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On Generalized Hybrid Oresme Numbers with Hybrid *k*-Oresme Coefficients

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Abstract

In current paper, we have defined a generalization of Oresme numbers with hybrid k- Oresme coefficients. We obtained the generating function of these newly defined numbers. Using generating function, we obtained some fundamental and important identities containing the elements of the newly defined number sequence. We also gave some special sum formulas of this sequence we are considering.

Keywords: *Hybrid numbers; Recurrence relations; k- Oresme numbers.*
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1. Introduction

We begin this section by summarizing some basic properties of the Horadam sequence W_n, $W_n = (a,b;p,q)$ and the Oresme sequence. The Horadam sequence is defined as follows [8].

$$W_{n+2} = pW_{n+1} - qW_n, W_0 = a, W_1 = b. \tag{1.1}$$

The characteristic equation of the sequence in equation (1.1) is

$$x^2 - px + q = 0. \tag{1.2}$$

Its characteristic roots are $\alpha = \frac{p+\sqrt{p^2-4pq}}{2}$ and $\beta = \frac{p-\sqrt{p^2-4pq}}{2}$. Thus, W_n can be expressed by

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $A = b - a\beta$ and $B = b - a\alpha$ [8]. As known, different integer sequences are obtained by changing the initial conditions. Different special number sequences are obtained according to different choices of W_0, W_1, p and q values, such as Fibonacci numbers $F_n = W_n(0,1;1,-1)$ and Pell numbers $P_n = W_n(0,1;2,-1)$. Nicole Oresme, expanded the values of p and q to be rational numbers and defined a new sequence of numbers as called Oresme [12]. Horadam studied the Oresme sequence by changing the initial conditions of equation such that (1.1) [9]:

$$W_n(0,\frac{1}{2};1,\frac{1}{4}) = O_n. \tag{1.3}$$

Using equation (1.1) one can write

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n, O_0 = 0, O_1 = \frac{1}{2}. \tag{1.4}$$

The closed formula for this sequence is

$$O_n = \frac{n}{2^n}. \tag{1.5}$$

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A generalization of these numbers was first made by Cook in 2004. For $k > 2$ and $n \geq 2$, this generalization is follows [1]:

$$O_{n+2}^{(k)} = O_{n+1}^{(k)} - \frac{1}{k^2} O_n^{(k)}, O_0 = 0, O_1 = \frac{1}{k}. \tag{1.6}$$

In the case $k = 2$, this sequence reduces to the classical Oresme sequence. For $k^2 - 4 > 0$, the implicit formula to the generalized $k-$ Oresme sequence $O_n^{(k)}$ is

$$O_n^{(k)} = \frac{1}{\sqrt{k^2-4}} \left[\left(\frac{k+\sqrt{k^2-4}}{2k} \right)^n - \left(\frac{k-\sqrt{k^2-4}}{2k} \right)^n \right]. \tag{1.7}$$

Hybrid numbers, first defined and studied by Ozdemir, are a set of numbers formed by the combination of complex, dual and hyperbolic numbers. Accordingly, any hybrid number can be written as follows [13].

$$\mathbb{K} = \{ a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a,b,c,d \in \mathbb{R} \}, \tag{1.8}$$

where, $\mathbf{i}^2 = -1$, $\varepsilon \neq 0$, $\mathbf{h}^2 = 1$ and $\mathbf{ih} = \textbf{-hi} = \varepsilon + \mathbf{i}$. The set $\mathbb{K}$ of hybrid numbers forms non-commutative ring with respect to the addition and multiplication operations. For any to hybrid numbers $z_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $z_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$, $z_1 = z_2$ if and only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$ and $d_1 = d_2$.
The product of basis elements is given in following table:

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h}$$

and

$$z_1 z_2 = (a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h})(a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}).$$

Here, addition is commutative and associative, with zero being the null element. By addition, the inverse element of z is $-z$. Therefore $(\mathbb{K}, +)$ is an Abelian group. The product of basis elements is given in the table below.

.	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	1	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$1+\mathbf{h}$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-(\varepsilon+\mathbf{i})$	ε	1

The characteristic of this number, where the conjugate of the hybrid number is $\bar{z}$, is given in the equation below.

$$\mathscr{C}(z) = z\bar{z} = \bar{z}z = a^2 + (b-c)^2 - c^2 - d^2.$$

This characteristic value is frequently used to find the generalized norm of hybrid numbers. For more detailed information, (see [13]). Szynal defined the *nth* Horadam numbers as follows: [16].

$$\mathscr{H}_n = W_n + W_{n+1}\mathbf{i} + W_{n+2}\varepsilon + W_{n+3}\mathbf{h}. \tag{1.9}$$

Recently, hybrid numbers are discussed by many authors. Tan and Ait-Amrane, in [18] introduced the bi-periodic Horadam hybrid numbers which generalize the classical Horadam hybrid numbers. For $n \geq 0$, Szynal and Iwona defined Oresme hybrid numbers as:

$$OH_n = O_n + O_{n+1}\mathbf{i} + O_{n+2}\varepsilon + O_{n+3}\mathbf{h}. \tag{1.10}$$

These authors also showed that the following equations are correct [17]:

$$\mathscr{C}(OH_n) = \frac{15}{16} O_n^2 + \frac{14}{16} O_n O_{n+1} - \frac{25}{16} O_{n+1}^2, \tag{1.11}$$

$$OH_n = \frac{n}{2^n} + \frac{n+1}{2^{n+1}}\mathbf{i} + \frac{n+2}{2^{n+2}}\varepsilon + \frac{n+3}{2^{n+3}}\mathbf{h}, \tag{1.12}$$

$$OH_{n+r}OH_{n-r} - OH_n^2 = \frac{-65r^2}{4^{n+3}} + \frac{-4r^2+4r}{4^{n+1}}\mathbf{i} + \frac{-8r^2+3r}{4^{n+2}}\varepsilon + \frac{-r^2-r}{4^{n+1}}\mathbf{h}, \tag{1.13}$$

$$OH_n^{(k)} = O_n^{(k)} + O_{n+1}^{(k)}\mathbf{i} + O_{n+2}^{(k)}\varepsilon + O_{n+3}^{(k)}\mathbf{h}. \tag{1.14}$$

