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RESEARCH ARTICLE

Simple Rotational Surfaces in Euclidean 4-Space with Generalized 1-type Gauss Map

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ABSTRACT

In this paper, we consider simple rotational surfaces in the Euclidean 4-space $\mathbb{E}^4$ with the profile curve contained in a 2-plane. In terms of having generalized 1-type Gauss map, we obtain some classification results of minimal surfaces, flat simple rotational surfaces and simple rotational surfaces with constant Gaussian curvature.

Keywords: *Submanifolds with finite type Gauss map; surfaces with finite type Gauss map; rotational surfaces with generalized 1-type Gauss map.*
AMS Subject Classification (2020): *Primary: 53B25 ; Secondary: 53C40.*

1. Introduction

The first study on rotational surfaces in the Euclidean 4-space $\mathbb{E}^4$ was studied by F.N. Cole in [10]. The obvious generalization of a surface of rotation in ordinary space is a surface left invariant by a rotation in four dimensions, a rotation being defined as a linear transformation of positive determinant preserving distance and leaving one point fixed [15]. Then, in the most general form in $\mathbb{E}^4$, a rotational surface is defined as

$$F(s,t) = (X_1(s,t), X_2(s,t), X_3(s,t), X_4(s,t)) \tag{1.1}$$

where $\beta(s) = (x(s), y(s), z(s), w(s))$ is the profile curve and

$$X_1(s,t) = x(s)\cos at - y(s)\sin at,$$
$$X_2(s,t) = x(s)\sin at + y(s)\cos at,$$
$$X_3(s,t) = z(s)\cos bt - w(s)\sin bt,$$
$$X_4(s,t) = z(s)\sin bt + w(s)\cos bt$$

is the position vector of the rotational surface. Specifically, if $a = 0, b = 1$ and $w(s) = 0$, the equation (1.1) is reduced to

$$F(s,t) = (x(s), y(s), z(s)\cos t, z(s)\sin t) \tag{1.2}$$

and (1.2) is defined as a simple rotational surface in the space $\mathbb{E}^4$.

On the other hand, the definition of finite type submanifold in Euclidean spaces was given for the first time by B.Y. Chen, while the subject of submanifolds in Euclidean space was studied, in order to define the total mean curvature of compact submanifolds in Euclidean spaces and the concept of degree, and the studies on this subject were published in his book [5]. By the definition, a map $\phi : M \to \mathbb{E}^n$ into a Euclidean space is said to be finite type if it can be expressed as

$$\phi = \phi_0 + \phi_1 + \phi_2 + \ldots + \phi_k$$

for some eigenvectors $\phi_0, \phi_1, \phi_2, \ldots, \phi_k$ of the Laplace operator Δ of M, where $\phi_0 \in \mathbb{E}^n$ is a constant vector. More precisely, if these eigenvectors are corresponding from k distinct eigenvalues of , then ϕ is said to be of

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k-type. Then, the definition of finite type map is given by using differentiable maps in the definition of finite type submanifold and firstly studied in [7]. Specifically, by considering the Gauss map, definition of 1-type Gauss map is defined as

$$\Delta\nu = \lambda(\nu + C)$$

After that, many studies on finite type Gauss map were studied in articles [3, 14, 19, 21, 23, 24]. Moreover, it has been seen that this equation is provided not only for the constant λ, but also for a differentiable function f in some cases, and therefore, in [18] Kim and Yoon definition of pointwise 1-type Gauss map is defined as

$$\Delta\nu = f(\nu + C).$$

Also, in this definition, it is defined as the first kind if $C = 0$ and as the second kind otherwise. Submanifolds in Euclidean spaces with pointwise 1-type Gauss map were studied in [2, 8, 9, 11, 13, 25]. In the articles [4, 6, 16, 17, 19], rotational surfaces were studied with pointwise 1-type Gauss map in Minkowski space. For example, in the paper [22], simple rotational surfaces in $\mathbb{E}^4$ were examined. Then, a complete classification of simple rotational surfaces with pointwise 1-type Gauss map of the first kind is made, and a classification of simple rotational surfaces with pointwise 1-type Gauss map of the second kind provided that one of the coordinate functions satisfies a third-order ordinary differential equation. In addition, in the article [12], the classification of space-like rotational surfaces in $\mathbb{E}^4_1$ Minkowski space was given, and in the article [2], necessary and sufficient conditions were determined for the flat Ganchev-Milousheva rotational surface to be of pointwise 1-type Gauss map.

Recent studies have shown that some surfaces in $\mathbb{E}^3$ satisfying

$$\Delta\nu = f_1\nu + f_2 C. \tag{1.3}$$

This situation is understood to be neither 1-type nor pointwise 1-type Gauss map, and the equation (1.3) is given to defined as a generalized 1-type Gauss map in [20, 26], where surfaces in $\mathbb{E}^3$ were studied. Note that generalized 1-type Gauss map includes considering that it includes both 1-type and pointwise 1-type Gauss map.

