of a graph G is denoted by $H(G,x)$ and it is computed by the following equation where $d(G,k)$ denotes the vertex pairs having distance k (Hosoya, 1988)

$$H(G,x) = \sum_{k \geq 1} d(G,k) x^k.$$

Theorem 2.4. The Hosoya polynomials of paths, stars, cycles with even order and odd order are presented as follows

$$i)\ H(P_n,x) = x^{n-1} + 2x^{n-2} + \cdots + (n-1)x$$

$$ii)\ H(S_n,x) = \binom{n-1}{2} x^2 + (n-1)x$$

$$iii)\ H(C_n,x) = n\left(x + x^2 + \cdots + x^{n/2-1}\right) + \frac{n}{2} x^{n/2}\ (n\ is\ even)$$

$$iv)\ H(C_n,x) = n\left(x + x^2 + \cdots + x^{(n-1)/2}\right)(n\ is\ odd).$$

Theorem 2.5. The Wiener indices of paths, stars and cycles are presented in the following equations

$$i)\ W(P_n) = \binom{n+1}{3} = \frac{(n+1)n(n-1)}{6}$$

$$ii)\ W(S_n) = (n-1)^2$$

$$iii)\ W(C_n) = \begin{cases} \dfrac{n^3}{8}, & n\ is\ even \\ \dfrac{n^3 - n}{8}, & n\ is\ odd \end{cases}$$

Definition 2.6. The Wiener index of a graph G is also computed by the following equation (Hosoya, 1988)

$$W(G) = \left(H(G,x)\right)'|_{x=1}$$

3. Level Index and Dendrimer Graphs

In a rooted tree, a vertex determined as a root or central vertex. The distance i from the central vertex is denoted by $D_i(T)$ (Balaji and Mahmoud, 2017). This distance (measured with edges) is called by level. The distance from the root to a vertex with the highest level is called height of the tree (Balaji and Mahmoud, 2017).

Balaji and Mahmoud introduced two distance based topological indices for rooted trees. The first one is called level index and level index of a tree is denoted by $L(T)$. Level index of a tree T is computed by the following equation

$$L(T) = \sum_{1 \le i < j \le n} \left| D_j(T) - D_i(T) \right|$$

such that $D_i(T)$ and $D_j(T)$ showing the vertices at distances i and j from the central vertex of the tree T.

In order to exemplify the level index, we use the example given in the paper (Balaji and Mahmoud, 2017).

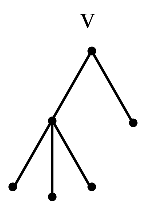

Figure 1. The tree T which is used in the following example

Example 3.1. The level index of T is computed by

$$L(T) = 1 + 1 + 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 = 14.$$

Now we can describe a level counting polynomial which is called level polynomial of the graphs.

Definition 3.2. The level polynomial of a rooted tree T is given by

$$L(T, x) = \sum_{k \ge 1} l(T, k) x^k$$

where $l(G, k)$ shows the number of vertex pairs having level difference k. It is understood that level index of a graph G equals to

$$L(T) = \left(L(T, x)\right)' |_{x=1}$$

Lemma 3.3. For a given dendrimer graph $T_{k,d}$ (depicted in Figure 2) with central vertex v, the following properties are hold (Şahin and Şener, 2020)

i) The order of $T_{k,d}$ is $1 + \frac{d[(d-1)^k - 1]}{d-2}$,

ii) $T_{k,d}$ contains d branches,

iii) Every branch of $T_{k,d}$ contains $\frac{(d-1)^k - 1}{d-2}$ vertices,

iv) Every branch of $T_{k,d}$ contains $(d-1)^{k-1}$ leaves,

v) Every branch of $T_{k,d}$ contains $\frac{(d-1)^{k-1} - 1}{d-2}$ non-leaf vertices,

vi) There are $d(d-1)^{k-1}$ vertices at distance k from v.