Halici and Sayin made a detailed study of k- Oresme hybrid numbers using equation (1.6) and (1.14). Also, the recurrence relation for $k-$ Oresme hybrid numbers is given by:

$$OH_{n+2}^{(k)} = OH_{n+1}^{(k)} - \frac{1}{k^2} OH_n^{(k)}, \tag{1.15}$$

where $OH_0^{(k)} = \frac{1}{k}\left(\mathbf{i} + \varepsilon + \frac{k^2-1}{k^2}\mathbf{h}\right)$ and $OH_1^{(k)} = \frac{1}{k}\left(1 + \mathbf{i} + \frac{k^2-1}{k^2}\varepsilon + \frac{k^2-2}{k^2}\mathbf{h}\right)$ [6].
The closed formula for the numbers $OH_n^{(k)}$ is given

$$OH_n^{(k)} = \frac{1}{\sqrt{k^2-4}} \left(\alpha^n \tilde{\alpha} - \beta^n \tilde{\beta} \right) \tag{1.16}$$

where $\tilde{\alpha} = 1+\alpha\mathbf{i}+\alpha^2\varepsilon+\alpha^3\mathbf{h}$ and $\tilde{\beta} = 1+\beta\mathbf{i}+\beta^2\varepsilon+\beta^3\mathbf{h}$ [6]. Here, $\tilde{\alpha}$ and $\tilde{\beta}$ are the roots of the Oresme numbers in equation (1.7). The author obtained some important identities and sum formulas for $k-$ Oresme hybrid numbers. In 2020, Dagdeviren defined and examine Fibonacci and Lucas numbers with hybrid coefficients [20]. For $n \geq 0$, Polatlı defined and examined, the *nth* term of hybrid numbers with Fibonacci and Lucas hybrid numbers coefficients as [14]:

$$\mathbb{F}_n = FH_n + FH_{n+1}\mathbf{i} + FH_{n+2}\varepsilon + FH_{n+3}\mathbf{h}, \tag{1.17}$$

$$\mathbb{L}_n = LH_n + LH_{n+1}\mathbf{i} + LH_{n+2}\varepsilon + LH_{n+3}\mathbf{h}. \tag{1.18}$$

There are also studies [11] and [15] on Fibonacci and Lucas hybrid numbers.
Additionally, different numbers such as hybrid numbers have been studied. Some of these are quaternions, bicomplex numbers and hyperbolic numbers. The difference between these numbers and hybrid numbers is the coefficients of consecutive terms. The equations of quaternion, bicomplex and hyperbolic numbers are given below respectively.

$$q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \;\; a_0, a_1, a_2, a_3 \in \mathbb{R}. \tag{1.19}$$

$$\mathbb{H} = \{q = a_0 + a_1 i + a_2 j + a_3 ij, \;\; a_0, a_1, a_2, a_3 \in \mathbb{R}\}. \tag{1.20}$$

$$z = x + hy, \;\; | \;\; h \notin \mathbb{R}, h^2 = 1, \;\; x, y \in \mathbb{R}. \tag{1.21}$$

There are also [2], [3], [4], [5], [7], [10] and [19] studies on these numbers.
In this study, we consider and examine $k-$ Oresme hybrid numbers with hybrid coefficients. Also, we derived a new generalization for Oresme numbers by examining existing studies in the literature.

2. On Generalized Hybrid Oresme Numbers with Hybrid *k*- Oresme Coefficients

In this section, we first defined hybrid *k*- Oresme numbers. Then, we examined some basic properties for the elements of *k*- Oresme number sequence and the equations they provide.

Definition 2.1. *For $n \geq 0$ and $k \geq 2$, we define hybrid k- Oresme numbers are defined by:*

$$\mathscr{HO}_n^{(k)} = OH_n^{(k)} + OH_{n+1}^{(k)}\mathbf{i} + OH_{n+2}^{(k)}\varepsilon + OH_{n+3}^{(k)}\mathbf{h}, \tag{2.1}$$

where $OH_n^{(k)} = O_n^{(k)} + O_{n+1}^{(k)}\mathbf{i} + O_{n+2}^{(k)}\varepsilon + O_{n+3}^{(k)}\mathbf{h}$.

In the following theorem, we give the identity will be used frequently in this study.

Theorem 2.2. *For $n \geq 1$ and $k \geq 2$, the recurrence relation of Hybrid k- Oresme numbers is*

$$\mathscr{HO}_{n+1}^{(k)} = \mathscr{HO}_n^{(k)} - \frac{1}{k^2}\mathscr{HO}_{n-1}^{(k)}, \tag{2.2}$$

where the initial conditions are $\mathscr{HO}_0^{(k)} = OH_0^{(k)} + OH_1^{(k)}\mathbf{i} + OH_2^{(k)}\varepsilon + OH_3^{(k)}\mathbf{h}$, $\mathscr{HO}_1^{(k)} = OH_1^{(k)} + OH_2^{(k)}\mathbf{i} + OH_3^{(k)}\varepsilon + OH_4^{(k)}\mathbf{h}$.