In this paper, the classification of simple rotational surfaces in space $\mathbb{E}^4$ is investigated in terms of having a generalized 1-type Gauss map. First, the necessary conditions for simple rotational surfaces to have a generalized 1-type Gauss map in space $\mathbb{E}^4$ are obtained. Then, it is shown that minimal surfaces do not have a generalized 1-type Gauss map, but a second kind of pointwise 1-type Gauss map. It is also shown that flat simple rotational surfaces whose profile curve is non-planar do not have a generalized 1-type Gauss map. Finally, by obtaining the necessary and sufficient conditions for simple rotational surfaces of constant Gaussian curvature for $K_G = 1$, it is shown that surfaces of constant Gaussian curvature have a generalized 1-type Gauss map.

2. Preliminaries

Let M be an oriented n-dimensional submanifold in $\mathbb{E}^{n+2}$. We choose an oriented local orthonormal frame $\{e_1, e_2, \ldots, e_{n+2}\}$ on M such that $e_1, e_2, \ldots, e_n$ are tangent to M and e_{n+1}, e_{n+2} are normal to M. We use the following convention on the range of indices, $1 \leq i, j, k, \ldots \leq n$, $n+1 \leq r, s, t, \ldots \leq n+2$. Let $\tilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^{n+2}$ and ∇ the induced connection on M. Also, let ω_{AB} be the connection forms defined as $\omega_{AB}(X) = \left\langle \tilde{\nabla}_X e_A, e_B \right\rangle$ and h^r_{ij} denote the components of the second principal form h of the M submanifold, i.e., we put

$$h^r_{ij} = \langle h(e_i, e_j), e_s \rangle.$$

The Gauss and Weingarten formulas for the M submanifold can be given as

$$\tilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{n+2} h^r_{ik} e_r$$

and

$$\tilde{\nabla}_{e_k} e_s = -\sum_{j=1}^{n} h^s_{jk} e_j + \sum_{r=n+1}^{n+2} \omega_{sr}(e_k) e_r.$$

Similarly, the normal connection $\nabla^{\perp}$ of M is given by

$$\nabla^{\perp}_{e_k} e_s = \sum_{r=n+1}^{n+2} \omega_{sr}(e_k) e_r.$$

The mean curvature vector H and the squared length $\|h\|^2$ of the second fundamental form h are defined as follows, respectively

$$H = \frac{1}{n}\sum_{i,r} h^r_{ii} e_r \qquad \text{and} \qquad \|h\|^2 = \sum_{i,j,r} h^r_{ij} h^r_{ji}.$$

The Codazzi equation of M in $\mathbb{E}^{n+2}$ is given by

$$h^r_{ij,k} = h^r_{jk,i},$$

$$h^r_{jk,i} = e_i(h^r_{jk}) + \sum_{s=n+1}^{n+2} h^s_{jk}\omega_{sr}(e_i) - \sum_{\gamma=1}^{n}\left(\omega_{j\gamma}(e_i) h^r_{\gamma k} + \omega_{k\gamma}(e_i) h^r_{j\gamma}\right).$$

The normal curvature tensor R^D of M in $\mathbb{E}^{n+2}$ is given by

$$R^D\left(e_j, e_k; e_r, e_s\right) = \langle\left[A_{e_r}, A_{e_s}\right](e_j), e_k\rangle = \sum_{i=1}^{n}\left(h^r_{ik}h^s_{ij} - h^r_{ij}h^s_{ik}\right).$$

2.1. *Simple Rotational Surfaces*

In $\mathbb{E}^4$, the parameterization of a general rotational surface and a simple rotational surface is defined by (1.1) and (1.2), respectively. Now, let M be a rotational surface given by (1.2) and we also consider the profile curve $\beta = (x, y, z, 0)$ of M and denote its curvature by κ. Without loss of generality, we assume that β is parametrized by its arc-length; that is, the equation

$$(x')^2 + (y')^2 + (z')^2 = 1 \tag{2.1}$$

is satisfied.

In this case, an orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ on M is defined as

$$e_1 = \frac{\partial}{\partial s}, \qquad e_2 = \frac{1}{z}\frac{\partial}{\partial t}, \tag{2.2a}$$

$$e_3 = \frac{1}{\kappa}(x'', y'', z''\cos t, z''\sin t), \tag{2.2b}$$

$$e_4 = \frac{1}{\kappa}\left(\rho_1, \rho_2, \rho_3\cos t, \rho_3 \sin t\right) \tag{2.2c}$$

where e_1, e_2 are tangent and e_3, e_4 are normal to M. Here, ρ_1, ρ_2 and ρ_3 are differentiable functions which are defined as,

$$\left(\rho_1, \rho_2, \rho_3\right) = \left(y'z'' - y''z', x''z' - x'z'', x'y'' - x''y'\right).$$