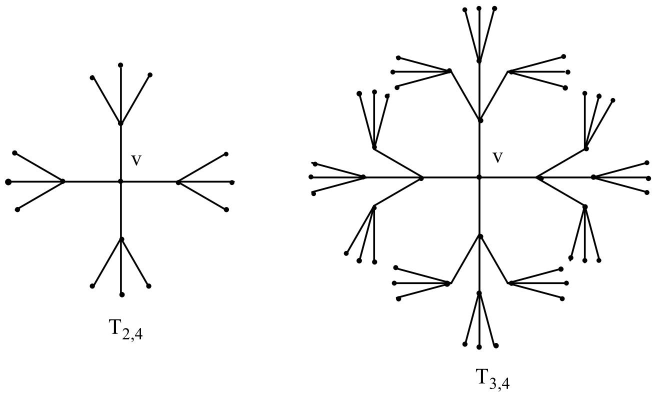

Figure 2. Dendrimers $T_{2,4}$ and $T_{3,4}$

4. Main Results

In this section we obtain the main results of the paper. We obtain the level polynomial of some classes of graphs. If a_i denotes the number of vertices on level i $(0 \leq i \leq n)$, we can show the level polynomials of rooted trees as in the following theorem. Even though there exists one vertex at first level $(a_0 = 1)$ in a rooted tree, the definition of level polynomial can be extended to other graphs and a_0 can take different values in the future.

Theorem 4.1. The level polynomial of a rooted tree T is obtained by the following equation such that the number of vertices on level i is denoted by a_i

$$L(T,x) = \sum_{j=1}^{n} \sum_{i=0}^{n-j} a_i a_{i+j} x^j$$

Proof. If the height of a rooted tree is showed by n**,** the exponents of x changes from 1 to n. Since the level polynomial of a rooted tree can be presented as

$$L(T,x) = b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1} + b_n a_n x^n.$$

The main problem is finding the coefficients of the level polynomial of T. Since a level has to be greater than 1, there is no constant term in the level polynomial.

The coefficient of x^n is a_n, because the vertex pairs which have level difference n are located on level 0 and level n. Similarly The coefficient of x^n is, $a_0 a_{n-1} + a_1 a_n$ because the vertex pairs which have level difference $n-1$ are located on levels $0, (n-1)$ and levels $1, n$.

By this way we obtain the coefficient of x^2 as $a_0 a_2 + a_1 a_3 + \cdots + a_{n-2} a_n$, because we want to obtain the number of vertices which have level difference 2.

Finally the coefficient of x is $a_0 a_1 + a_1 a_2 + \cdots + a_{n-1} a_n$. Because the vertices which have level difference 1 are located at consecutive levels. It means that the level polynomial of a rooted tree T is presented as follows

$$L(T,x) = (a_0 a_1 + a_1 a_2 + \cdots + a_{n-1} a_n)x + (a_0 a_2 + a_1 a_3 + \cdots + a_{n-2} a_n)x^2 + \cdots$$

$$+(a_0 a_{n-1} + a_1 a_n)x^{n-1} + a_0 a_n x^n$$

$$L(T,x) = \sum_{j=1}^{n} \sum_{i=0}^{n-j} a_i a_{i+j} x^j.$$

Remark 4.2. The level index of a rooted tree equals to following equation by Definition 3.2

$$L(T) = \left(L(T,x)\right)'|_{x=1} = \sum_{j=1}^{n} \sum_{i=0}^{n-j} j a_i a_{i+j}.$$

We can find the level polynomials of trees which represent the triangular numbers as in the following figure.