Proof. Using mathematical induction, it is easily seen that the equation is true for 1. If we accept it as true for n, the following equations can be written for $n+1$.

$$\mathscr{HO}_{n+2}^{(k)} = \mathscr{HO}_{n+1}^{(k)} - \frac{1}{k^2}\mathscr{HO}_n^{(k)},$$

$$\mathscr{HO}_{n+2}^{(k)} = \left(OH_{n+1}^{(k)} - \frac{1}{k^2}OH_n^{(k)}\right) + \left(OH_{n+2}^{(k)} - \frac{1}{k^2}OH_{n+1}^{(k)}\right)\mathbf{i} + \left(OH_{n+3}^{(k)} - \frac{1}{k^2}OH_{n+2}^{(k)}\right)\varepsilon + \left(OH_{n+4}^{(k)} - \frac{1}{k^2}OH_{n+3}^{(k)}\right)\mathbf{h}.$$

If the necessary adjustments and calculations are made in the last equation, the following equation is obtained:

$$\mathscr{HO}_{n+2}^{(k)} = OH_{n+2}^{(k)} + OH_{n+3}^{(k)}\mathbf{i} + OH_{n+4}^{(k)}\varepsilon + OH_{n+5}^{(k)}\mathbf{h},$$

which is desired result. Thus, the proof is completed. □

We give some terms of these numbers below.

$$\mathscr{HO}_0^{(k)} = \frac{(4k^4 - 6k^2 + 3) + 2k^4\varepsilon + 2k^2(k^2-1)\mathbf{h}}{k^5}, \mathscr{HO}_1^{(k)} = \frac{(3k^6 - 8k^4 + 6k^2 - 1) + 2k^6\mathbf{i} - 2k^4(1-k^2)\varepsilon + 2k^4(k^2-1)\mathbf{h}}{k^7}, \ldots$$

In the next Theorem, we give the closed formula for hybrid *k*- Oresme numbers.

Theorem 2.3. *For $n \geq 0$ and $k \geq 2$, we have*

$$\mathscr{HO}_n^{(k)} = \frac{1}{\sqrt{k^2-4}}\left[\alpha^n(\tilde{\alpha})^2 - \beta^n(\tilde{\beta})^2\right], \tag{2.3}$$

where $\tilde{\alpha} = 1+\alpha\mathbf{i}+\alpha^2\varepsilon+\alpha^3\mathbf{h}$, $\tilde{\beta} = 1+\beta\mathbf{i}+\beta^2\varepsilon+\beta^3\mathbf{h}$.

Proof. Let us write the closed formula of k- Oresme hybrid number $OH_n^{(k)}$ into equation (2.1).

$$\mathscr{H}\mathscr{O}_n^{(k)} = \frac{1}{\sqrt{k^2-4}}\left[\left(\alpha^n\tilde{\alpha}-\beta^n\tilde{\beta}\right)+\left(\alpha^{n+1}\tilde{\alpha}-\beta^{n+1}\tilde{\beta}\right)\mathbf{i}+\left(\alpha^{n+2}\tilde{\alpha}-\beta^{n+2}\tilde{\beta}\right)\varepsilon+\left(\alpha^{n+3}\tilde{\alpha}-\beta^{n+3}\tilde{\beta}\right)\mathbf{h}\right],$$

$$\mathscr{H}\mathscr{O}_n^{(k)} = \frac{1}{\sqrt{k^2-4}}\left[\alpha^n\tilde{\alpha}\left(1+\alpha\mathbf{i}+\alpha^2\varepsilon+\alpha^3 h\right)-\beta^n\tilde{\beta}\left(1+\beta i+\beta^2\varepsilon+\beta^3\mathbf{h}\right)\right],$$

$$\mathscr{H}\mathscr{O}_n^{(k)} = \frac{1}{\sqrt{k^2-4}}\left[\alpha^n(\tilde{\alpha})^2-\beta^n(\tilde{\beta})^2\right].$$

Thus, the desired result is obtained. □

We can give the following power series obtained using the elements of this sequence we have considered.

Theorem 2.4. *For $n \geq 1$ and $k \geq 2$, the generating function for k- Oresme numbers with hybrid coefficients is*

$$f(x) = \sum_{n=1}^{\infty}\mathscr{H}\mathscr{O}_n^{(k)}x^n = \frac{k^2(1-x)\mathscr{H}\mathscr{O}_0^{(k)}+(k^2x)\mathscr{H}\mathscr{O}_1^{(k)}}{x^2+k^2(1-x)}. \tag{2.4}$$

Proof. Let us calculate the functions $f(x)$, $-xf(x)$ and $\frac{x^2}{k^2}f(x)$:

$$f(x) = \mathscr{H}\mathscr{O}_0^{(k)}+\mathscr{H}\mathscr{O}_1^{(k)}x+\mathscr{H}\mathscr{O}_2^{(k)}x^2+\mathscr{H}\mathscr{O}_3^{(k)}x^3+\cdots$$

$$-xf(x) = -\mathscr{H}\mathscr{O}_0^{(k)}x-\mathscr{H}\mathscr{O}_1^{(k)}x^2-\mathscr{H}\mathscr{O}_2^{(k)}x^3-\mathscr{H}\mathscr{O}_3^{(k)}x^4+\cdots$$

$$\frac{x^2}{k^2}f(x) = \frac{x^2}{k^2}\mathscr{H}\mathscr{O}_0^{(k)}+\frac{x^3}{k^2}\mathscr{H}\mathscr{O}_1^{(k)}+\frac{x^4}{k^2}\mathscr{H}\mathscr{O}_2^{(k)}+\frac{x^5}{k^2}\mathscr{H}\mathscr{O}_3^{(k)}+\cdots,$$

Then, we get

$$f(x)\left(1-x+\frac{x^2}{k^2}\right) = \mathscr{H}\mathscr{O}_0^{(k)}+x\left(\mathscr{H}\mathscr{O}_1^{(k)}-\mathscr{H}\mathscr{O}_0^{(k)}\right)+x^2\left(\mathscr{H}\mathscr{O}_2^{(k)}-\mathscr{H}\mathscr{O}_1^{(k)}+\frac{1}{k^2}\mathscr{H}\mathscr{O}_0^{(k)}\right)+\cdots,$$

$$f(x)\left(1-x+\frac{x^2}{k^2}\right) = \mathscr{H}\mathscr{O}_0^{(k)}+x\left(\mathscr{H}\mathscr{O}_1^{(k)}-\mathscr{H}\mathscr{O}_0^{(k)}\right),$$

$$f(x) = \frac{(1-x)\mathscr{H}\mathscr{O}_0^{(k)}+x\mathscr{H}\mathscr{O}_1^{(k)}}{\left(1-x+\frac{x^2}{k^2}\right)}.$$

Thus, the desired result is obtained. □

We gave some important sum formulas in the rest of the study.