With a direct calculation, we obtain the connection forms of M and components of the second fundamental form of M as

$$h^3_{11} = \kappa, \quad h^3_{22} = -\frac{z''}{\kappa z}, \quad h^3_{12} = 0, \tag{2.3}$$

$$h^4_{11} = 0, \quad h^4_{12} = 0, \quad h^4_{22} = -\frac{\rho_3}{\kappa z},$$

$$\omega_{12}(e_1) = 0, \quad \omega_{12}(e_2) = \frac{z'}{z},$$

$$\omega_{34}(e_1) = \tau, \quad \omega_{34}(e_2) = 0. \tag{2.4}$$

Consequently, the shape operators of M has the matrix representation

$$A_3 = \begin{pmatrix} \kappa & 0 \\ 0 & -\frac{z''}{\kappa z} \end{pmatrix} \qquad \text{and} \qquad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\rho_3}{\kappa z} \end{pmatrix}.$$

From these matrices, the mean curvature vector H and the Gaussian curvature K of M are calculated as, respectively

$$H = \frac{(h_{11}^3 + h_{22}^3)}{2} e_3 + \frac{h_{22}^4}{2} e_4,$$

$$K = -\frac{z''}{z}.$$

The Codazzi and Gauss equations of a simple surface M are as follows [22]

$$e_1(h_{22}^3) = \omega_{21}(e_2)(h_{22}^3 - \kappa) + \tau h_{22}^4,$$
$$e_1(h_{22}^4) = h_{22}^4 \omega_{21}(e_2) - \tau h_{22}^3,$$
$$(\omega_{21}(e_2))' = (\omega_{21}(e_2))^2 + \kappa h_{22}^3.$$

3. Generalized 1-type Gauss map in Euclidean space $\mathbb{E}^m$

Let M be an n-dimensional submanifold of the Euclidean space $\mathbb{E}^m$. Consider the orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ of the tangent bundle of M. Let us denote the space spanned by n vectors in the Euclidean space $\mathbb{E}^m$ by $\Lambda^n(\mathbb{E}^m)$. Note that the dimension of the space $\Lambda^n(\mathbb{E}^m)$ is

$$N = \begin{pmatrix} m \\ n \end{pmatrix}.$$

Now, for any $p \in M$ the n-vector given by $(e_1 \wedge e_2 \wedge \cdots \wedge e_n)_p$ represents the n-plane in the space $\mathbb{E}^m$ spanned by the vectors $\{e_1, e_2, \cdots, e_n\}$. The Gauss map ν of M is defined as the mapping that assigns to each point of M the tangent plane at that point. In expression, we have

$$\begin{array}{lll} \nu: & M \to \mathbb{E}^N & \\ & p \to \nu(p) & = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)_p . \end{array}$$

Laplacian with respect to the reduced metric on the M submanifold is defined

$$\Delta = \sum_{i=1}^{n} (\nabla_{e_i} e_i - e_i e_i)$$

According to this definition, pointwise 1-type Gauss map and generalized 1-type Gauss map are defined as follows.

Definition 3.1. Let M be an n-dimensional submanifold of the Euclidean space $\mathbb{E}^m$, and ν be the Gauss map of this submanifold. M is said to have a pointwise 1-type Gauss map if the Gauss map, on M satisfies the equation

$$\Delta \nu = f(\nu + C) \tag{3.1}$$

for a smooth function f and has a constant vector, $C \in \mathbb{E}^N$. Additionally, if the equation (3.1) satisfies for $C = 0$, it has pointwise 1-type Gauss map of the first kind, if not, it has pointwise 1-type Gauss map of the second kind.

Definition 3.2. Let M be an n-dimensional submanifold of the $\mathbb{E}^m$ Euclidean space, and ν be the Gauss map of this submanifold. M is said to have a generalized 1-type Gauss map if the Gauss map, on M satisfies the equation

$$\Delta \nu = f_1 \nu + f_2 C \tag{3.2}$$

for some smooth functions (f_1, f_2) and has a constant vector, $C \in \mathbb{E}^N$.

Now, we are going to consider the case when M is a surface in the Euclidean 4-space $\mathbb{E}^4$. Note that in this Definition 3.2, the constant vector $C \in \Lambda(4,2) \equiv \mathbb{E}^6$, we define c_{ij} by

$$c_{ij} = \langle C, e_i \wedge e_j \rangle. \tag{3.3}$$

Consequently, we have

$$C = \sum_{1\leq i<j\leq 4} c_{ij} e_i \wedge e_j,$$

Also, the calculation of the Laplacian of the Gauss map used in this definition is explained by the lemma given below [7, Lemma 3.1].

Lemma 3.1. *The Gauss map $\nu = e_1 \wedge e_2$ of an oriented surface M in $\mathbb{E}^4$ satisfies the equation*

$$\Delta\nu = \|h\|^2 \nu + 2R^D(e_1, e_2; e_3, e_4)e_3 \wedge e_4 - 2(D_{e_1}H \wedge e_2 + e_1 \wedge D_{e_2}H), \tag{3.4}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal moving frame.

Theorem 3.1. *A non-planar minimal oriented surface M in the Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the second kind if and only if, with respect to some suitable local orthonormal frame $\{e_1, e_2, e_3, e_4\}$ on M, the shape operators of M are given by $A_3 = diag\,(\rho, -\rho)$ and $A_4 = adiag\,(\pm\rho, \pm\rho)$, where ρ is a smooth non-zero function on M and $adiag\,(a, b)$ means a 2×2 anti-diagonal matrix [11].*

Now that we will classify minimal surfaces in $\mathbb{E}^4$ with generalized 1-type Gauss map by using the above theorem.

