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RESEARCH ARTICLE

Warped Product Pointwise Quasi Bi-slant Submanifolds of Kaehler Manifolds

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ABSTRACT

In this paper, we have examined the idea of warped product pointwise quasi bi-slant submanifolds in Kaehler manifolds. We have shown that every warped product pointwise quasi bi-slant submanifold in a Kaehler manifold is either a Riemannian product or a warped product pointwise quasi hemi slant submanifold. Furthermore, we offer examples of both cases.

Keywords: *Quasi bi-slant submanifolds; pointwise bi-slant submanifolds; warped products.*
AMS Subject Classification (2020): *Secondary:53C15; 53C25; 53C40; 53C42.*

1. Introduction

In [7], Chen introduced the notion of slant submanifolds, the initial findings on slant submanifolds were collected in his book [8]. Numerous geometer groups continue to study and conduct research on this idea of submanifolds. Recently, the related literature of slant submanifolds has been compiled in the form of two books by Chen, Shahid and Solamy (see [17, 18]). Since the introduction of slant submanifolds, many generalizations and extensions of slant submanifolds have been introduced, like: semi-slant, pointwise slant, hemi-slant, pointwise hemi-slant and many more. The related literature of these kind of generalizations can be be found in (see, [13, 19, 20, 22, 24, 26]). A more generic class of submanifolds in the form of bi-slant submanifolds was introduced by Cabrerizo and Cariazo [6]. This class of submanifolds acts as a natural generalization of CR, semi-slant, slant, hemi-slant submanifolds [20, 22, 27]. Further the extended notion of pointwise bi-slant submanifolds of Kaehler manifolds can be found in [16]. Etayo [19] introduced the idea of pointwise slant submanifolds as an extension of slant submanifolds and gave them the label quasi-slant submanifolds. Prasad, Shukla, and Haseeb [28] recently proposed the notion of quasi hemi-slant submanifolds of Kaehler manifolds. This notion of quasi hemi-slant submanifolds was generalized by Prasad, Akyol, Verma, and Kumar [29] to a more generic class of submanifolds in the form of quasi bi-slant submanifolds of Kaehler manifolds. They established the prerequisites for the integrability of the distributions used in the definition of such submanifolds. More recently, Akyol, Beyendi, Fatima and Ali [1] introduced a new class of submanifolds known as pointwise quasi bi-slant submanifolds in almost Hermitian manifolds.

Bishop and O'Neill [4] in $1960s$ introduced the concept of warped product manifolds. These manifolds find their applications both in physics as well as in mathematics. Since then the study of warped product submanifolds has been investigated by many geometers (see, [2, 3, 11, 12, 14, 23, 25]). In particular, Chen started looking these warped products as submanifolds of different kinds of manifolds (see, [9, 10]). In this connection, in Kaehlerian settings, he proved besides CR- products the non-existence of warped products of the form $N^{\perp} \times_f N^T$, where $N^{\perp}, N^T$ is a totally real and holomorphic submanifold, respectively. Now from the past two decades, this research area has been actively pursued by numerous geometry groups. For the overall development of the subject we refer the readers to see Chen's book on it [15].

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Now while importing the survey of warped products to slant cases, Şahin in [31] proved the non-existence of semi-slant warped products in any Kaehler manifold. Then in [32] he extended the study to pointwise semi-slant warped products of Kaehlerian manifolds. Uddin, Chen and Solamy [33] studied warped product bi-slant submanifolds in Kaehler manifolds. In this paper we have studied the notion of warped product pointwise quasi bi-slant submanifolds in Kaehler manifolds, we proved that every warped product pointwise quasi bi-slant submanifold in a Kaehler manifold is either a Riemannian product or a warped product pointwise quasi hemi-slant submanifold. Moreover, we provide the examples of both the cases.

2. Preliminaries

Let $(\bar{N}, J, g)$ be an almost Hermitian manifold with an almost complex structure J and a Riemannian metric g such that

$$J^2 = -I, \qquad g(JU, JV) = g(U, V), \tag{2.1}$$

for any $U, V \in \Gamma(T\bar{N})$, where I is the identity map and $\Gamma(T\bar{N})$ denotes the set of all vector fields of $\bar{N}$, $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{N}$ with respect to the Riemannian metric g. If the almost complex structure J satifies

$$(\bar{\nabla}_U J)V = 0, \tag{2.2}$$

for any vector $U, V \in \Gamma(T\bar{N})$, then $\bar{N}$ is called a Kaehler manifold.
Let N be a Riemannian manifold isometrically immersed in $\bar{N}$ and we denote by the symbol g the Riemannian metric induced on N. Let $\Gamma(TN)$ denote the Lie algebra of vector fields in N and $\Gamma(T^\perp N)$, the set of all vector fields normal to N. If ∇ is the induced Levi-Civita connection on N, the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_U V = \nabla_U V + \sigma(U, V), \tag{2.3}$$