Theorem 2.5. *For $n \geq 0$, the following equation is satisfied:*

$$\sum_{n\geq 0}\mathscr{H}\mathscr{O}_n^{(k)}\frac{x^n}{n!} = \frac{(\tilde{\alpha})^2e^{\alpha x}-(\tilde{\beta})^2e^{\beta x}}{k(\alpha-\beta)}. \tag{2.5}$$

Proof. Let us write the closed formula for k- Oresme numbers with hybrid coefficients in the sum formula. Then, we have:

$$\sum_{n\geq 0}\mathscr{H}\mathscr{O}_n^{(k)}\frac{x^n}{n!} = \sum_{n\geq 0}\left(\frac{\alpha^n(\tilde{\alpha})^2-\beta^n(\tilde{\beta})^2}{\sqrt{k^2-4}}\right)\frac{x^n}{n!},$$

$$LHS = \frac{(\tilde{\alpha})^2}{\sqrt{k^2-4}}\sum_{n\geq 0}\frac{(\alpha x)^n}{n!}-\frac{\left(\tilde{\beta}\right)^2}{\sqrt{k^2-4}}\sum_{n\geq 0}\frac{(\beta x)^n}{n!},$$

$$LHS = \frac{1}{\sqrt{k^2-4}}\left((\tilde{\alpha})^2e^{\alpha x}-\left(\tilde{\beta}\right)^2e^{\beta x}\right).$$

When the equation $\alpha-\beta = \frac{\sqrt{k^2-4}}{k}$ is written,

$$\sum_{n\geq 0}\mathscr{H}\mathscr{O}_n^{(k)}\frac{x^n}{n!} = \frac{(\tilde{\alpha})^2e^{\alpha x}-(\tilde{\beta})^2e^{\beta x}}{k(\alpha-\beta)}$$

is obtained. Thus, the proof is completed. □

In the next theorem, we will give the sum formula.

Theorem 2.6. *For $n \geq 1$, the following sum formula is satisfied:*

$$\sum_{n\geq 1} \mathscr{H}\mathscr{O}_n^{(k)} = k^2 \left(\mathscr{H}\mathscr{O}_1^{(k)} - \mathscr{H}\mathscr{O}_{n+1}^{(k)} \right) - \left(\mathscr{H}\mathscr{O}_{-1}^{(k)} - \mathscr{H}\mathscr{O}_n^{(k)} \right). \tag{2.6}$$

Proof. Let us write the closed formula for $\mathscr{H}\mathscr{O}_n^{(k)}$ into the sum formula.

$$\sum_{n\geq 1} \mathscr{H}\mathscr{O}_n^{(k)} = \sum_{n\geq 1} \left(\frac{\alpha^n (\tilde{\alpha})^2 - \beta^n (\tilde{\beta})^2}{\sqrt{k^2-4}} \right),$$

$$LHS = \frac{1}{\sqrt{k^2-4}} \left[(\tilde{\alpha})^2 \sum_{n\geq 1} (\alpha)^n - (\tilde{\beta})^2 \sum_{n\geq 1} (\beta)^n \right],$$

$$LHS = \frac{1}{\sqrt{k^2-4}} \left[(\tilde{\alpha})^2 \left(\frac{1-\alpha^{n+1}}{1-\alpha} \right) - (\tilde{\beta})^2 \left(\frac{1-\beta^{n+1}}{1-\beta} \right) \right],$$

$$LHS = \frac{1}{\sqrt{k^2-4}} \left[\frac{(\tilde{\alpha})^2 \left(1-\alpha^{n+1}-\beta+\alpha^{n+1}\beta\right) - (\tilde{\beta})^2 \left(1-\beta^{n+1}-\alpha+\beta^{n+1}\alpha\right)}{1-\alpha-\beta+\alpha\beta} \right].$$

Here, let us write the equations $\alpha+\beta=1$, $\alpha\beta=\frac{1}{k^2}$ and $\alpha-\beta=\frac{\sqrt{k^2-4}}{k}$. It is also use the equations $\alpha=\frac{1}{k^2\beta}$ and $\beta=\frac{1}{k^2\alpha}$.

$$LHS \frac{k^2}{\sqrt{k^2-4}} \left[\left((\tilde{\alpha})^2 - (\tilde{\beta})^2 \right) - \left((\tilde{\alpha})^2\alpha^{n+1} - (\tilde{\beta})^2\beta^{n+1} \right) - \frac{1}{k^2} \left((\tilde{\alpha})^2\alpha^{-1} - (\tilde{\beta})^2\beta^{-1} \right) + \frac{1}{k^2} \left((\tilde{\alpha})^2\alpha^{n} - (\tilde{\beta})^2\beta^{n} \right) \right],$$

$$\sum_{n\geq 1} \mathscr{H}\mathscr{O}_n^{(k)} = k^2 \left(\mathscr{H}\mathscr{O}_1^{(k)} - \mathscr{H}\mathscr{O}_{n+1}^{(k)} \right) - \left(\mathscr{H}\mathscr{O}_{-1}^{(k)} - \mathscr{H}\mathscr{O}_n^{(k)} \right).$$

Thus, the desired result is obtained. □

In the next theorem, we give the sum of the even terms of k- Oresme numbers with hybrid coefficients.