Theorem 3.2. *Let M be a minimal surface in $\mathbb{E}^4$. If M has a generalized 1-type Gauss map, then it is pointwise 1-type.*

Proof. Let's assume that M is the minimal surface in $\mathbb{E}^4$. In this case, $trA_3 = 0$ and $trA_4 = 0$, where the shape operators A_3 and A_4 are defined as follows

$$A_3 = \begin{pmatrix} h^3_{11} & 0 \\ 0 & h^3_{22} \end{pmatrix} \quad and \quad A_4 = \begin{pmatrix} h^4_{11} & h^4_{12} \\ h^4_{12} & h^4_{22} \end{pmatrix}$$

In this case, for $\nu = e_1 \wedge e_2$, equation (3.4) is reduced to

$$\Delta\nu = \|h\|^2 \nu + 2R^D(e_1, e_2; e_3, e_4)e_3 \wedge e_4$$

Suppose, M has a generalized 1-type Gauss map. Since the M will provide the equation (1.3), with f_1, f_2 differentiable functions and C being a constant vector, the following expressions can be written

$$\begin{aligned} f_1 + f_2 c_{12} &= \|h\|^2, \\ f_2 c_{34} &= 2h^4_{12}\left(h^3_{22} - h^3_{11}\right) \end{aligned}$$

and

$$C = c_{12} e_1 \wedge e_2 + c_{34} e_3 \wedge e_4. \tag{3.5}$$

With the help of equation (3.5), it is understood that $c_{13} = c_{14} = c_{23} = c_{24} = 0$. In addition, the equations and equalities obtained with the help of the conditions that the C vector must meet to be constant are given as,

$$e_i(c_{12}) = 0, \qquad e_i(c_{34}) = 0 \tag{3.6}$$

$$h^4_{11} c_{12} = 0, \qquad h^4_{22} c_{12} = 0$$

$$h^3_{11} c_{34} = h^4_{12} c_{12} \tag{3.7}$$

$$h^4_{12} c_{34} = h^3_{11} c_{12} \tag{3.8}$$

Since the coefficients c_{12} and c_{34} must be nonzero constants from equations (3.6), it is understood that $h^4_{11} = h^4_{22} = 0$. Also, since $h^3_{11} = \epsilon h^4_{12}$ from equations (3.7) and (3.8), the shape operators of surface M are as follows

$$A_3 = \begin{pmatrix} h^3_{11} & 0 \\ 0 & -h^3_{11} \end{pmatrix} \quad and \quad A_4 = \begin{pmatrix} 0 & \epsilon h^3_{11} \\ \epsilon h^3_{11} & 0 \end{pmatrix}$$

From this, it is understood that the surface M has a pointwise-type 1-type Gauss map. □

4. Simple rotational surfaces with Generalized 1-type Gauss map

In this section, it will be investigated whether the Gauss map $\nu = e_1 \wedge e_2$ of simple rotational surfaces satisfies the condition (1.3). For this, assume that M is a rotational surface in $\mathbb{E}^4$ parametrized by (1.2) and that it has the orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ given in (2.2).

Lemma 4.1. *Let M be an oriented simple rotational surface given by parametrization (4.0.1) with Gauss map $\nu = e_1 \wedge e_2$ at $\mathbb{E}^4$. In this case, the surface M satisfies the equation*

$$\begin{aligned} \triangle\nu = \|h\|^2 \nu + (e_1(h^3_{11}) + h^3_{11}\omega_{12}(e_2) - h^3_{22}\omega_{12}(e_2))e_2 \wedge e_3 \\ +(h^3_{11}\omega_{34}(e_1) - h^4_{22}\omega_{12}(e_2))e_2 \wedge e_4 \end{aligned} \tag{4.1}$$

Here, the vector fields e_1, e_2, e_3 and e_4 are the orthormal moving frame of the surface M.

Proof. In the equation (3.4), (4.1) is obtained by direct calculation using the parametric values (2.3)-(2.4). □

Remark 4.1. From lemma (4.1), the normal curvatures of simple rotational surfaces in $\mathbb{E}^4$ are always equal to zero. Therefore, if a surface M is a minimal surface or a surface whose mean curvature vector is parallel, $\triangle\nu = \|h\|^2 \nu$, and since the surface M has a pointwise 1-type Gauss map, these cases will not be studied in this paper.

