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RESEARCH ARTICLE

On HD- Split Euler-Rodrigues Equations

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ABSTRACT

In this work, we introduce HD- split Euler-Rodrigues equations. First, we include the basic concepts of dual numbers, dual vectors, HD- numbers, HD- vectors and HD- split vectors, which form the basis of the study. Then we obtain HD- split Euler-Rodrigues relations for HD- unit spacelike axes and HD- unit timelike axes. Thanks to these relations, we obtain HD- split rotation matrices and we examine the relationships with the E.Study transformation defined for HD- split vectors. We also reconstruct Euler's fixed point theorem with HD- split rotation matrices. Finally, we provide extensive and interesting examples that support the theory.

Keywords: ***HD****- split vectors,* ***HD****- split Euler-Rodrigues,* ***HD****- split Euler Theorem.*
AMS Subject Classification (2020): *11R52; 53A35; 70B10; 70E17.*

1. Introduction

The dual numbers were introduced by Clifford in 1873 [3]. After Clifford's discovery, Study and Kotelnikov used dual vectors to express the lines in kinematics and rigid body motions [21, 14]. Then, the E.Study transform was defined using dual split vectors in Lorentzian-3 space [13, 22]. Dual numbers and their applications have a wide place in the literature. Examples such as dual matrices, dual Fibonacci numbers, dual space curves, and automatic differentiation can be given.

HD- numbers were defined by Fike as an extension of dual numbers [9]. Fike and Alonso used the **HD**- numbers in the first and second derivative calculations due to the reduction of the calculation time and errors [10]. The expression of a **HD**- number in terms of two dual numbers was perfoermed by Cohen and Shoham [4]. In this way, physical applications of **HD**- numbers, such as **HD**- velocity, **HD**- momentum and **HD**- inertia operators, were defined and rigid body movements were examined [5]. Cohen and Shoham interpreted the **HD**- vectors in the sense of Study and Kotelnikov.

In [1], **HD**- split vectors are defined with the help of **HD**- numbers. The E. Study transformation for **HD**- split vectors was examined by expressing the Lorentzian unit **HD**- sphere and the hyperbolic unit **HD**- sphere. Corresponding spacelike or timelike lines in the 3-dimensional Minkowski space were examined. In addition, some structures in kinematics are given by introducing **HD**- quaternions and **HD**- split quaternions in three-dimensional Euclidean and Minkowski spaces, respectively [2, 6].

The geometry of rigid body dynamics has led scientists to study Lie algebras and Lie groups. Fixed point theories are studied in many sub-branches of mathematics. Geometrically, Euler realized that any displacement of a rigid body by a point on the rigid body will remain constant in a three-dimensional space [8, 19]. Therefore, Euler's theorem shows the existence of the axis of rotation. The Euler-Rodrigues formula expresses the matrix representation of a rotation with the angle of rotation and the axis of rotation [7, 16, 18]. The dual Euler-Rodrigues formulas are given using the dual axis and the dual angle in [12], and **HD**- Euler Rodrigues formulas are given using **HD**- axis and **HD**- angle in [20].

In this study, we introduce **HD**- split Euler-Rodrigues equations by using **HD**- split matrices. Then we obtain **HD**- split Euler-Rodrigues relations for **HD**- unit spacelike axes and **HD**- unit timelike axes. Thanks to these relations, we obtain **HD**- split rotation matrices, and we examine the relationships with the E.Study

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transformation defined for **HD**- split vectors. We also reconstruct Euler's fixed-point theorem with **HD**- split rotation matrices. Finally, we provide extensive and interesting examples that support the theory.
The table below shows the symbols that will be used throughout the article.