and

$$\bar{\nabla}_V M = -A_M U + \nabla^\perp_U M, \tag{2.4}$$

for any $U, V \in \Gamma(TN)$ and $M \in \Gamma(T^\perp N)$, where $\nabla^\perp$ is the normal connection on $T^\perp N$ and A is the shape operator. The shape operator A and the second fundamental form σ of N are related by

$$g(A_M U, V) = g(\sigma(U, V), M), \tag{2.5}$$

for any $U, V \in \Gamma(TN)$ and $M \in \Gamma(T^\perp N)$, and g denotes the induced metric on N as well as the metric on $\bar{N}$.
For a tangent vector field U, we can write

$$JU = \phi U + \omega U, \tag{2.6}$$

where ϕU and ωU are the tangential and normal components of JU on N respectively. Similarly for $M \in \Gamma(T^\perp M)$, we have

$$JM = BM + CM, \tag{2.7}$$

where BM and CM are tangential and normal components of JM on N respectively. Moreover, from (2.1), (2.6) and (2.7), we have

$$g(\phi U, V) = g(U, \phi V), \tag{2.8}$$

for any $U, V \in \Gamma(TN)$.
We can now specify the following classes of submanifolds of an almost Hermitian manifolds for later use:
(1) A submanifold N of an almost Hermitian manifold $\bar{N}$ is said to be slant (see [7]), if for each non-zero vector U tangent to N, the angle $\theta(U)$ between JU and T_pN is a constant, i.e., it does not depend on the choice of $p \in N$ and $U \in T_pN$. In this case, the angle θ is called the slant angle of the submanifold. A slant submanifold N is called proper slant submanifold if $\theta \neq 0, \frac{\pi}{2}$.

(2) A submanifold N of an almost Hermitian manifold $\bar{N}$ is said to be invariant(holomorphic or complex) submanifold (see [7]), if $J(T_pN) \subseteq T_p(N)$ for every point $p \in N$.

(3) A submanifold N of an almost Hermitian manifold $\bar{N}$ is said to be anti-invariant (totally real) submanifold (see [21]), if $J(T_pN) \subseteq T_p^{\perp}(N)$ for every point $p \in N$.

(4) A submanifold N of an almost Hermitian manifold $\bar{N}$ is said to be semi-invariant (see [5]), if there exist two orthogonal complementary distributions $\mathfrak{D}$ and $\mathfrak{D}^{\perp}$ on N such that

$$TN = \mathfrak{D} \oplus \mathfrak{D}^{\perp},$$

where $\mathfrak{D}$ is invariant and $\mathfrak{D}^{\perp}$ is anti-invariant.

(5) A submanifold N of an almost Hermitian manifold $\bar{N}$ is said to be semi-slant [27], if there exist two orthogonal complementary distributions $\mathfrak{D}$ and $\mathfrak{D}_{\theta}$ on N such that

$$TN = \mathfrak{D} \oplus \mathfrak{D}_{\theta},$$

where $\mathfrak{D}$ is invariant and $\mathfrak{D}_{\theta}$ is slant with slant angle θ. In this case, the angle θ is called semi-slant angle.

(6) A submanifold N of an almost Hermitian manifold $\bar{N}$ is said to be hemi-slant (see, [22, 26]), if there exist two orthogonal complementary distributions $\mathfrak{D}_{\theta}$ and $\mathfrak{D}^{\perp}$ on N such that

$$TN = \mathfrak{D}_{\theta} \oplus \mathfrak{D}^{\perp},$$

where $\mathfrak{D}_{\theta}$ is slant with slant angle θ and $\mathfrak{D}^{\perp}$ is anti-invariant. In this case, the angle θ is called hemi-slant angle.

Definition 2.1. Let N be a submanifold of an almost Hermitian manifold $\bar{N}$. Then, we say N is a pointwise bi-slant submanifold of $\bar{N}$ if there exists a pair of orthogonal distributions $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of N, at a point $p \in N$ such that
(a) $TN = \mathfrak{D}_1 \oplus \mathfrak{D}_2$;
(b) $J\mathfrak{D}_1 \perp \mathfrak{D}_2$ and $J\mathfrak{D}_2 \perp \mathfrak{D}_1$;
(c) The distributions $\mathfrak{D}_1, \mathfrak{D}_2$ are pointwise slant with slant functions θ_1, θ_2, respectively.

The pair $\{\theta_1, \theta_2\}$ of slant functions is called the bi-slant function. A pointwise bi-slant submanifold N is called proper if its bi-slant function satisfies $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$ and both θ_1, θ_2 are not constant on N.