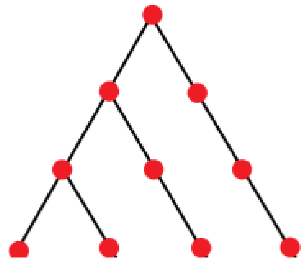

Figure 3. The tree S for $n = 4$

Let S be a tree which has $i + 1$ vertices on the level i (depicted in Figure 3 for $n = 4$). It means that there is a central vetrex, two vertices on first level, three vertices on the second level and n vertices on $(n-1)$-th level (triangular numbers). The sum of coefficient of level polynomial of S gives a new application of the integer sequence A000914 from OEIS (Sloane and Ploufe, 1995)

Theorem 4.3. Assume that S is defined above. Then its level polynomial is defined as follows

$$L(S,x) = \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} i(i+j)x^j$$

Proof. For a given sequence $i = 1,2,\dots,n$, the level partitions are defined as in the following phrases,

$$1 \times n \text{ (Level } n-1)$$

$$1 \times (n-1) + 2 \times n \text{ (Level } n-2)$$

$$1 \times (n-2) + 2 \times (n-1) + 3 \times n \text{ (Level } n-3)$$

$$\vdots$$

$$1 \times 3 + 2 \times 4 + \cdots + (n-3) \times (n-1) + (n-2) \times n \text{ (Level 2)}$$

$$1 \times 2 + 2 \times 3 + \cdots + (n-2) \times (n-1) + (n-1) \times n \text{ (Level 1).}$$

By these phrases for a given level $i = 1,2,\dots,n-1$, the coefficients are ordered. Then the level polynomial of S is presented as in the following equation

$$L(S,x) = \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} i(i+j)x^j.$$

Let S be as in the previous theorem. Now it is denoted the sum of coefficients of level polynomials by $\mathcal{L}(n)$ such that

$$\mathcal{L}(n) = \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} i(i+j)$$

Theorem 4.4. For a positive integer n, the number of $\mathcal{L}(n)$ is computed as in the following equation

$$\mathcal{L}(n)=\sum_{j=1}^{n-1}\sum_{i=1}^{n-j} i(i+j)=\sum_{i=1}^{n-1}\frac{i(n-i)(n+i+1)}{2}$$

Proof. In order to find the level number of a positive integer n, we use Therorem 4.3 in obtaining the sum of coefficients of Level polynomials of the tree S.

$$\mathcal{L}(n)=\sum_{j=1}^{n-1}\sum_{i=1}^{n-j} i(i+j)$$

$$=\sum_{j=1}^{n-1}\sum_{i=1}^{n-j} i^2+\sum_{j=1}^{n-1}\sum_{i=1}^{n-j} ij$$

$$=\sum_{i=1}^{n-1}(n-i)i^2+\sum_{i=1}^{n-1}\frac{i(n-i)(n-i+1)}{2}$$

$$=\sum_{i=1}^{n-1}\frac{i(n-i)(n+i+1)}{2}.$$

Now we obtain the initial terms of the sequence of $\mathcal{L}(n)$. For a positive integer n, there is a tree S which has $n-1$ levels and there are $i+1$ vertices on level i $(1\le i\le n-1)$. By this way initial terms are obtained as

$$\mathcal{L}(1)=0, \mathcal{L}(2)=2, \mathcal{L}(3)=11, \mathcal{L}(4)=35, \mathcal{L}(5)=85, \mathcal{L}(6)=175, \mathcal{L}(7)=322.$$

Theorem 4.5. The level index of the tree S is defined in the following equation

$$L(S)=\sum_{j=1}^{n-1}\sum_{i=1}^{n-j} ij(i+j)=2\sum_{i=1}^{n-1}\frac{i^2(n-i)(n-i+1)}{2}$$

Proof. In order to find the level index of a positive integer n, we use Remark 4.2 in obtaining the sum of coefficients of Level polynomials of the tree S.