List of symbols		List of symbols	
x_0, x_1	Real scalars	ξ	Dual unit
$\mathbf{x_0}, \mathbf{x_1}$	Real vectors	$\overline{\theta}$	Dual angle
X	Dual number	X^*	**HD**- number
$\overline{X}$	Dual split vector	$\mathcal{D}^*$	The set of all **HD**- numbers
$\mathcal{D}$	The set of all dual numbers	$\overline{X}^*$	**HD**- split vector
$\mathcal{D}^3$	Dual space	$\mathcal{D}^{*3}$	**HD**- space
$\mathcal{D}_1^3$	Dual Lorentzian space	$\mathcal{D}_1^{*3}$	**HD**- Lorentzian space
S_1^2	Lorentzian unit sphere	S_1^{*2}	Lorentzian unit **HD**- sphere
H_0^2	Hyperbolic unit sphere	H_0^{*2}	Hyperbolic unit **HD**- sphere
$\overline{S}_1^2$	Lorentzian unit dual sphere	θ^*	**HD**- angle
$\overline{H}_0^2$	Hyperbolic unit dual sphere		

2. Basic concepts

The Lorentz-Minkowski space is the metric space $\mathbb{R}_1^3 = (\mathbb{R}^3, \langle , \rangle_L)$ where the metric $\langle , \rangle_L$ is

$$\langle , \rangle_L = -dx_1^2 + dx_2^2 + dx_3^2$$

which is called the Lorentzian metric. A non-zero vector $\rho \in \mathbb{R}_1^3$ is to be spacelike, timelike, lightlike if $\langle \rho, \rho \rangle_L > 0$, $\langle \rho, \rho \rangle_L < 0$ or $\langle \rho, \rho \rangle_L = 0$ respectively. The norm of the vector $\rho \in \mathbb{R}_1^3$ is denoted by $\|\rho\|_L = \sqrt{|\langle \rho, \rho \rangle_L|}$. The vector product $\rho \wedge_L \varrho$ of $\rho \in \mathbb{R}_1^3$ and $\varrho \in \mathbb{R}_1^3$ is given by

$$\rho \wedge_L \varrho = \begin{vmatrix} -e_1 & e_2 & e_3 \\ \rho_1 & \rho_2 & \rho_3 \\ \varrho_1 & \varrho_2 & \varrho_3 \end{vmatrix}$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}_1^3$. On the other hand, Lorentzian and hyperbolic unit spheres are defined by,

$$\begin{aligned} S_1^2 &= \{\rho \in \mathbb{R}_1^3 | \quad \langle \rho, \rho \rangle_L = 1\}, \\ H_0^2 &= \{\rho \in \mathbb{R}_1^3 | \quad \langle \rho, \rho \rangle_L = -1\}. \end{aligned}$$

respectively [15].
The set of all dual numbers is

$$\mathcal{D} = \{X = x_0 + \xi x_1 : x_0, x_1 \in \mathbb{R}, \xi^2 = 0, \xi \neq 0\}. \tag{2.1}$$

The sum and product of two dual numbers are defined as follows, respectively:

$$\begin{aligned} X + Y &= (x_0 + y_0) + \xi(x_1 + y_1), \\ XY &= x_0 y_0 + \xi(x_0 y_1 + y_0 x_1). \end{aligned}$$

The set

$$\mathcal{D}^3 = \{\overline{X} = \mathbf{x_0} + \xi \mathbf{x_1} : \mathbf{x_0}, \mathbf{x_1} \in \mathbb{R}^3\}. \tag{2.2}$$

is a module on dual numbers, and each element of $\mathcal{D}^3$ is called a dual vector.
The set

$$\mathcal{D}_1^3 = \{\overline{X} = \mathbf{x_0} + \xi \mathbf{x_1} : \mathbf{x_0}, \mathbf{x_1} \in \mathbb{R}_1^3\} \tag{2.3}$$

is the dual Lorentzian space, and each element of $\mathcal{D}_1^3$ is called a dual split vector. If $\mathbf{x_0}$ is spacelike, timelike or lightlike, a dual split vector $\overline{X} = \mathbf{x_0} + \xi \mathbf{x_1} \in \mathcal{D}_1^3$ is called as spacelike, timelike or lightlike, respectively.