3. Pointwise quasi bi-slant submanifolds of Kaehler manifolds

In this section, we study pointwise quasi bi-slant submanifolds of Kaehler manifolds.

Definition 3.1. A submanifold N of an almost Hermitian manifold $\bar{N}$ is called a quasi bi-slant submanifold if there exist distributions $\mathfrak{D}$, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ such that:
(a) TN admits the orthogonal direct decomposition as

$$TN = \mathfrak{D} \oplus \mathfrak{D}_1 \oplus \mathfrak{D}_2;$$

(b) $J(\mathfrak{D}) = \mathfrak{D}$ i.e., $\mathfrak{D}$ is invariant;
(c) $J(\mathfrak{D}_1) \perp \mathfrak{D}_2$ and $J(\mathfrak{D}_2) \perp \mathfrak{D}_1$;
(d) For any non-zero vector field $U \in (\mathfrak{D}_1)_u$; $u \in N$; the angle θ_1 between JU and $(\mathfrak{D}_1)_u$ is constant and independent of the choice of point u and U in $(\mathfrak{D}_1)_u$;
(e) For any non-zero vector field $Z \in (\mathfrak{D}_2)_v$; $v \in N$; the angle θ_2 between JZ and $(\mathfrak{D}_2)_v$ is constant and independent of the choice of point v and Z in $(\mathfrak{D}_2)_v$;
The angles θ_1 and θ_1 are called slant angles of quasi bi-slant submanifold.

Remark 3.1. We can generalize the above definition by taking $TN = \mathfrak{D} \oplus \mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2} ... \oplus \mathfrak{D}_{\theta_n}$. Hence we can define multi-slant submanifolds, quasi multi-slant submanifolds etc.

Definition 3.2. [1] Let N be an isometrically immersed submanifold in a Kaehler manifold $\bar{N}$. Then, we say that N is a pointwise quasi bi-slant submanifold if there exists three orthogonal distributions $(\mathfrak{D}, \mathfrak{D}_{\theta_1}, \mathfrak{D}_{\theta_2})$ satisfying the following conditions:
(i) $TN = \mathfrak{D} \oplus \mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2}$,
(ii) The distribution $\mathfrak{D}$ is invariant, i.e. $J\mathfrak{D} = \mathfrak{D}$,

(iii) $J\mathfrak{D}_{\theta_1} \perp \mathfrak{D}_{\theta_2}$ and $J\mathfrak{D}_{\theta_2} \perp \mathfrak{D}_{\theta_1}$,
(iv) The distributions $\mathfrak{D}_{\theta_1}$, $\mathfrak{D}_{\theta_2}$ are pointwise slant distributions with slant functions θ_1, θ_2, respectively.
The pair $\{\theta_1, \theta_2\}$ of slant functions is called the bi-slant functions. A pointwise quasi bi-slant submanifold N is called proper if its bi-slant function satisfies $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$, and both θ_1, θ_2 are not constant on N.

Let N be a pointwise quasi bi-slant submanifold of an almost Hermitian manifold $\bar{N}$. We denote the projections of $U \in \Gamma(TN)$ on the distributions $\mathfrak{D}$, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ by P, Q and R, respectively. Then we can write, for any $U \in \Gamma(TN)$

$$U = PU + QU + RU, \tag{3.1}$$

we can write

$$JU = \phi U + \omega U, \tag{3.2}$$

where ϕU and ωU are tangential and normal components of JU on N, respectively.
Using (3.1) and (3.2), we obtain

$$\begin{aligned} JU &= JPU + JQU + JRU \\ &= \phi PU + \omega PU + \phi QU + \omega QU + \phi RU + \omega RU. \end{aligned} \tag{3.3}$$

Since $J\mathfrak{D} = \mathfrak{D}$, we have $\omega PU = 0$. Therefore, we get

$$JU = \phi PU + \phi QU + \omega QU + \phi RU + \omega RU. \tag{3.4}$$

This means, for any $U \in \Gamma(TN)$, we have

$$\phi U = \phi PU + \phi QU + \phi RU \text{ and } \omega U = \omega QU + \omega RU.$$

Thus, we have the following decomposition

$$J(TN) \subset \mathfrak{D} \oplus \phi\mathfrak{D}_1 \oplus \omega\mathfrak{D}_1 \oplus \phi\mathfrak{D}_2 \oplus \omega\mathfrak{D}_2.$$

Since $\omega\mathfrak{D}_1 \in (T^\perp N)$ and $\omega\mathfrak{D}_2 \in (T^\perp N)$, we have

$$T^\perp N = \omega\mathfrak{D}_1 \oplus \omega\mathfrak{D}_2 \oplus \mu, \tag{3.5}$$

where μ is the orthogonal complement of $\omega\mathfrak{D}_1 \oplus \omega\mathfrak{D}_2$ in $(T^\perp N)$ and it is invariant with respect to J.
For any $Z \in \Gamma(T^\perp N)$, we put

$$JZ = BZ + CZ,$$

where $BZ \in \Gamma(TN)$ and $CZ \in \Gamma(T^\perp N)$.
Now, we have the following results on pointwise quasi bi-slant submanifolds.