Theorem 2.7. *For $n \geq 1$, the following equation is satisfied:*

$$\sum_{n\geq 1} \mathscr{H}\mathscr{O}_{2n}^{(k)} = k^2 \left(\mathscr{H}\mathscr{O}_n^{(k)} - \mathscr{H}\mathscr{O}_{2n+1}^{(k)} \right) - \left(\mathscr{H}\mathscr{O}_{-1}^{(k)} - \mathscr{H}\mathscr{O}_{2n}^{(k)} \right). \tag{2.7}$$

Proof. Let us write the Binet formula of hybrid$k$- Oresme numbers in the sum formula of the even terms.

$$\sum_{n\geq 1} \mathscr{H}\mathscr{O}_{2n}^{(k)} = \sum_{n\geq 1} \left(\frac{\alpha^{2n} (\tilde{\alpha})^2 - \beta^{2n} (\tilde{\beta})^2}{\sqrt{k^2-4}} \right),$$

$$LHS = \frac{1}{\sqrt{k^2-4}} \left[(\tilde{\alpha})^2 \sum_{n\geq 1} (\alpha)^{2n} - (\tilde{\beta})^2 \sum_{n\geq 1} (\beta)^{2n} \right],$$

$$LHS = \frac{1}{\sqrt{k^2-4}} \left[(\tilde{\alpha})^2 \left(\frac{1-\alpha^{2n+1}}{1-\alpha} \right) - (\tilde{\beta})^2 \left(\frac{1-\beta^{2n+1}}{1-\beta} \right) \right],$$

$$LHS = \frac{1}{\sqrt{k^2-4}} \left[\frac{(\tilde{\alpha})^2 \left(1-\alpha^{2n+1}-\beta+\alpha^{2n+1}\beta\right) - (\tilde{\beta})^2 \left(1-\beta^{2n+1}-\alpha+\beta^{2n+1}\alpha\right)}{\frac{1}{k^2}} \right],$$

$$LHS = \frac{k^2}{\sqrt{k^2-4}} \left[(\tilde{\alpha})^2 - \frac{(\tilde{\alpha})^2}{k^2\alpha} - (\tilde{\alpha})^2 \alpha^{2n+1} + \frac{(\tilde{\alpha})^2 \alpha^{2n}}{k^2} - \left(\tilde{\beta}\right)^2 + \frac{\left(\tilde{\beta}\right)^2}{k^2\beta} + \left(\tilde{\beta}\right)^2 \beta^{2n+1} - \frac{\left(\tilde{\beta}\right)^2 \beta^{2n}}{k^2} \right],$$

$$LHS = \frac{k^2}{\sqrt{k^2-4}} \left[\left((\tilde{\alpha})^2 - (\tilde{\beta})^2 \right) - \left((\tilde{\alpha})^2\alpha^{2n+1} - (\tilde{\beta})^2\beta^{2n+1} \right) - \frac{1}{k^2} \left((\tilde{\alpha})^2\alpha^{-1} - (\tilde{\beta})^2\beta^{-1} \right) + \frac{1}{k^2} \left((\tilde{\alpha})^2\alpha^{2n} - (\tilde{\beta})^2\beta^{2n} \right) \right],$$

$$LHS = k^2 \left(\mathscr{H}\mathscr{O}_n^{(k)} - \frac{1}{k^2} \mathscr{H}\mathscr{O}_{-1}^{(k)} - \mathscr{H}\mathscr{O}_{2n+1}^{(k)} + \frac{1}{k^2} \mathscr{H}\mathscr{O}_{2n}^{(k)} \right),$$

$$\sum_{n\geq 1} \mathscr{H}\mathscr{O}_{2n}^{(k)} = k^2 \left(\mathscr{H}\mathscr{O}_n^{(k)} - \mathscr{H}\mathscr{O}_{2n+1}^{(k)} \right) - \left(\mathscr{H}\mathscr{O}_{-1}^{(k)} - \mathscr{H}\mathscr{O}_{2n}^{(k)} \right).$$

Thus, the desired result is obtained. □

We have given the sums of the odd terms of k- Oresme numbers with hybrid coefficients in the equation below.

Theorem 2.8. *For $n>0$, we have*

$$\sum_{n\geq 1}\mathscr{H}\mathscr{O}_{2n+1}^{(k)}=k^{2}\left(\mathscr{H}\mathscr{O}_{1}^{(k)}-\mathscr{H}\mathscr{O}_{2n+2}^{(k)}\right)-\left(\mathscr{H}\mathscr{O}_{-1}^{(k)}-\mathscr{H}\mathscr{O}_{2n+1}^{(k)}\right). \tag{2.8}$$

Proof. Let us write the closed formula for k- Oresme numbers with hybrid coefficients odd terms.

$$\sum_{n\geq 1}\mathscr{H}\mathscr{O}_{2n+1}^{(k)}=\sum_{n\geq 1}\left(\frac{\alpha^{2n+1}(\tilde{\alpha})^{2}-\beta^{2n+1}(\tilde{\beta})^{2}}{\sqrt{k^{2}-4}}\right),$$

$$LHS=\frac{1}{\sqrt{k^{2}-4}}\left[(\tilde{\alpha})^{2}\sum_{n\geq 1}(\alpha)^{2n+1}-(\tilde{\beta})^{2}\sum_{n\geq 1}(\beta)^{2n+1}\right],$$

$$LHS=\frac{1}{\sqrt{k^{2}-4}}\left[(\tilde{\alpha})^{2}\left(\frac{1-\alpha^{2n+2}}{1-\alpha}\right)-(\tilde{\beta})^{2}\left(\frac{1-\beta^{2n+2}}{1-\beta}\right)\right],$$

$$LHS=\frac{k^{2}}{\sqrt{k^{2}-4}}\left[\left((\tilde{\alpha})^{2}-(\tilde{\beta})^{2}\right)-\left((\tilde{\alpha})^{2}\alpha^{2n+2}-(\tilde{\beta})^{2}\beta^{2n+2}\right)-\frac{1}{k^{2}}\left((\tilde{\alpha})^{2}\alpha^{-1}-(\tilde{\beta})^{2}\beta^{-1}\right)+\frac{1}{k^{2}}\left((\tilde{\alpha})^{2}\alpha^{2n+1}-(\tilde{\beta})^{2}\beta^{2n+1}\right)\right],$$

$$\sum_{n\geq 1}\mathscr{H}\mathscr{O}_{2n+1}^{(k)}=k^{2}\left(\mathscr{H}\mathscr{O}_{1}^{(k)}-\mathscr{H}\mathscr{O}_{2n+2}^{(k)}\right)-\left(\mathscr{H}\mathscr{O}_{-1}^{(k)}-\mathscr{H}\mathscr{O}_{2n+1}^{(k)}\right).$$

Thus, the desired result is obtained. □

We give the binomial sum in the lemma below.