Let $\overline{X} = \mathbf{x_0} + \xi \mathbf{x_1}$ and $\overline{Y} = \mathbf{y_0} + \xi \mathbf{y_1}$ be dual split vectors. The Lorentzian scalar product, Lorentzian vector product, norm and modulus of dual split vectors are defined by

$$\begin{aligned} \langle \overline{X}, \overline{Y} \rangle_L &= \langle \mathbf{x_0}, \mathbf{y_0} \rangle_L + \xi(\langle \mathbf{x_0}, \mathbf{y_1} \rangle_L + \langle \mathbf{y_0}, \mathbf{x_1} \rangle_L) \in \mathcal{D}, \\ \overline{X} \wedge_L \overline{Y} &= \mathbf{x_0} \wedge_L \mathbf{y_0} + \xi(\mathbf{x_0} \wedge_L \mathbf{y_1} + \mathbf{y_0} \wedge_L \mathbf{x_1}) \in \mathcal{D}_1^3, \\ N_{\overline{X}} &= \langle \overline{X}, \overline{X} \rangle_L = \langle \mathbf{x_0}, \mathbf{x_0} \rangle_L + 2\xi \langle \mathbf{x_0}, \mathbf{x_1} \rangle_L, \\ \|\overline{X}\|_L &= \|\mathbf{x_0}\|_L + \xi \frac{\langle \mathbf{x_0}, \mathbf{x_1} \rangle_L}{\|\mathbf{x_0}\|_L}, \quad (\|\mathbf{x_0}\|_L \neq 0), \end{aligned}$$

respectively. If $\langle \overline{X}, \overline{X} \rangle_L = \pm 1$ $(\langle \mathbf{x_0}, \mathbf{x_0} \rangle_L \pm 1$ and $\langle \mathbf{x_0}, \mathbf{x_1} \rangle_L = 0)$, then $\overline{X}$ is called a unit dual split vector. Moreover, Lorentzian unit dual sphere and hyperbolic unit dual sphere are defined by

$$\begin{aligned} \overline{S}_1^2 &= \{\overline{X} = \mathbf{x_0} + \xi \mathbf{x_1} \in \mathcal{D}_1^3 | \quad \langle \overline{X}, \overline{X} \rangle_L = 1\}, \\ \overline{H}_0^2 &= \{\overline{X} = \mathbf{x_0} + \xi \mathbf{x_1} \in \mathcal{D}_1^3 | \quad \langle \overline{X}, \overline{X} \rangle_L = -1\}, \end{aligned}$$

respectively [3, 21, 14, 13, 22].

Theorem 2.1 (E. Study Mapping for Dual Split Vectors). *Each point of the Lorentzian unit dual sphere $\overline{S}_1^2$ corresponds to a directed spacelike line and each point of the hyperbolic unit dual sphere $\overline{H}_0^2$ corresponds to a directed timelike line [21].*

Definition 2.1. a.) The Lorentzian scalar product of unit dual spacelike vectors $\overline{X}$ and $\overline{Y}$ is defined by

$$\langle \overline{X}, \overline{Y} \rangle_L = \cos\theta_0 - \xi\theta_1 \sin\theta_0 = \cos\overline{\theta}$$

where $\overline{\theta} = \theta_0 + \xi\theta_1 \in \mathcal{D}$. The dual number $\overline{\theta} = \theta_0 + \xi\theta_1 \in \mathcal{D}$ is called the dual angle.

b.) The Lorentzian scalar product of unit dual timelike vectors $\overline{X}$ and $\overline{Y}$ is defined by

$$\langle \overline{X}, \overline{Y} \rangle_L = -\cosh\theta_0 - \xi\theta_1 \sinh\theta_0 = -\cosh\overline{\theta}$$

where $\overline{\theta} = \theta_0 + \xi\theta_1 \in \mathcal{D}$. The dual number $\overline{\theta} = \theta_0 + \xi\theta_1 \in \mathcal{D}$ is called the dual hyperbolic angle [13, 22].

2.1. *HD-split vectors*

Before defining **HD-** split vectors in this section, let's give **HD-** numbers, **HD-** vectors and examine their algebraic properties:
The set of all **HD-** numbers is defined by

$$\mathcal{D}^* = \{X^* = x_0 + \xi_1 x_1 + \xi_2 x_2 + \xi_1\xi_2 x_3 : x_0, x_1, x_2, x_3 \in \mathbb{R}\}$$

where $\xi_1^2 = \xi_2^2 = (\xi_1\xi_2)^2 = 0$ and $\xi_1 \neq \xi_2$, $\xi_1 \neq 0$, $\xi_2 \neq 0$, $\xi_1\xi_2 = \xi_2\xi_1 \neq 0$.
A **HD-** number $X^* = x_0 + \xi_1 x_1 + \xi_2 x_2 + \xi_1\xi_2 x_3$ can be constructed with two dual numbers as follows [4, 5]:

$$\begin{aligned} X^* &= x_0 + \xi_1 x_1 + \xi_2 x_2 + \xi_1\xi_2 x_3, \\ &= (x_0 + \xi_1 x_1) + \xi_2(x_2 + \xi_1 x_3), \\ &= (x_0 + \xi x_1) + \varepsilon^*(x_2 + \xi x_3), \\ &= X_0 + \varepsilon^* X_1. \end{aligned}$$