Lemma 3.1. *[1] Let N be a pointwise quasi bi-slant submanifold of an almost Hermitian manifold $\bar{N}$, Then*
(i) $\phi^2 U = -(\cos^2\theta_1)U$,
(ii) $g(\phi U, \phi V) = (\cos^2\theta_1)g(U, V)$,
(iii) $g(\omega U, \omega V) = (\sin^2\theta_1)g(U, V)$
for any $U, V \in \Gamma(\mathfrak{D}_1)$, where θ_1 denotes the slant angle of $\mathfrak{D}_1$.

Lemma 3.2. *[1] Let N be a pointwise quasi bi-slant submanifold of an almost Hermitian manifold $\bar{N}$, Then*
(i) $\phi^2 Z = -(\cos^2\theta_2)Z$,
(ii) $g(\phi Z, \phi W) = (\cos^2\theta_2)g(Z, W)$,
(iii) $g(\omega Z, \omega W) = (\sin^2\theta_2)g(Z, W)$
for any $Z, W \in \Gamma(\mathfrak{D}_2)$, where θ_2 denotes the slant angle of $\mathfrak{D}_2$.

Lemma 3.3. *[1] Let N be a pointwise quasi bi-slant submanifold of a Kaehler manifold $\bar{N}$. Then, we have*
(i) $B\omega U_1 = -\sin^2\theta_1 U_1$, *(ii)* $B\omega U_2 = -\sin^2\theta_2 U_2$,
(iii) $\phi^2 U_1 + B\omega U_1 = -U_1$, *(iv)* $\phi^2 U_2 + B\omega U_2 = -U_2$,
(v) $\omega\phi U_1 + C\omega U_1 = 0$ *(vi)* $\omega\phi U_2 + C\omega U_2 = 0$,
for any $U_1 \in \mathfrak{D}_{\theta_1}$ and $U_2 \in \mathfrak{D}_{\theta_2}$.

Lemma 3.4. *[1] Let N be a pointwise quasi bi-slant submanifold of a Kaehler manifold $\bar{N}$ with invariant distribution $\mathfrak{D}$ and pointwise slant distributions $\mathfrak{D}_1$ and $\mathfrak{D}_2$ with distinct slant functions θ_1 and θ_2, respectively. Then*

(i) *For any $U, V \in \Gamma(\mathfrak{D}_{\theta_1})$ and $Z \in \Gamma(\mathfrak{D}_{\theta_2})$, we have*

$$\begin{aligned}(\sin^2\theta_1 - \sin^2\theta_2)g(\nabla_U V, Z) &= g(A_{\omega T_2 Z}V - A_{\omega Z}T_1 V, U)\\ &\quad + g(A_{\omega T_1 V}Z - A_{\omega V}T_2 Z, U).\end{aligned}$$

(ii) *For any $Z, W \in \Gamma(\mathfrak{D}_{\theta_2})$ and $U \in \Gamma(\mathfrak{D}_{\theta_1})$, we have*

$$\begin{aligned}(\sin^2\theta_1 - \sin^2\theta_2)g(\nabla_Z W, U) &= g(A_{\omega T_2 W}U - A_{\omega W}T_1 U, Z)\\ &\quad + g(A_{\omega T_1 U}W - A_{\omega U}T_2 W, Z).\end{aligned}$$

4. Warped product pointwise quasi bi-slant submanifolds

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and $f > 0$, be a positive differentiable function on N_1. Consider the product manifold $N_1 \times N_2$ with its canonical projections $\pi : N_1 \times N_2 \to N_1$ and $\rho : N_1 \times N_2 \to N_2$. The warped product $N = N_1 \times_f N_2$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian metric g such that

$$g(U, V) = g_1(\pi_*(U), \pi_*(V)) + (f \circ \pi)^2 g_2(\rho_*(U), \rho_*(V))$$

for any tangent vector $U, V \in TN$, where $*$ is the symbol for the tangent maps. It was proved in [4] that for any $U \in TN_1$ and $Z \in TN_2$, the following holds

$$\nabla_U Z = \nabla_Z U = (U ln f)Z, \tag{4.1}$$

where ∇ denotes the Levi-Civita connection of g on N. A warped product manifold $N = N_1 \times_f N_2$ is said to be trivial if the warping function f is constant. If $N = N_1 \times_f N_2$ is a warped product manifold then N_1 is totally geodesic and N_2 is a totally umbilical (see [4, 10]).