Proposition 4.1. *Let M be a rotational surface in $\mathbb{E}^4$ given by (1.2) with generalized 1-type Gauss map. Then, the functions f_1, f_2 and the constant vector C in (1.3) satisfy*

$$f_1 + f_2 c_{12} = \|h\|^2, \tag{4.2}$$

$$f_2 c_{23} = -e_1(h^3_{11}) + h^3_{11}\omega_{12}(e_2) + h^3_{22}\omega_{12}(e_2), \tag{4.3}$$

$$f_2 c_{24} = -h^3_{11}\omega_{34}(e_1) + h^4_{22}\omega_{12}(e_2), \tag{4.4}$$

$$e_1(c_{12}) = -h^3_{11}c_{23}, \tag{4.5}$$

$$e_1(c_{23}) = h^3_{11}c_{12} + \omega_{34}(e_1)c_{24}, \tag{4.6}$$

$$e_1(c_{24}) = -\omega_{34}(e_1)c_{23}, \tag{4.7}$$

and

$$-h^3_{22}c_{12} + \omega_{12}(e_2)c_{23} = 0, \tag{4.8}$$

$$-h^4_{22}c_{12} + \omega_{12}(e_2)c_{24} = 0, \tag{4.9}$$

$$h^4_{22}c_{23} - h^3_{22}c_{24} = 0 \tag{4.10}$$

where the functions c_{ij}, $1 \le i < j \le 4$ are defined by (3.3).

Proof. Assume that the Gauss map $\nu = e_1 \wedge e_2$ of M is generalized 1-type, i.e., the equation (1.3) is satisfied for some differentiable functions f_1, f_2 and a constant vector C. Then, by combining (1.3) and (3.4) and inner product both sides of this equation with $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$, $e_2 \wedge e_3$, $e_2 \wedge e_4$ and $e_3 \wedge e_4$ at the same time considering that the functions h^3_{11}, h^3_{22} depend only on s and $R^D = 0$, we get (4.2)-(4.4). Also, by applying the e_1 derivative to equation (3.3), the equations (4.5)-(4.7) are obtained and by applying the e_2 derivative, we get

$$\omega_{12}(e_2)c_{23} = h^3_{22}c_{12}$$

$$\omega_{12}(e_2)c_{24} = h^4_{22}c_{12}$$

and

$$h^4_{22}c_{23} = h^3_{22}c_{24}$$

Thus, equations (4.8) and (4.9) are obtained. □

Remark 4.2. It was shown by theorem (3.2) that in Euclid space $\mathbb{E}^4$, minimal surfaces do not have a generalized 1-type Gauss map, but a second kind of pointwise 1-type Gauss map. Therefore, since minimal general general rotational surfaces in Euclid space $\mathbb{E}^4$ will not have a generalized 1-type Gauss map, the minimality will not be studied.

Now, it will be shown that the conditions (4.5)-(4.10) that must be satisfied for the vector C in the proposition (4.1) to be constant are equivalent to the lemma given below. The proof of the lemma given below will be given in a similar way to the proof given in N. C. Turgay's doctoral thesis [22].

"The surface M remain completely in $\mathbb{E}^4$" means that M is not a surface of $\mathbb{E}^3$, that is, any component of the position vector of M is not constant.

Lemma 4.2. *Let M be a simple rotational surface with a generalized 1-type Gauss map given by the parameterization (1.2) which the surface M remain completely in $\mathbb{E}^4$. In this case, for the vector C in (1.3) to be constant*

$$h^4_{22}c_{23} - h^3_{22}c_{24} = 0, \tag{4.11}$$

$$-h^4_{22}c_{12} + \omega_{12}(e_2)c_{24} = 0, \tag{4.12}$$

$$e_1(c_{24}) = -\omega_{34}(e_1)c_{23} \tag{4.13}$$

equations must be satisfied.

Proof. Since the surface M remain completely in $\mathbb{E}^4$, we have $h^4_{22} \neq 0$. Moreover, since the surface M has a generalized 1-type Gauss map, the differentiable functions f_1 and f_2 and the constant vector C satisfy the conditions (4.2)-(4.10). Here, if the equations (4.9) and (4.10) are multiplied by h^3_{22} and $\omega_{12}(e_2)$, respectively and the resulting equations are subtracted side by side

$$h^4_{22}(h^3_{22}c_{12} - \omega_{12}(e_2)c_{23}) = 0 \tag{4.14}$$

equation is found. Since $h^4_{22} \neq 0$, from (4.14) we have

$$h^3_{22}c_{12} - \omega_{12}(e_2)c_{23} = 0. \tag{4.15}$$

Thus, from (4.11) and (4.12) we obtain (4.8).

If we take the derivative of (4.11) with respect to s we get

$$e_1(h^4_{22})c_{23} + h^4_{22}e_1(c_{23}) = e_1(h^3_{22})c_{24} + h^3_{22}e_1(c_{24}).$$

If the Codazzi equations and (4.13) are used in the above equation, we have the following equation

$$(-h^4_{22}\omega_{12}(e_2) - h^3_{22}\omega_{34}(e_1))c_{23} + h^4_{22}e_1(c_{23}) = (\omega_{12}(e_2)(h^3_{11} - h^3_{22}) + h^4_{22}\omega_{34}(e_1))c_{24} - h^3_{22}\omega_{34}(e_1)c_{23}$$

is obtained. If we use equations (4.12) and (4.13) and make the necessary simplifications

$$h^4_{22}(e_1(c_{23}) - h^3_{11}c_{12} - \omega_{34}(e_1)c_{24}) = 0$$

is obtained. Hence, since $h^4_{22} \neq 0$, the equation (4.6) is obtained.