The sum and product of two **HD-** dual numbers $X^* = X_0 + \varepsilon^* X_1$ and $Y^* = Y_0 + \varepsilon^* Y_1$ are defined as follows, respectively:

$$\begin{aligned} X^* + Y^* &= (X_0 + Y_0) + \varepsilon^*(X_1 + Y_1), \\ X^* Y^* &= X_0 Y_0 + \varepsilon^*(X_0 Y_1 + Y_0 X_1). \end{aligned}$$

The set of all **HD-** vectors is defined by

$$\mathcal{D}^{*3} = \{\overline{X}^* = \overline{X}_0 + \varepsilon^* \overline{X}_1 : \overline{X}_0, \overline{X}_1 \in \mathcal{D}^3\}. \tag{2.4}$$

and each element of $\mathcal{D}^{*3}$ is called a **HD**- vector.
The set

$$\mathcal{D}_1^{*3} = \{\overline{X}^* = \overline{X}_0 + \varepsilon^* \overline{X}_1 : \overline{X}_0, \overline{X}_1 \in \mathcal{D}_1{}^3\}. \tag{2.5}$$

is the **HD**- Lorentzian space, and each element of $\mathcal{D}_1^{*3}$ is called a **HD**- split vector. If $\overline{X}_0$ is spacelike, timelike or lightlike, a **HD**- split vector $\overline{X}^* = \overline{X}_0 + \varepsilon^* \overline{X}_1 \in \mathcal{D}_1^{*3}$ is called as spacelike, timelike or lightlike, respectively. Let $\overline{X}^* = \overline{X}_0 + \varepsilon^* \overline{X}_1$ and $\overline{X}^* = \overline{Y}_0 + \varepsilon^* \overline{Y}_1$ are **HD**- split vectors. The Lorentzian scalar product, Lorentzian vector product, norm and modulus of **HD**- split vectors are defined by

$$\begin{aligned}
\langle \overline{X}^*, \overline{Y}^* \rangle_L &= \langle \overline{X}_0, \overline{Y}_0 \rangle_L + \xi(\langle \overline{X}_0, \overline{Y}_1 \rangle_L + \langle \overline{Y}_0, \overline{X}_1 \rangle_L) \in \mathcal{D}^*, \\
\overline{X}^* \wedge_L \overline{Y}^* &= \overline{X}_0 \wedge_L \overline{Y}_0 + \xi(\overline{X}_0 \wedge_L \overline{Y}_1 + \overline{Y}_0 \wedge_L \overline{X}_1) \in \mathcal{D}_1^{*3} \\
N_{\overline{X}^*} &= \langle \overline{X}^*, \overline{X}^* \rangle_L = \langle \overline{X}_0, \overline{X}_0 \rangle_L + 2\varepsilon^* \langle \overline{X}_0, \overline{X}_1 \rangle_L, \\
\|\overline{X}^*\|_L &= \|\overline{X}_0\|_L + \varepsilon^* \frac{\langle \overline{X}_0, \overline{X}_1 \rangle_L}{\|\overline{X}_0\|_L}, \quad (\|\overline{X}_0\|_L \neq 0),
\end{aligned}$$

respectively. If $\langle \overline{X}^*, \overline{X}^* \rangle_L = \pm 1$ ($\langle \overline{X}_0, \overline{X}_0 \rangle_L \pm 1$ and $\langle \overline{X}_0, \overline{X}_1 \rangle_L = 0$), then $\overline{X}^*$ is called a unit **HD**- split vector. Moreover, Lorentzian unit **HD**- sphere and hyperbolic unit **HD**- sphere are defined by

$$\begin{aligned}
S_1^{*2} &= \{\overline{X}^* = \overline{X}_0 + \varepsilon^* \overline{X}_1 \in \mathcal{D}_1^{*3} | \quad \langle \overline{X}^*, \overline{X}^* \rangle_L = 1\}, \\
H_0^{*2} &= \{\overline{X}^* = \overline{X}_0 + \varepsilon^* \overline{X}_1 \in \mathcal{D}_1^{*3} | \quad \langle \overline{X}^*, \overline{X}^* \rangle_L = -1\},
\end{aligned}$$

respectively [1].