From now onwards, we assume the ambient manifold $\bar{N}$ is Kaehler manifold and N is pointwise quasi bi-slant submanifold in $\bar{N}$.

Now, we give the following useful lemma for later use.

Lemma 4.1. *Let $N = N_1 \times_f N_2$ be a warped product pointwise quasi bi-slant submanifold of a Kaehler manifold $\bar{N}$ such that N_1 and N_2 are pointwise slant submanifolds with slant functions θ_1 and θ_2, respectively of $\bar{N}$. Then, we have*

$$g(\sigma(U, W), \omega\phi RZ) - g(\sigma(U, \phi PZ), \omega W) - g(\sigma(U, \phi RZ), \omega W) = \sin 2\theta_2 U(\theta_2) g(RZ, W) \tag{4.2}$$

for any $U \in TN_1$ and $Z, W \in TN_2$ such that $Z = PZ + RZ$ and $W = PW + RW$.

Proof. For any $U \in TN_1$ and $Z, W \in TN_2$, we have

$$g(\bar{\nabla}_U Z, W) = g(\nabla_U Z, W) = U(ln f)g(Z, W). \tag{4.3}$$

On the otherhand, we can write

$$g(\bar{\nabla}_U Z, W) = g(J\bar{\nabla}_U Z, JW) = g(\bar{\nabla}_U JZ, JW)$$

for any $U \in TN_1$ and $Z, W \in TN_2$. Using (3.2), we obtain

$$\begin{aligned}g(\bar{\nabla}_U Z, W) &= g(\bar{\nabla}_U \phi PZ + \phi RZ, \phi W) + g(\bar{\nabla}_U \phi PZ + \phi RZ, \omega W)\\ &\quad + g(\bar{\nabla}_U \omega RZ, JW).\end{aligned}$$

Then from (2.1), (2.2), (2.3) and (4.1), we can derive

$$\begin{aligned}g(\bar{\nabla}_U Z, W) &= g(\nabla_U PZ, W) + \cos^2\theta_2 U(ln f)g(RZ, W) + g(\sigma(U, \phi PZ), \omega W)\\ &\quad g(\sigma(U, \phi RZ), \omega W) - g(\bar{\nabla}_U J\omega RZ, W)\\ &= U(ln f)g(PZ, W) + \cos^2\theta_2 U(ln f)g(RZ, W) + g(\sigma(U, \phi PZ), \omega W)\\ &\quad + g(\sigma(U, \phi RZ), \omega W) - g(\bar{\nabla}_U B\omega RZ, W) - g(\bar{\nabla}_U C\omega RZ, W).\end{aligned}$$

Using lemma 3.3, we find

$$\begin{aligned} g(\bar{\nabla}_U Z, W) &= U(lnf)g(PZ, W) + \cos^2\theta_2 U(lnf)g(RZ, W) + g(\sigma(U, \phi PZ), \omega W) \\ &\quad + g(\sigma(U, \phi RZ), \omega W) + \sin^2\theta_2 g(\bar{\nabla}_U RZ, W) + \sin 2\theta_2 U(\theta_2)g(RZ, W) \\ &\quad + g(\bar{\nabla}_U \omega\phi RZ, W). \end{aligned} \tag{4.4}$$

Thus, lemma follows from (4.3) and (4.4) by using (2.4) and (4.1). □

Lemma 4.2. *Let $N = N_1 \times_f N_2$ be a warped product pointwise quasi bi-slant submanifold of a Kaehler manifold $\bar{N}$ such that N_1 and N_2 are pointwise slant submanifolds with slant functions θ_1 and θ_2,respectively of $\bar{N}$. Then*

$$g(\sigma(U, PZ), \omega RW) + g(\sigma(U, RZ), \omega RW) - g(\sigma(U, W), \omega RZ) = 2(\tan\theta_2)U(\theta_2)g(\phi RZ, W) \tag{4.5}$$

for any $U \in TN_1$ and $Z, W \in TN_2$ such that $Z = PZ + RZ$ and $W = PW + RW$.

Proof. By Interchanging RZ by ϕRZ in (4.2) for any $Z \in TN_2$ and PZ by ϕPZ and by using lemma 3.2(i), we obtain the following result. □

Lemma 4.3. *Let $N = N_1 \times_f N_2$ be a warped product pointwise quasi bi-slant submanifold of a Kaehler manifold $\bar{N}$ such that N_1 and N_2 are pointwise slant submanifolds with slant functions θ_1 and θ_2,respectively of $\bar{N}$. Then*

$$g(\sigma(U, W), \omega\phi RZ) - g(\sigma(U, \phi PZ), \omega W) - g(\sigma(U, \phi RZ), \omega W) = 2U(lnf)(\cos^2\theta_2)g(RZ, W) \tag{4.6}$$

for any $U \in TN_1$ and $Z, W \in TN_2$.