Lemma 2.9. *For $x,y\in\mathbb{N}$, the following equality is satisfied:*

$$\sum_{y=1}^{x}\binom{x}{y}\mathscr{H}\mathscr{O}_{n}^{(k)}=\frac{1}{\sqrt{k^{2}-4}}\left[(\tilde{\alpha})^{2}(1+\alpha)^{n}-\left(\tilde{\beta}\right)^{2}(1+\beta)^{n}\right]. \tag{2.9}$$

Proof. Let us write the closed formula for k- Oresme numbers with hybrid coefficients in the sum formula.

$$\sum_{y=1}^{x}\binom{x}{y}\mathscr{H}\mathscr{O}_{n}^{(k)}=\sum_{y=1}^{x}\binom{x}{y}\left(\frac{(\tilde{\alpha})^{2}\alpha^{n}-\left(\tilde{\beta}\right)^{2}\beta^{n}}{\sqrt{k^{2}-4}}\right),$$

$$LHS=\frac{(\tilde{\alpha})^{2}}{\sqrt{k^{2}-4}}\sum_{y=1}^{x}\binom{x}{y}\alpha^{n}-\frac{\left(\tilde{\beta}\right)^{2}}{\sqrt{k^{2}-4}}\sum_{y=1}^{x}\binom{x}{y}\beta^{n}.$$

Let us use the binomial equation $\sum_{k=1}^{n}\binom{n}{k}x^{n}=(1+x)^{n}$. Then,

$$\sum_{y=1}^{x}\binom{x}{y}\mathscr{H}\mathscr{O}_{n}^{(k)}=\frac{1}{\sqrt{k^{2}-4}}\left[(\tilde{\alpha})^{2}(1+\alpha)^{n}-\left(\tilde{\beta}\right)^{2}(1+\beta)^{n}\right]$$

is obtained. Thus, the proof is completed. □

Now, we give important identities in the rest of this section. We show the Cassini identity in the theorem below.

Theorem 2.10. *For $n\geq 1$, we have*

$$\mathscr{H}\mathscr{O}_{n-1}^{(k)}\mathscr{H}\mathscr{O}_{n+1}^{(k)}-\left(\mathscr{H}\mathscr{O}_{n}^{(k)}\right)^{2}=\frac{k^{-2n}}{k^{2}-4}\left[\left(\tilde{\alpha}\tilde{\beta}\right)^{2}\left(2-(k\alpha)^{-2}\right)-\left(\tilde{\beta}\tilde{\alpha}\right)^{2}(k\alpha)^{2}\right]. \tag{2.10}$$

Proof. From the closed formula, we can write $\mathscr{H}\mathscr{O}_{n-1}^{(k)}=\frac{\alpha^{n-1}(\tilde{\alpha})^{2}-\beta^{n-1}(\tilde{\beta})^{2}}{\sqrt{k^{2}-4}}$, $\mathscr{H}\mathscr{O}_{n}^{(k)}=\frac{\alpha^{n}(\tilde{\alpha})^{2}-\beta^{n}(\tilde{\beta})^{2}}{\sqrt{k^{2}-4}}$ and $\mathscr{H}\mathscr{O}_{n+1}^{(k)}=\frac{\alpha^{n+1}(\tilde{\alpha})^{2}-\beta^{n+1}(\tilde{\beta})^{2}}{\sqrt{k^{2}-4}}$. If these values are substituted on the left side of equation (2.10), the following equation is obtained.

$$\mathscr{H}\mathscr{O}_{n-1}^{(k)}\mathscr{H}\mathscr{O}_{n+1}^{(k)}-\left(\mathscr{H}\mathscr{O}_{n}^{(k)}\right)^{2}=\frac{1}{k^{2}-4}\left[\left((\tilde{\alpha})^{2}\alpha^{n-1}-\left(\tilde{\beta}\right)^{2}\beta^{n-1}\right)\left((\tilde{\alpha})^{2}\alpha^{n+1}-\left(\tilde{\beta}\right)^{2}\beta^{n+1}\right)-\left((\tilde{\alpha})^{2}\alpha^{n}-\left(\tilde{\beta}\right)^{2}\beta^{n}\right)^{2}\right],$$

$$LHS=\frac{1}{k^{2}-4}\left[(\tilde{\alpha})^{4}\alpha^{2n}-\left(\tilde{\alpha}\tilde{\beta}\right)^{2}\alpha^{n-1}\beta^{n+1}-\left(\tilde{\beta}\tilde{\alpha}\right)^{2}\beta^{n-1}\alpha^{n+1}+\left(\tilde{\beta}\right)^{4}\beta^{2n}-\left((\tilde{\alpha})^{4}\alpha^{2n}+\left(\tilde{\beta}\right)^{4}\beta^{2n}-2\left(\tilde{\alpha}\tilde{\beta}\right)^{2}(\alpha\beta)^{n}\right)\right],$$

$$LHS=\frac{1}{k^2-4}\left[-(\tilde{\alpha}\beta)^2\alpha^{n-1}\beta^{n+1}-\left(\tilde{\beta}\tilde{\alpha}\right)^2\beta^{n-1}\alpha^{n+1}+2\left(\tilde{\alpha}\tilde{\beta}\right)^2(\alpha\beta)^n\right],$$

$$LHS=\frac{(\alpha\beta)^n}{k^2-4}\left[-\left(\tilde{\alpha}\tilde{\beta}\right)^2\frac{\beta}{\alpha}-\left(\tilde{\beta}\tilde{\alpha}\right)^2\frac{\alpha}{\beta}+2(\alpha\beta)^2\right],$$

$$LHS=\frac{k^{-2n}}{k^2-4}\left[\left(\tilde{\alpha}\tilde{\beta}\right)^2\left(2-\frac{\beta}{\alpha}\right)-\left(\tilde{\beta}\tilde{\alpha}\right)^2\frac{\alpha}{\beta}\right].$$