Theorem 2.2 (E. Study Mapping for **HD**- Split Vectors)**.** *Each point of the Lorentzian unit **HD**- sphere S_1^{*2} corresponds to a directed dual spacelike line and each point of the hyperbolic unit **HD**- sphere H_0^{*2} corresponds to a directed dual timelike line [1].*

Theorem 2.3.

1.) *Let $\overline{X}^* = \overline{X}_0 + \varepsilon^* \overline{X}_1$ be a **HD**- split vector with $\overline{X}_0, \overline{X}_1 \in \overline{S_1}^2$. There exists a one to one correspondence between the points of S_1^{*2} and any two intersecting perpendicular directed spacelike lines in $\mathbb{R}_1^3$.*

2.) *Let $\overline{X}^* = \overline{X}_0 + \varepsilon^* \overline{X}_1$ be a **HD**- split vector with $\overline{X}_0 \in \overline{S}_1^2, \overline{X}_1 \in \overline{H}_0^2$. There exists a one-to-one correspondence between the points of S_1^{*2} and any two intersecting perpendicular directed lines in $\mathbb{R}_1^3$ such that one of these lines (corresponding to the unit dual spacelike vector $\overline{X}_0$) is spacelike and the other line (corresponding to the unit dual timelike vector $\overline{X}_1$) is timelike.*

3.) *Let $\overline{X}^* = \overline{X}_0 + \varepsilon^* \overline{X}_1$ be a **HD**- split vector with $\overline{X}_0 \in \overline{H_0}^2, \overline{X}_1 \in \overline{S_1}^2$. There exists a one-to-one correspondence between the points of H_0^{*2} and any two intersecting perpendicular directed lines in $\mathbb{R}_1^3$ such that one of these lines (corresponding to unit dual timelike vector $\overline{X}_0$) is timelike and the other line (corresponding to unit dual spacelike vector $\overline{X}_1$) is spacelike [1].*

Definition 2.2. a.) The Lorentzian scalar product of unit **HD**- spacelike vectors $\overline{X}^*$ and $\overline{Y}^*$ is defined by

$$\langle \overline{X}^*, \overline{Y}^* \rangle_L = \cos \overline{\theta}_0 + \varepsilon^* \overline{\theta}_1 \sin \overline{\theta}_0 = \cos \theta^*$$

where $\theta^* = \overline{\theta}_0 + \varepsilon^* \overline{\theta}_1 \in \mathcal{D}^*$. The **HD**- number $\theta^* = \overline{\theta}_0 + \varepsilon^* \overline{\theta}_1 \in \mathcal{D}^*$ is called **HD**- angle.

b.) The Lorentzian scalar product of unit **HD**- timelike vectors $\overline{X}^*$ and $\overline{Y}^*$ is defined by

$$\langle \overline{X}^*, \overline{Y}^* \rangle_L = -\cosh \overline{\theta}_0 - \varepsilon^* \overline{\theta}_0 \sinh \overline{\theta}_0 = -\cosh \theta^*$$

where $\theta^* = \overline{\theta}_0 + \varepsilon^* \overline{\theta}_1 \in \mathcal{D}^*$. The **HD**- number $\theta^* = \overline{\theta}_0 + \varepsilon^* \overline{\theta}_1 \in \mathcal{D}^*$ is called **HD**- hyperbolic angle [1].