Proof. For any $U \in TN_1$ and $Z \in TN_2$, we have

$$\begin{aligned} g(\sigma(U, Z), \omega W) &= g(\bar{\nabla}_Z U, \omega W) \\ &= g(\bar{\nabla}_Z U, JW) - g(\bar{\nabla}_Z U, \phi W). \end{aligned}$$

Using (2.1), (2.2) and (3.2), we arrive at

$$g(\sigma(U, Z), \omega W) = -g(\bar{\nabla}_Z \phi U, W) - g(\bar{\nabla}_Z \omega U, W) - g(\bar{\nabla}_Z U, \phi W).$$

On further simplification and using (4.1), we obtain

$$g(\sigma(U, Z), \omega W) = -g(\bar{\nabla}_Z \phi U, W) - g(\bar{\nabla}_Z \omega U, W) - U(lnf)g(Z, \phi W).$$

Using (2.5) and (4.1), we obtain

$$g(\sigma(U, Z), \omega W) = -\phi U(lnf)g(Z, W) + g(\sigma(Z, W), \omega U) + U(lnf)g(\phi RZ, W). \tag{4.7}$$

Then, by Polarization, we can derive

$$g(\sigma(U, W), \omega Z) = -\phi U(lnf)g(Z, W) + g(\sigma(Z, W), \omega U) - U(lnf)g(\phi RZ, W). \tag{4.8}$$

Subtracting (4.8) from (4.7), we obtain

$$g(\sigma(U, Z), \omega W) - g(\sigma(U, W), \omega Z) = 2U(lnf)g(\phi RZ, W). \tag{4.9}$$

Interchanging RZ by ϕRZ in (4.9) and using lemma 3.2(i), we get (4.6), which proves the lemma completely. □

Remark 4.1. A warped product submanifold $N_1 \times_f N_2$ of a Kaehler manifold $\bar{N}$ is called *mixed totally geodesic* if $\sigma(U, Z) = 0$ for any $U \in TN_1$ and $Z \in TN_2$.

Now, the consequences of lemma 4.3, we obtain the following theorem.

Theorem 4.1. *Let $N = N_1 \times_f N_2$ be a warped product pointwise quasi bi-slant submanifold of a Kaehler manifold $\bar{N}$ such that N_1 and N_2 are pointwise slant submanifolds with slant functions θ_1 and θ_2, respectively of $\bar{N}$. Then, if N is mixed totally geodesic warped product submanifold, then the following cases occurs:*

(i) either N is Riemannian product submanifold of N_1 and N_2,

(ii) or $\theta_2 = \frac{\pi}{2}$, i.e., N is a warped product submanifold of the form $N_1 \times_f N_\perp$, where $N_\perp$ is a totally real submanifold of $\bar{N}$.

Proof. From lemma 4.3 and using the mixed totally geodesic condition, we have

$$2U(lnf)(\cos^2\theta_2)g(Z,W) = 0,$$

for $Z, W \in \Gamma(TN_2)$ and $U \in \Gamma(TN_1)$. Since $N_2 \neq \{0\}$, we obtain

$$U(lnf) = 0 \qquad or \qquad \cos^2\theta_2 = 0,$$

which imply that either f is constant on N or $\theta_2 = \frac{\pi}{2}$. Hence, either N is simply a Riemannian product or $\theta_2 = \frac{\pi}{2}$. Hence, the proof is compltete. □

Remark 4.2. In theorem 4.1, if N is mixed totally geodesic and f is not constant on N, then N is a warped product pointwise quasi hemi-slant submanifold of the form $N_{\theta_1} \times_f N_\perp$, where N_{θ_1} is a pointwise slant submanifold and $N_\perp$ is a totally real submanifold of $\bar{N}$.

Theorem 4.2. *Let $N = N_1 \times_f N_2$ be a warped product pointwise quasi bi-slant submanifold of a Kaehler manifold $\bar{N}$ such that N_1 and N_2 are pointwise slant submanifolds with slant functions θ_1 and θ_2,respectively of $\bar{N}$. Then*

$$U(lnf) = U(\theta_2)\tan\theta_2, \tag{4.10}$$

for any $U \in TN_1$.