$$L(S)=\big(L(S,x)\big)'|_{x=1}$$

Since the level polynomial of S is given in the Theorem 4.2

$$L(S,x)=\sum_{j=1}^{n-1}\sum_{i=1}^{n-j} i(i+j)x^j.$$

$$L(S)=\big(L(S,x)\big)'|_{x=1}=\sum_{j=1}^{n-1}\sum_{i=1}^{n-j} i^2 j+\sum_{j=1}^{n-1}\sum_{i=1}^{n-j} ij^2$$

$$=2\sum_{i=1}^{n-1}\frac{i^2(n-i)(n-i+1)}{2}.$$

If we obtain the initial terms of the sequence which is obtained in the Theorem 4.5

$$L(1)=0, L(2)=2, L(3)=14, L(4)=54, L(5)=154, L(6)=364.$$

This sequence is appeared in the OEIS with reference number A067056 (Sloane and Ploufe, 1995) for level index of greater than 1.

Theorem 4.6. Let T be a tree with level ℓ. Assume that T' is a tree which is obtained from T by attaching a new vertex u to k-th level of T. Then the difference between the level polynomials of T' and T is

$$L(T',x) - L(T,x) = \sum_{i=0}^{k-1} a_i x^{k-i} + \sum_{i=k+1}^{\ell} a_i x^{i-k}$$

Proof. Assume that a vertex u is attached the k-th level of T. Then difference between the level polynomials of T' and T is

$$L(T',x) - L(T,x) = a_0 x^k + a_1 x^{k-1} + \cdots + a_{k-1}x + a_{k+1}x + a_{k+2}x^2 + \cdots + a_\ell x^{\ell-k}$$

with the open form. We can write this equation by

$$L(T',x) - L(T,x) = \sum_{i=0}^{k-1} a_i x^{k-i} + \sum_{i=k+1}^{\ell} a_i x^{i-k}.$$

By the last equation, we can compute the difference of level indices of T' and T.

$$\begin{aligned} L(T') - L(T) &= (L(T',x) - L(T,x))'|_{x=1} \\ &= \sum_{i=0}^{k-1} a_i(k-i) + \sum_{i=k+1}^{\ell} a_i(i-k) \end{aligned}$$

Theorem 4.7. Let T be a rooted tree. Then the level polynomial of T equals to Hosoya polynomial of T if and only if $T = P_n$.

Proof. Since a path $P_n: v_1 v_2 \ldots v_n$ has one vertex at each level, there exists one vertex for each distance from v_1. Then the level polynomais of P_n equals to Hosoya polynomial of P_n as in the following equation

$$L(P_n,x) = H(P_n,x) = x^{n-1} + 2x^{n-2} + \cdots + (n-1)x.$$

Since the polynomials equal, we obtain that

$$L(P_n) = W(P_n) = \binom{n+1}{3} = \frac{(n+1)n(n-1)}{6}.$$

Now we assume that $T \neq P_n$. It means that $\ell \leq n-2$ and there are at least two vertices at a level. Let such a level be k-th level and two vertices u and v be two vertices at this level. Therefore, u and v are at the same level and the difference of level equals to zero but the distance between u and v is two. For the vertices which are located at the levels greater than k, distances from v equal to level difference plus two. Then the Hosoya polynomial of T is greater than Level polynomial of T as in the following equation.

$$\begin{aligned} H(T,x) - L(T,x) &= x^2 + x^3 + \cdots + x^{\ell-k+2} - x - x^2 - \cdots - x^{\ell-k} \\ &= x^{\ell-k+2} + x^{\ell-k+1} - x \end{aligned}$$

If the number of vertices which are the same level increases, then the difference $H(T,x) - L(T,x)$ also increases.