Similarly, if we take the derivative of the equation (4.12) with respect to the parameter s

$$e_1(h^4_{22})c_{12} + h^4_{22}e_1(c_{12}) = e_1(\omega_{12}(e_2))c_{24} + \omega_{12}(e_2)e_1(c_{24})$$

$$(-h^4_{22}\omega_{12}(e_2) - h^3_{22}\omega_{34}(e_1))c_{12} + h^4_{22}e_1(c_{12}) = (-(\omega_{12}(e_2))^2 - h^3_{11}h^3_{22})c_{24} + \omega_{12}(e_2)(-\omega_{34}(e_1)c_{23})$$

is obtained. In this last equation, if the necessary operations and simplifications are made using equations (4.12) and (4.15), we have

$$h^4_{22}(e_1(c_{12}) + h^3_{11}c_{23}) = 0$$

and since $h^4_{22} \neq 0$, the equation (4.5) is obtained. Thus the proof is completed. □

Theorem 4.1. *Let M be a flat simple rotational surface which is the surface M remain completely in $\mathbb{E}^4$, parametrized by (1.2) and with a profile curve that is not planar. In this case, the surface M can't have a generalized 1-type Gauss map.*

Proof. Since the simple rotational surface M is flat, $K_G = 0$, so $z'' = 0$ and can be expressed as $z(s) = a_1s + a_2$.

Moreover, since $z(s) = a_1s + a_2$, the β profile curve and its derivative are expressed as

$$\beta(s) = (x(s), y(s), a_1s + a_2, 0)$$

and

$$\beta'(s) = (x'(s), y'(s), a_1, 0)\,.$$

Since β curve is arc-length, we obtain that
$$x'^2 + y'^2 + a_1^2 = 1 \tag{4.16}$$
and since $z \neq 0$ it must be $0 < |a_1| < 1$, the equation (4.16) can be expressed as
$$x'^2 + y'^2 = 1 - a_1^2. \tag{4.17}$$
From the equation (4.17), with $\theta = \theta(s)$ a differentiable function, x' and y' are written as
$$x' = \sqrt{1 - a_1^2}\cos\theta,$$
$$y' = \sqrt{1 - a_1^2}\sin\theta.$$
Thus, β', β'' and β''' are written as
$$\beta' = \left(\sqrt{1 - a_1^2}\cos\theta, \sqrt{1 - a_1^2}\sin\theta, a_1\right),$$
$$\beta'' = \left(-\sqrt{1 - a_1^2}\theta'\sin\theta, \sqrt{1 - a_1^2}\theta'\cos\theta, 0\right)$$
and
$$\begin{aligned}\beta''' &= \left(-\sqrt{1 - a_1^2}\theta''\sin\theta, \sqrt{1 - a_1^2}\theta''\cos\theta, 0\right)\\ &\quad + \left(-\sqrt{1 - a_1^2}(\theta')^2\cos\theta, -\sqrt{1 - a_1^2}(\theta')^2\sin\theta, 0\right)\end{aligned}$$
The curvature and torsion of curve β are calculated as
$$\kappa = -\sqrt{1 - a_1^2}\theta',$$
$$\tau = a_1\theta'.$$
So we have the following relation between τ and κ
$$\tau = \frac{a_1}{\sqrt{1 - a_1^2}}\kappa.$$
Thus, the second fundemantal form and the connection forms of M can be written as
$$h^3_{11} = c(a_1 s + a_2), \quad h^3_{22} = 0, \quad h^4_{22} = -\frac{\sqrt{1 - a_1^2}}{a_1 s + a_2}, \tag{4.18}$$
$$\omega_{12}(e_2) = \frac{a_1}{a_1 s + a_2}, \qquad \omega_{34}(e_1) = \frac{a_1 c(a_1 s + a_2)}{\sqrt{1 - a_1^2}}. \tag{4.19}$$
In this case, it is found as $c_{23} = 0$ from the equation (4.3). Then, the (4.5)-(4.10) equations are reduced to the following equations
$$e_i(c_{12}) = 0, \quad e_i(c_{24}) = 0,$$
$$\omega_{12}(e_2)c_{24} - h^4_{22}c_{12} = 0, \tag{4.20}$$
$$h^3_{11}c_{12} + \omega_{34}(e_1)c_{24} = 0. \tag{4.21}$$
Accordingly, it is sufficient to provide (4.20) and (4.21). This system of two equations can be expressed in matrix form
$$\begin{pmatrix} -h^4_{22} & \omega_{12}(e_2) \\ h^3_{11} & \omega_{34}(e_1) \end{pmatrix}\begin{pmatrix} c_{12} \\ c_{24} \end{pmatrix} = 0.$$