Proof. From lemma 4.1 and lemma 4.3, we obtain

$$2U(lnf)(\cos^2\theta_2)g(RZ,W) = \sin 2\theta_2 U(\theta_2)g(RZ,W),$$

for any $U \in TN_1$ and $Z, W \in TN_2$. Now, after using the trigonometric identities, we obtain $\{U(lnf) - \tan\theta_2 U(\theta_2)\}g(RZ,W) = 0$, this implies $U(lnf) = \tan\theta_2 U(\theta_2)$. Hence, the proof is complete. □

Example 4.1. For $a, b \neq 0, 1$ and $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$. Consider a submanifold N of a Kaehler manifold $\mathbb{R}^{12}$ defined by

$$\begin{aligned}\chi(a,b,\theta_1,\theta_2,u,v) \;&=\; (b\cos\theta_2, a\cos\theta_1, b\sin\theta_2, a\sin\theta_1, b\cos\theta_1, a\cos\theta_2,\\ &\quad\; b\sin\theta_1, a\sin\theta_2, \sqrt{3}\theta_1 + \sqrt{2}\theta_2, \sqrt{2}\theta_1 + \sqrt{3}\theta_2, u, v).\end{aligned}$$

The complex structure J is defined as

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \qquad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, 1 \leq i, j \leq 6.$$

It is easy to see that the tangent bundle TN of N is spanned by the following vectors

$$v_1 = \cos\theta_1\frac{\partial}{\partial y_1} + \sin\theta_1\frac{\partial}{\partial y_2} + \cos\theta_2\frac{\partial}{\partial y_3} + \sin\theta_2\frac{\partial}{\partial y_4},$$

$$v_2 = \cos\theta_2\frac{\partial}{\partial x_1} + \sin\theta_2\frac{\partial}{\partial x_2} + \cos\theta_1\frac{\partial}{\partial x_3} + \sin\theta_1\frac{\partial}{\partial x_4},$$

$$v_3 = -a\sin\theta_1\frac{\partial}{\partial y_1} + a\cos\theta_1\frac{\partial}{\partial y_2} - b\sin\theta_1\frac{\partial}{\partial x_3} + b\cos\theta_1\frac{\partial}{\partial x_4} + \sqrt{3}\frac{\partial}{\partial x_5} + \sqrt{2}\frac{\partial}{\partial y_5},$$

$$v_4 = -b\sin\theta_2\frac{\partial}{\partial x_1} + b\cos\theta_2\frac{\partial}{\partial x_2} - a\sin\theta_2\frac{\partial}{\partial y_3} + a\cos\theta_2\frac{\partial}{\partial y_4} + \sqrt{2}\frac{\partial}{\partial x_5} + \sqrt{3}\frac{\partial}{\partial y_5},$$

$$v_5 = \frac{\partial}{\partial x_6}, \qquad v_6 = \frac{\partial}{\partial y_6}.$$

Then, clearly we obtain

$$Jv_1 = -\cos\theta_1\frac{\partial}{\partial x_1} - \sin\theta_1\frac{\partial}{\partial x_2} - \cos\theta_2\frac{\partial}{\partial x_3} - \sin\theta_2\frac{\partial}{\partial x_4},$$

$$Jv_2 = \cos\theta_2 \frac{\partial}{\partial y_1} + \sin\theta_2 \frac{\partial}{\partial y_2} + \cos\theta_1 \frac{\partial}{\partial y_3} + \sin\theta_1 \frac{\partial}{\partial y_4},$$

$$Jv_3 = a\sin\theta_1 \frac{\partial}{\partial x_1} - a\cos\theta_1 \frac{\partial}{\partial x_2} - b\sin\theta_1 \frac{\partial}{\partial y_3} + b\cos\theta_1 \frac{\partial}{\partial y_4} + \sqrt{3}\frac{\partial}{\partial y_5} - \sqrt{2}\frac{\partial}{\partial x_5},$$

$$Jv_4 = -b\sin\theta_2 \frac{\partial}{\partial y_1} + b\cos\theta_2 \frac{\partial}{\partial y_2} + a\sin\theta_2 \frac{\partial}{\partial x_3} - a\cos\theta_2 \frac{\partial}{\partial x_4} + \sqrt{2}\frac{\partial}{\partial y_5} - \sqrt{3}\frac{\partial}{\partial x_5},$$

$$Jv_5 = \frac{\partial}{\partial y_6}, \qquad Jv_6 = -\frac{\partial}{\partial x_6}.$$

Then, clearly $\mathfrak{D} = \{v_5, v_6\}$ is holomorphic distribution and $\mathfrak{D}_{\theta_1} = span\{v_1, v_2\}$, $\mathfrak{D}_{\theta_2} = spna\{v_3, v_4\}$ are pointwise bi-slant functions with slant functions $\cos^{-1}[\cos(\theta_1 - \theta_2)]$ and $\cos^{-1}(\frac{1}{a^2+b^2+5})$, respectively. Hence the submanifold N defined by χ is a proper 6-dimensional pointwise quasi bi-slant submanifold of $\mathbb{R}^{12}$.