Let us write the equation $\beta=\frac{1}{k^2\alpha}$.

$$LHS=\frac{k^{-2n}}{k^2-4}\left[\left(\tilde{\alpha}\tilde{\beta}\right)^2\left(2-\frac{1}{k^2\alpha^2}\right)-\left(\tilde{\beta}\tilde{\alpha}\right)^2k^2\alpha^2\right],$$

$$\mathscr{HO}_{n-1}^{(k)}\mathscr{HO}_{n+1}^{(k)}-\left(\mathscr{HO}_n^{(k)}\right)^2=\frac{k^{-2n}}{k^2-4}\left[\left(\tilde{\alpha}\tilde{\beta}\right)^2\left(2-(k\alpha)^{-2}\right)-\left(\tilde{\beta}\tilde{\alpha}\right)^2(k\alpha)^2\right].$$

Thus, the desired result is obtained. □

In the next theorem, we give the Catalan identity.

Theorem 2.11. *For $n\geq r$ and $k\geq 2$, the following equation is true:*

$$\mathscr{HO}_{n-r}^{(k)}\mathscr{HO}_{n+r}^{(k)}-\left(\mathscr{HO}_n^{(k)}\right)^2=\frac{k^{-2n}}{k^2-4}\left[\left(\tilde{\alpha}\tilde{\beta}\right)^2\left(2-(k\alpha)^{2r}\right)-\left(\tilde{\beta}\tilde{\alpha}\right)^2(k\alpha)^{-2r}\right].\quad(2.11)$$

Proof. Let us use the closed formula of k- Oresme hybrid number $\mathscr{HO}_n^{(k)}$.

$$\mathscr{HO}_{n-r}^{(k)}\mathscr{HO}_{n+r}^{(k)}-\left(\mathscr{HO}_n^{(k)}\right)^2=\frac{1}{k^2-4}\left[\left((\tilde{\alpha})^2\alpha^{n-r}-\left(\tilde{\beta}\right)^2\beta^{n-r}\right)\left((\tilde{\alpha})^2\alpha^{n+r}-\left(\tilde{\beta}\right)^2\beta^{n+r}\right)-\left((\tilde{\alpha})^2\alpha^{n}-\left(\tilde{\beta}\right)^2\beta^{n}\right)^2\right],$$

$$LHS=\frac{1}{k^2-4}\left[-\left(\tilde{\alpha}\tilde{\beta}\right)^2(\alpha\beta)^n\left(\frac{\alpha}{\beta}\right)^r-\left(\tilde{\beta}\tilde{\alpha}\right)^2(\alpha\beta)^n\left(\frac{\beta}{\alpha}\right)^2+2\left(\tilde{\alpha}\tilde{\beta}\right)^2(\alpha\beta)^n\right],$$

$$LHS=\frac{k^{-2n}}{k^2-4}\left[\left(\tilde{\alpha}\tilde{\beta}\right)^2\left(2-\left(\frac{\alpha}{\beta}\right)^r\right)-\left(\tilde{\beta}\tilde{\alpha}\right)^2\left(\frac{\beta}{\alpha}\right)^r\right],$$

$$\mathscr{HO}_{n-r}^{(k)}\mathscr{HO}_{n+r}^{(k)}-\left(\mathscr{HO}_n^{(k)}\right)^2=\frac{k^{-2n}}{k^2-4}\left[\left(\tilde{\alpha}\tilde{\beta}\right)^2\left(2-(k\alpha)^{2r}\right)-\left(\tilde{\beta}\tilde{\alpha}\right)^2(k\alpha)^{-2r}\right].$$

Thus, the proof is completed. When $r=1$ is written in the last equation, it turns into Cassini identity. □

In the following Theorem, we give the d'Ocagne's identity.

Theorem 2.12. *For $n,m\in\mathbb{Z}^+$, we have*

$$\mathscr{HO}_{n+1}^{(k)}\mathscr{HO}_m^{(k)}-\mathscr{HO}_n^{(k)}\mathscr{HO}_{m-1}^{(k)}=-\frac{1}{k\sqrt{k^2-4}}\left[\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^n\beta^m-\left(\tilde{\beta}\tilde{\alpha}\right)^2\alpha^m\beta^n\right].\quad(2.12)$$

Proof. Let us calculate the left side of the equation.

$$\begin{aligned}\mathscr{HO}_{n+1}^{(k)}\mathscr{HO}_m^{(k)}-\mathscr{HO}_n^{(k)}\mathscr{HO}_{m-1}^{(k)}&=\frac{1}{k^2-4}\left[\left((\tilde{\alpha})^2\alpha^{n+1}-\left(\tilde{\beta}\right)^2\beta^{n+1}\right)\left((\tilde{\alpha})^2\alpha^{m}-\left(\tilde{\beta}\right)^2\beta^{m}\right)\right]\\&-\frac{1}{k^2-4}\left[\left((\tilde{\alpha})^2\alpha^{n}-\left(\tilde{\beta}\right)^2\beta^{n}\right)\left((\tilde{\alpha})^2\alpha^{m+1}-\left(\tilde{\beta}\right)^2\beta^{m+1}\right)\right],\end{aligned}$$