In this matrix, since both coefficient functions c_{12} and c_{24} can not be zero, $-h^4_{22}\omega_{34}(e_1) - h^3_{11}\omega_{12}(e_2) = 0$ must be. In this case, if (4.18) and (4.19) are substituted in (4.20), we have

$$c_{24} = -\frac{\sqrt{1-a_1^2}}{a_1}c_{12}.$$

So, it is understood that both the coefficient functions c_{12} and c_{24} are nonzero constants. In this case, the differentiable functions f_1 and f_2 are found as

$$\begin{aligned} f_1 &= (1-2a_1^2)\left[\frac{c^2(a_1 s + a_2)^2}{1-a_1^2} + \frac{1}{(a_1 s + a_2)^2}\right], \\ f_2 &= a_1^2\left[\frac{c^2(a_1 s + a_2)^2}{1-a_1^2} + \frac{1}{(a_1 s + a_2)^2}\right]. \end{aligned}$$

Since $f_2 = c_0 f_1$ is obtained from these equations, the surface M has not the generalized 1-type Gauss map. $\square$

Now, we will investigate the case where simple surfaces of revolution with constant Gaussian curvature have a generalized type-1 Gaussian transformation. In [2] Arslan et al. showed that for a simple rotation surface which is the surface M remain completely in $\mathbb{E}^4$ to have a second kind of pointwise 1-type Gauss map in $\mathbb{E}^4$, it must be congruent to a helicoidal surface of revolution with the position vector

$$\begin{aligned} F(s,t) = \Bigg(& \frac{\sqrt{1-z_0^2}}{z_0(1+q_0^2)} z(s)\left(q_0 \sin(q_0 \ln|z(s)|) + \cos(q_0 \ln|z(s)|)\right), \\ & \frac{\sqrt{1-z_0^2}}{z_0(1+q_0^2)} z(s)\left(\sin(q_0 \ln|z(s)|) - q_0\cos(q_0 \ln|z(s)|)\right)s, \\ & z(s)\cos t, z(s)\sin t\Big) \end{aligned} \tag{4.22}$$

where $z(s) = z_0 s + z_1$ and $q_0 \neq 0$, z_0, $|z_0| < 1$ and z_1 are real constants. We note that this surface is flat.

Lemma 4.3. *Let M be a simple rotational surface of constant Gauss curvature, given by the position vector (1.2) which is the surface M remain completely in $\mathbb{E}^4$. A necessary and sufficient condition for the surface M to have a pointwise 1-type Gauss map of the second kind is that the rotational surface M is congruent to the surface given by the position vector (4.22). Furthermore, for $f = \frac{q_0^2 z_0^2 + 1}{(z(s))^2}$ and $C = -\sqrt{1-z_0^2}z_0 e_1 \wedge e_3 - z_0^2 e_3 \wedge e_4$, the helical rotational surface given by the position vector (4.22) satisfies the equation $\Delta\nu = f(\nu + C)$ with $\nu = e_3 \wedge e_4$ [22].*

It will now be shown by the following theorem that for a value of $K_G \neq 0$, the simple rotational surface will have a generalized 1-type Gauss map. For this, since the Gauss curvature of M surface is constant, we can take $K_G = K_0$. Hence, it can be written as $z'' = K_0 z$. Also, if necessary simplifications are made in equation (4.10) by taking the parametric values of h^3_{22} and h^4_{22}

$$\rho_3 c_{23} = z'' c_{24} \tag{4.23}$$

is obtained. Both sides of this equation are multiplied by the function f_2 and the obtained equation is used in equations (4.3) and (4.4),

$$\rho_3\left(e_1(h^3_{11}) + h^3_{11}\omega_{12}(e_2) - h^3_{22}\omega_{12}(e_2)\right) = z''\left(h^3_{11}\omega_{34}(e_1) - h^4_{22}\omega_{12}(e_2)\right) \tag{4.24}$$

is obtained. In this obtained equation (4.24), if (2.3)-(2.4) values are replaced and necessary adjustments are made,

$$\tau = \frac{(\kappa z' + \kappa' z)}{\kappa z}\frac{\rho_3}{z''} \tag{4.25}$$

is obtained. Also, using equation (4.23)

$$\tau = \frac{(\kappa z' + \kappa' z)}{\kappa z}\frac{c_{24}}{c_{23}} \tag{4.26}$$

is obtained. Thus, in equation (4.7), using equation (4.26), we have

$$e_1(c_{24}) = -\omega_{34}(e_1)c_{23} = -\frac{(\kappa z' + \kappa' z)}{\kappa z}\frac{c_{24}}{c_{23}}c_{23},$$

or

$$\frac{e_1(c_{24})}{c_{24}} = -\frac{(\kappa z' + \kappa' z)}{\kappa z} = -\left(\frac{z'}{z} + \frac{\kappa'}{\kappa}\right) \tag{4.27}$$

and c_{24} is obtained by solving the differential equation (4.27). Also, using c_{24} in equation (4.23), we obtain c_{23} and using equation (4.9), we obtain c_{12} as follows

$$c_{12} = -\frac{c_1 z'}{\rho_3 z}, \qquad c_{23} = \frac{c_1 z''}{\rho_3 \kappa z}, \qquad c_{24} = \frac{c_1}{\kappa z} \tag{4.28}$$

is obtained and the functions f_2 and f_1 are obtained by using equations (4.4) and (4.2), respectively

$$f_2 = \frac{\kappa^2 z^2 \tau + \rho_3 z'}{c_1 z} \tag{4.29}$$

and

$$f_1 = \|h\|^2 + \frac{(\kappa^2 z^2 \tau + \rho_3 z')z'}{\rho_3 z^2}. \tag{4.30}$$

In this case, when the torsion of the profile curve $\beta(s)$ is equal to (4.25), the simple rotational surface of constant Gaussian curvature has generalized 1-type Gauss map.