Theorem 4.8. The level polynomial of a star S_n of order n equals to following equation

$$L(S_n,x) = (n-1)x.$$

Proof. The star graph S_n is consisted of a root and $n-1$ leaves at distance one from the root. Then we obtain the level polynomial and level index of S_n as follows

$$L(S_n,x) = (n-1)x$$
$$L(S_n) = (n-1).$$

Let $C_h(1+X_1, 1+X_2, \ldots, 1+X_h)$ be a caterpillar graph which is defined in (Balaji and Mahmoud, 2017). $C_h(1+X_1, 1+X_2, \ldots, 1+X_h)$ is obtained from a path P_h $v_0 v_1 \ldots v_{h-1}$ by attaching leaves to vertices of paths as the leaves located at consecutive level. It means that v_0 is root, at level i for $1 \leq i \leq h-1$ there are $1+X_i$ vertices, and at the level h there are X_h leaves. To easify the notation we can write C_h instead of $C_h(1+X_1, 1+X_2, \ldots, 1+X_h)$. In the next theorem we give the level index of caterpillar graphs by the level polynomial of the caterpillar graphs.

Theorem 4.9. The level polynomial of a caterpillar graph C_h equals to

$$L(C_h, x) = \sum_{i=1}^{h-1} \left((X_i + 1) + \sum_{j=1}^{h-i-1} (X_j + 1)(X_{j+i} + 1) + (X_{h-i} + 1) X_h \right) x^i + X_h x^h$$

Proof. The distance h can be obtained between the root v_0 and

$$L(C_h, x) = X_h x^h + (X_{h-1} + 1 + (X_1 + 1) X_h) x^{h-1} + (X_{h-2} + 1 + (X_1 + 1)(X_{h-1} + 1) + (X_2 + 1) X_h) x^{h-2} + \cdots +$$
$$[X_1 + 1 + (X_1 + 1)(X_2 + 1) + (X_2 + 1)(X_3 + 1) + \cdots + (X_{h-2} + 1)(X_{h-1} + 1) + (X_{h-1} + 1) X_h] x$$

Then we can write the level polynomial of C_h caterpillar graph

$$L(C_h, x) = \sum_{i=1}^{h-1} \left((X_i + 1) + \sum_{j=1}^{h-i-1} (X_j + 1)(X_{j+i} + 1) + (X_{h-i} + 1) X_h \right) x^i + X_h x^h$$

We compute the level index of caterpillar C_h as in the following equation

$$\left(L(C_h, x)\right)'|_{x=1} = \sum_{i=1}^{h-1} i(X_i + 1) + \sum_{i=1}^{h-1} \sum_{j=1}^{h-i-1} i(X_j + 1)(X_{j+i} + 1) + \sum_{i=1}^{h-1} i(X_{h-i} + 1) X_h + hX_h.$$

Corollary 4.10. If it is taken $X_1 = X_2 = \cdots = X_h = X$, the level polynomial and level index of a caterpillar graph $C_h(1+X, 1+X, \ldots, 1+X)$ are given in the following equations

$$L(C_h(1+X, 1+X, \ldots, 1+X), x) = X x^h + (X+1)^2 \sum_{i=1}^{h-1} i \, x^{h-i}.$$

$$L\left(C_h(1+X, 1+X, \ldots, 1+X)\right) = hX + (X+1)^2 \sum_{i=1}^{h-1} i \, (h-i)$$

$$= hX + (X+1)^2 \binom{h+1}{3}$$

Corollary 4.11. If it is taken as $X = 1$, the following equations are obtained

$$L(C_h(2,2, \ldots, 2), x) = x^h + 4 \sum_{i=1}^{h-1} i \, x^{h-i}$$

$$L\left(C_h(2,2, \ldots, 2)\right) = h + 4 \sum_{i=1}^{h-1} i \, (h-i)$$

Corollary 4.12. If it is taken $X_1 = X_2 = \cdots = X_h = 0$, the level polynomial and level index of a caterpillar graph $C_h(1,1, \ldots, 1) = P_h$ are given in the following equations

$$L(C_h(1,1,\dots,1),x)=\sum_{i=1}^{h-1} i\,x^{h-i}.$$

$$L\big(C_h(1,1,\dots,1)\big)=\sum_{i=1}^{h-1} i\,(h-i)=\frac{(h+1)h(h-1)}{6}=\binom{h+1}{3}$$