$$LHS=\frac{1}{k^2-4}\left[-\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^{n+1}\beta^m-\left(\tilde{\beta}\tilde{\alpha}\right)^2\beta^{n+1}\alpha^m+\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^n\beta^{m+1}+\left(\tilde{\beta}\tilde{\alpha}\right)^2\beta^n\alpha^{m+1}\right],$$

$$LHS=\frac{1}{k^2-4}\left[\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^n\beta^m(\beta-\alpha)-\left(\tilde{\beta}\tilde{\alpha}\right)^2\alpha^m\beta^n(\beta-\alpha)\right],$$

$$LHS=\frac{-(\alpha-\beta)}{k^2-4}\left[\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^n\beta^m-\left(\tilde{\beta}\tilde{\alpha}\right)^2\alpha^m\beta^n\right],$$

$$\mathscr{HO}_{n+1}^{(k)}\mathscr{HO}_m^{(k)}-\mathscr{HO}_n^{(k)}\mathscr{HO}_{m-1}^{(k)}=-\frac{1}{k\sqrt{k^2-4}}\left[\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^n\beta^m-\left(\tilde{\beta}\tilde{\alpha}\right)^2\alpha^m\beta^n\right].$$

Thus, the desired result is obtained. □

In the following theorem, we give the Honsberger's identity.

Theorem 2.13. *For $n,m \in \mathbb{Z}^{+}$, we have*

$$\mathscr{H}\mathscr{O}_{n-1}^{(k)}\mathscr{H}\mathscr{O}_{m}^{(k)}+\mathscr{H}\mathscr{O}_{n}^{(k)}\mathscr{H}\mathscr{O}_{m+1}^{(k)}=\frac{1}{k^2-4}\left[(\tilde{\alpha})^4\alpha^{n+m}+\left(\tilde{\beta}\right)^4\beta^{n+m}-\left(k^2+1\right)\left(\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^n\beta^{m+1}+\left(\tilde{\beta}\tilde{\alpha}\right)^2\alpha^{m+1}\beta^n\right)\right]. \quad (2.13)$$

Proof. Let us calculate the left side of the equation.

$$\mathscr{H}\mathscr{O}_{n-1}^{(k)}\mathscr{H}\mathscr{O}_{m}^{(k)}+\mathscr{H}\mathscr{O}_{n}^{(k)}\mathscr{H}\mathscr{O}_{m+1}^{(k)}=\frac{1}{k^2-4}\left[\left((\tilde{\alpha})^2\alpha^{n-1}-\left(\tilde{\beta}\right)^2\beta^{n-1}\right)\left((\tilde{\alpha})^2\alpha^{m}-\left(\tilde{\beta}\right)^2\beta^{m}\right)\right]$$
$$+\frac{1}{k^2-4}\left[\left((\tilde{\alpha})^2\alpha^{n}-\left(\tilde{\beta}\right)^2\beta^{n}\right)\left((\tilde{\alpha})^2\alpha^{m+1}-\left(\tilde{\beta}\right)^2\beta^{m+1}\right)\right],$$

$$LHS=\frac{1}{k^2-4}\left[(\tilde{\alpha})^4\alpha^{n+m-1}-\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^{n-1}\beta^{m}-\left(\tilde{\beta}\tilde{\alpha}\right)^2\beta^{n-1}\alpha^{m}+\left(\tilde{\beta}\right)^4\beta^{n+m-1}\right]$$
$$+\frac{1}{k^2-4}\left[(\tilde{\alpha})^4\alpha^{n+m+1}-\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^{n}\beta^{m+1}-\left(\tilde{\beta}\tilde{\alpha}\right)^2\beta^{n}\alpha^{m+1}+\left(\tilde{\beta}\right)^4\beta^{n+m+1}\right],$$

$$LHS=\frac{1}{k^2-4}\left[(\tilde{\alpha})^4\alpha^{n+m}\left(\alpha+\frac{1}{\alpha}\right)+\left(\tilde{\beta}\right)^4\beta^{n+m}\left(\beta+\frac{1}{\beta}\right)-\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^{n}\beta^{m}\left(\frac{1}{\alpha}+\beta\right)-\left(\tilde{\beta}\tilde{\alpha}\right)^2\alpha^{m}\beta^{n}\left(\frac{1}{\beta}+\alpha\right)\right].$$

Now, let us write the equations $\alpha+\frac{1}{\alpha}=1$, $\beta+\frac{1}{\beta}=1$, $\frac{1}{\alpha}=k^2\beta$ and $\frac{1}{\beta}=k^2\alpha$. Then, we get

$$LHS=\frac{1}{k^2-4}\left[(\tilde{\alpha})^4\alpha^{n+m}+\left(\tilde{\beta}\right)^4\beta^{n+m}-\left(k^2+1\right)\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^n\beta^{m+1}+\left(k^2+1\right)\left(\tilde{\beta}\tilde{\alpha}\right)^2\alpha^{m+1}\beta^n\right],$$

$$\mathscr{H}\mathscr{O}_{n-1}^{(k)}\mathscr{H}\mathscr{O}_{m}^{(k)}+\mathscr{H}\mathscr{O}_{n}^{(k)}\mathscr{H}\mathscr{O}_{m+1}^{(k)}=\frac{1}{k^2-4}\left[(\tilde{\alpha})^4\alpha^{n+m}+\left(\tilde{\beta}\right)^4\beta^{n+m}-\left(k^2+1\right)\left(\left(\tilde{\alpha}\tilde{\beta}\right)^2\alpha^n\beta^{m+1}+\left(\tilde{\beta}\tilde{\alpha}\right)^2\alpha^{m+1}\beta^n\right)\right].$$

Thus, the desired result is obtained. □

3. Conclusion

In this study, we defined Hybrid k- Oresme numbers with k- Oresme hybrid coefficients. We gave the closed formula and generating function for this new number sequence. Then, we obtained important identities. We gave some sum formulas for $\mathscr{H}\mathscr{O}_{n}^{(k)}$.

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