Now, with the help of the following theorem, it will be shown that since the torsion of the profile curve is equal to (4.25), the simple rotational surface with constant Gaussian curvature obtained with this profile curve has a generalized 1-type Gauss map.

Theorem 4.2. *Let M be a simple rotational surface in space $\mathbb{E}^4$ with $K_G = 1$ constant Gaussian curvature given by parameterization (1.2). Also, the profile curve of the surface M is a arc-length curve $\beta(s) = (x(s), y(s), \cos s, 0)$ and the coordinate functions $x(s)$ and $y(s)$ are*

$$x'^2 + y'^2 = \cos^2 s$$

and with the differentiable function $\theta = \theta(s)$ be defined as

$$x' = \cos s \cos\theta$$
$$y' = \cos s \sin\theta.$$

In this case, the necessary and sufficient condition for the surface M to have a generalized 1-type Gauss map is that the function $\theta(s)$ satisfies the differential equation

$$(\theta'^2 \cos^2 s - 1)(\theta'' \cos s - \theta' \sin s) = 0.$$

Proof. By assumption, $K_G = 1$. For this, we can take the profile curve of the surface M as $\beta(s) = (x(s), y(s), \cos s, 0)$. In this case, $\beta'(s) = (x'(s), y'(s), -\sin s, 0)$ and since the curve β has unit speed

$$x'^2 + y'^2 + \sin^2 s = 1$$
$$x'^2 + y'^2 = \cos^2 s$$

is obtained. Hence, for $\theta = \theta(s)$, x' and y' are expressed as

$$x' = \cos s \cos\theta$$
$$y' = \cos s \sin\theta.$$

In this case,

$$\begin{aligned}
\beta'(s) &= (\cos s\cos\theta, \cos s\sin\theta, -\sin s, 0)\\
\beta''(s) &= (-\sin s\cos\theta - \theta'\cos s\sin\theta, -\sin s\sin\theta + \theta'\cos s\cos\theta, -\cos s, 0)\\
\beta'''(s) &= (-\cos s\cos\theta(1+\theta'^2) + \sin\theta(2\theta'\sin s - \theta''\cos s),\\
&\quad -\cos s\sin\theta(1+\theta'^2) + \cos\theta(-2\theta'\sin\theta + \theta''\cos\theta), \sin s, 0)
\end{aligned}$$

and

$$\kappa(s) = \sqrt{1 + \theta'^2\cos^2 s}.$$

Since the torsion of the β curve is calculated as follows

$$\tau(s) = -\frac{(\beta' \wedge \beta''(s)) \cdot \beta'''(s)}{|\kappa|^2}$$

torsion of the β curve is obtained as

$$\tau(s) = \frac{\theta' \sin s(2 + \theta'^2 \cos^2 s) - \theta'' \cos s}{1 + \theta'^2 \cos^2 s}. \tag{4.31}$$

Also, according to equation (4.25), the torsion is calculated as follows

$$\tau(s) = \frac{\theta'(\sin s + \theta' \cos^2 s(2\theta' \sin s - \theta'' \cos s))}{1 + \theta'^2 \cos^2 s}. \tag{4.32}$$

By equating (4.31) and (4.32), the differential equation is obtained as follows

$$(\theta'^2 \cos^2 s - 1)(\theta'' \cos s - \theta' \sin s) = 0.$$

From the solution of this differential equation, $\theta(s)$ is found as follows

$$\theta(s) = 2arc\tanh\left(\tan\frac{s}{2}\right). \tag{4.33}$$

Thus, for the value of $\theta(s)$ in (4.33), the torsion of the profile curve is equal to (4.25), the surface M satisfies the conditions (4.28)-(4.30), and for the vector C is a constant vector, the coefficient functions c_{12}, c_{23}, c_{24} are as follows

$$c_{12} = c_1 \sec s \tan s, \quad c_{23} = -\frac{c_1 \sec s}{\sqrt{2}}, \quad c_{24} = \frac{c_1 \sec s}{\sqrt{2}} \tag{4.34}$$

and also the vector C is expressed as follows,

$$C = c_{12}e_1 \wedge e_2 + c_{23}e_2 \wedge e_3 + c_{24}e_2 \wedge e_4 \tag{4.35}$$

Since for the vector C to be constant, the coefficient functions c_{12}, c_{23} and c_{24} must satisfy the conditions (4.11)-(4.13) according to lemma (4.2). It can be easily shown that these conditions are satisfied. Thus, for the coefficients in (4.34), the constant vector C of the form (4.35) and

$$f_1 = 4 - (\sec s)^2$$
$$f_2 = \frac{\sin s}{c_1}$$

for differential functions f_1 and f_2 the simple rotational surface with Gaussian curvature $K_G = 1$ has a generalized 1-type Gauss map. □

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