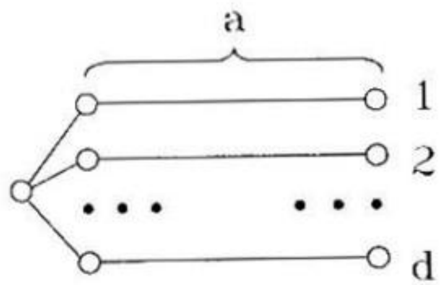

Figure 4. The tree H (Subdivisions of star graph)

Theorem 4.13. The level polynomial of tree H of order n is computed by the following equation

$$L(H,x)=\sum_{i=0}^{a-1}(id^2+d)\,x^{a-i}$$

Proof. The tree H is consisted a central vertex v and d paths P_a which are attached to v (see Figure 4). It means that $n=da+1$.

$$L(H,x)=dx^a+(d^2+d)x^{a-1}+(2d^2+d)x^{a-2}+\cdots+\big((a-1)d^2+d\big)x$$

$$L(H,x)=\sum_{i=0}^{a-1}(id^2+d)\,x^{a-i}$$

The level index of H can be computed from the first derivative of $L(H,x)$.

$$\big(L(H,x)\big)'=dax^{a-1}+(d^2+d)(a-1)x^{a-2}+\cdots+\big((a-1)d^2+d\big)$$

$$\big(L(H,x)\big)'|_{x=1}=da+(d^2+d)(a-1)+(2d^2+d)(a-2)\dots+\big((a-1)d^2+d\big)$$

By this equation we obtain the level index of H as in the following equation.

$$L(H)=\sum_{i=0}^{a-1}(id^2+d)\,(a-i)$$

$$=(ad^2-d)\sum_{i=0}^{a-1} i-d^2\sum_{i=0}^{a-1} i^2+da^2$$

$$=\frac{da(a+1)(da-d+3)}{6}$$

Theorem 4.14. The level polynomial of dendrimer (depicted in Figure 2) graph $T_{k,d}$ of order n is computed by the following equation

$$L\big(T_{k,d},x\big)=d(d-1)^{k-1}x^k+\sum_{i=1}^{k-1}\left(d(d-1)^{i-1}+d^2\sum_{j=0}^{k-i-1}(d-1)^{i+2j}\right)x^i$$

Proof. We use Theorem 4.1 for the level polynomial of dendrimer graph $T_{k,d}$

$$L(T_{k,d},x) = d(d-1)^{k-1}x^k + d(d-1)^{k-2}[1+d(d-1)]x^{k-1} +$$
$$\dots + d[1+d(d-1)+d(d-1)^3+\dots+d(d-1)^{2k-4}]x$$

If the previous equation is written in a closed form, we obtain that

$$L(T_{k,d},x) = d(d-1)^{k-1}x^k + \sum_{i=1}^{k-1}\left(d(d-1)^{i-1} + d^2\sum_{j=0}^{k-i-1}(d-1)^{i+2j}\right)x^i.$$

To compute the level index of dendrimer graph $T_{k,d}$, we can take the first derivative of level polynomial of $T_{k,d}$. Then we obtain that

$$L(T_{k,d}) = \left(L(T_{k,d},x)\right)'|_{x=1} = dk(d-1)^{k-1} + d\sum_{i=1}^{k-1} i(d-1)^{i-1} + d^2\sum_{i=1}^{k-1}\sum_{j=0}^{k-i-1} i(d-1)^{i+2j}$$

This equation can be restated as follows.