It is easy to verify that $\mathfrak{D} \oplus \mathfrak{D}_2$ and $\mathfrak{D}_1$ are integrable. If we denote the integrable manifolds of $\mathfrak{D}$, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ by N_T, N_{θ_1} and N_{θ_2}, respectively. Then the metric tensor g of product manifold N is given by

$$\begin{aligned} ds^2 &= du^2 + dv^2 + (a^2+b^2+5)d\theta_1^2 + (a^2+b^2+5)d\theta_2^2 + 2da^2 + 2db^2 \\ &= g_{N_1} + 2g_{N_\theta}, \end{aligned}$$

such that,

$$g_{N_1} = du^2 + dv^2 + (a^2+b^2+5)d\theta_1^2 + (a^2+b^2+5)d\theta_2^2 \quad and \quad g_{N_\theta} = da^2 + db^2,$$

where $N_1 = N_T \times N_{\theta_2}$, are the metric tensor of N_1 and N_θ, respectively. Consequently, $N = N_1 \times_f N_\theta$ is a warped product pointwise quasi bi-slant submanifold of $\mathbb{R}^{12}$ with warping function $f = \sqrt{2}$ a constant function and whose bi-slant angles $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$. Hence, N is simply a Riemannian product submanifold of $\mathbb{R}^{12}$.

Example 4.2. Consider a submanifold N of $\mathbb{R}^{10}$ defined by

$$\chi(u, v, w, a, b) = (v\cos u, w\cos u, v\sin u, w\sin u, -v+w, v+w, 0, 0, a, b),$$

with almost complex structure J defined by

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \qquad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, 1 \leq i, j \leq 5.$$

It is easy to see that its tangent space TN of N is spanned by the following vectors

$$v_1 = -v\sin u\frac{\partial}{\partial x_1} + v\cos u\frac{\partial}{\partial x_2} - w\sin u\frac{\partial}{\partial y_1} + w\cos u\frac{\partial}{\partial y_2},$$

$$v_2 = \cos u\frac{\partial}{\partial x_1} + \sin u\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3},$$

$$v_3 = \frac{\partial}{\partial x_3} + \cos u\frac{\partial}{\partial y_1} + \sin u\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3},$$

$$v_4 = \frac{\partial}{\partial x_5}, \qquad v_6 = \frac{\partial}{\partial y_5}.$$

Then, we have

$$Jv_1 = -v\sin u\frac{\partial}{\partial y_1} + v\cos u\frac{\partial}{\partial y_2} + w\sin u\frac{\partial}{\partial x_1} - w\cos u\frac{\partial}{\partial x_2},$$

$$Jv_2 = \cos u \frac{\partial}{\partial y_1} + \sin u \frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3} - \frac{\partial}{\partial x_3},$$

$$Jv_3 = \frac{\partial}{\partial y_3} - \cos u \frac{\partial}{\partial x_1} - \sin u \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}.$$

$$Jv_4 = \frac{\partial}{\partial y_5}, \qquad Jv_5 = -\frac{\partial}{\partial x_5}.$$

Let us put $\mathfrak{D} = span\{v_4, v_5\}$ is an invariant distribution, $\mathfrak{D}_1 = span\{v_2, v_3\}$ a proper slant distribution with slant angle $\theta_1 = \cos^{-1}\left(\frac{1}{3}\right)$ and $\mathfrak{D}_2 = span\{v_1\}$ an anti-invariant distribution with slant angle $\theta_2 = \frac{\pi}{2}$. Hence the submanifold N defined by χ is a pointwise quasi hemi-slant submanifold of $\mathbb{C}^5$.

It is easy to verify that $\mathfrak{D} \oplus \mathfrak{D}_1$ and $\mathfrak{D}_2$ are integrable. If we denote the integrable manifolds of $\mathfrak{D}$, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ by N_T, N_{θ_1} and $N_\perp$, respectively. Then the metric tensor g of product manifold N is given by

$$ds^2 = da^2 + db^2 + 3(dv^2 + dw^2) + (v^2 + w^2)du^2,$$

such that

$$g_{N_1} = da^2 + db^2 + 3(dv^2 + dw^2) \quad and \quad g_{N_\perp} = du^2,$$

where $N_1 = N_T \times N_\theta$. In this case the warping function $f = \sqrt{v^2 + w^2}$ and hence, N is a case of warped product pointwise quasi hemi-slant submanifold. So we have discussed both the case of theorem 4.1.

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