The first term and second term of $L(T_{k,d})$ are showed by the following equation.

$$d[1+2(d-1)+3(d-1)^2+\dots+k(d-1)^{k-1}] \quad (*)$$

The third term of the $L(T_{k,d})$ is restated by the following equations

$$d^2[(d-1)+(d-1)^3+\dots+(d-1)^{2k-3}] \text{ (for } i=1)$$
$$2d^2[(d-1)^2+(d-1)^4+\dots+(d-1)^{2k-4}] \text{ (for } i=2)$$
$$3d^2[(d-1)^3+(d-1)^5+\dots+(d-1)^{2k-5}] \text{ (for } i=3)$$
$$\vdots$$
$$(k-2)d^2[(d-1)^{k-2}+(d-1)^k] \text{ (for } i=k-2)$$
$$(k-1)d^2(d-1)^{k-1} \text{(for } i=k-1)$$

We can take $x = d-1$ for easy writing of the equations. By this way we obtain the equation $(*)$ as in the short equation

$$d[1+2(d-1)+3(d-1)^2+\dots+k(d-1)^{k-1}]$$
$$= (x+1)(1+2x+3x^2+\dots+kx^{k-1})$$
$$= (x+1)\times\frac{kx^{k+1}-(k+1)x^k+1}{(x-1)^2}$$

The third term of the $L(T_{k,d})$ can be written as follows

$$(x+1)^2[x+2x^2+(1+3)x^3+(2+4)x^4+(1+3+5)x^5+\dots+(1+3)x^{2k-5}+2x^{2k-4}+x^{2k-3}]$$
$$= (x+1)^2P(x)$$

such that

$$P(x) = \sum_{n=1}^{2k-3} a_n x^n = \sum_{n=1}^{k-1}\left\lfloor\frac{(n+1)^2}{4}\right\rfloor x^n + \sum_{n=1}^{k-2}\left\lfloor\frac{(n+1)^2}{4}\right\rfloor x^{2k-2-n}$$

It follows from the fact that the coefficients of $P(x)$ are symmetric around a_{k-1}, $a_n = a_{2k-2-n}$ and from the fact that the coefficients of odd powers sum to squares and of even powers to twice the triangular numbers.

The first ten coefficients of $P(x)$ are 1, 2, 4, 6, 9, 12, 16, 20, 25, 30 which are the first terms of an interesting integer sequence which is appeared in OEIS by reference number A002620 (Sloane and Ploufe, 1995).

Finally the level index of the dendrimer graph $T_{k,d}$ equals to

$$L(T_{k,d}) = (x+1) \times \frac{kx^{k+1} - (k+1)x^k + 1}{(x-1)^2} + (x+1)^2 \left(\sum_{n=1}^{k-1} \left\lfloor \frac{(n+1)^2}{4} \right\rfloor x^n + \sum_{n=1}^{k-2} \left\lfloor \frac{(n+1)^2}{4} \right\rfloor x^{2k-2-n} \right)$$

such that $x = d - 1$.

5. Conclusion

In this paper, we define a new graph polynomial which is based on the level index of rooted trees. The level index was defined by Balaji and Mahmoud for statistical analysis of graphs. It is used to measure balancing of rooted trees.

We show that level index can be calculated by level polynomials of graphs. We obtain the level polynomial and level index of trees which represent the triangular numbers. The sum of coefficients of level polynomials and level index of triangular numbers correspond some integer sequences appeared in OEIS (Sloane and Ploufe, 1995). Moreover, we compute the level polynomial and level index of caterpillar graphs, subdivision of star graphs and dendrimer graphs.

It is clear that level polynomial concept can be applied to rooted trees which represent the square numbers, pentagonal numbers, hexagonal numbers and others. We know that Pascal triangle can be represented by a perfect binary tree. Then, level polynomials can be applied to many integer objects. We finished our paper with an open problem. It is known that average distance or mean distance is a well-studied graph invariant. Similar to the average distance, we can define the average level as in the following equation

$$avl = \frac{\left(L(T,x)\right)'|_{x=1}}{L(T,x)|_{x=1}}$$

The average level can be studied for rooted trees and the relations between the average level and average distance can be investigated.

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