3. HD-Split Euler-Rodrigues equations

The basic concepts in this section are based on studies [17] and [11]. Let us examine **HD**- semi-orthogonal matrices using the structure of Minkowski space to give the Euler-Rodrigues equations for **HD**- split vectors. Let

$$O_1^*(3) = \{\overline{Y}^* \in \mathcal{D}_3^{*3} | \quad (\overline{Y}^*)^T \varepsilon \overline{Y}^* \varepsilon = \overline{Y}^* \varepsilon (\overline{Y}^*)^T \varepsilon = I_3\}$$

be the set of all **HD**- semi-orthogonal matrices where

$$\varepsilon = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here $\overline{Y}^*$ is a **HD**- matrix represented by $\overline{Y}^* = \left[\overline{Y}^*_{ij}\right] : \overline{Y}^*_{ij} = \overline{Y}_{ij} + \varepsilon^* \overline{Z}_{ij} \in \mathcal{D}^*$ where $\overline{Y}, \overline{Z}$ are dual matrices. If the **HD**- semi-orthogonal matrix $\overline{Y}^*$ satisfies the condition $det\overline{Y}^* = 1$, it forms a Lorentzian Lie group structure and is denoted by $SO_1^*(3)$. From the structure of Lorentzian Lie algebra, **HD**- semi-skew symmetric matrices are explained by

$$so_1^*(3) = \{Z^* \in \mathcal{D}_3^{*3} | \quad (Z^*)^T = -\varepsilon Z^* \varepsilon\}.$$

An isomorphism between $so_1^*(3)$ and the set of **HD**- split vectors can be given as follows:

$$\begin{aligned} g \quad &: \quad so_1^*(3) \to \mathcal{D}_1^{*3}, \\ \mathbf{S}^* \quad &= \quad \begin{pmatrix} 0 & s_z^* & -s_y^* \\ s_z^* & 0 & -s_x^* \\ -s_y^* & s_x^* & 0 \end{pmatrix} \to g(\mathbf{S}^*) = S^* = (s_x^*, s_y^*, s_z^*) \end{aligned} \tag{3.1}$$

Using the Lorentzian vector product of **HD**- split vectors, $\mathbf{S}^*\overline{Y}^* = S^* \otimes_L \overline{Y}^*$ can be written. Now let us define $\mathbf{R}^*$, which we will call **HD**- split rotation, for the angles and axes defined for **HD**- split vectors. Let us examine the cases where the rotation axis is a **HD**- spacelike unit axis and a **HD**- timelike unit axis:

1. If S^* is a **HD**- spacelike unit axis and θ^* is a **HD**- hyperbolic angle, we get

$$\mathbf{R}^* = I_3 + \sinh\theta^* \mathbf{S}^* + (-1 + \cosh\theta^*)\mathbf{S}^{*2} \tag{3.2}$$

   where $\mathbf{S}^*$ is a semi-skew symmetric matrix of the axis $S^* = (s_x^*, s_y^*, s_z^*)$. Then, we have

$$\mathbf{R}^* = \begin{pmatrix} 1 + (-1+\cosh\theta^*)(s_y^{*2} + s_z^{*2}) & (1-\cosh\theta^*)s_x^* s_y^* + \sinh\theta^* s_z^* & (1-\cosh\theta^*)s_x^* s_z^* - \sinh\theta^* s_y^* \\ (-1+\cosh\theta^*)s_x^* s_y^* + \sinh\theta^* s_z^* & 1 + (-1+\cosh\theta^*)(-s_x^{*2} + s_z^{*2}) & (1-\cosh\theta^*)s_y^* s_z^* - \sinh\theta^* s_x^* \\ (-1+\cosh\theta^*)s_x^* s_z^* - \sinh\theta^* s_y^* & (1-\cosh\theta^*)s_y^* s_z^* + \sinh\theta^* s_x^* & 1 + (-1+\cosh\theta^*)(-s_x^{*2} + s_y^{*2}) \end{pmatrix}$$

2. If S^* is a **HD**- timelike unit axis and θ^* is a **HD**- angle, we get

$$\mathbf{R}^* = I_3 + \sin\theta^* \mathbf{S}^* + (1 - \cos\theta^*)\mathbf{S}^{*2} \tag{3.3}$$

   where $\mathbf{S}^*$ is a semi-skew symmetric matrix of the axis $S^* = (s_x^*, s_y^*, s_z^*)$. Then, we have

$$\mathbf{R}^* = \begin{pmatrix} 1 + (1-\cos\theta^*)(s_y^{*2} + s_z^{*2}) & (-1+\cos\theta^*)s_x^* s_y^* + \sin\theta^* s_z^* & (-1+\cos\theta^*)s_x^* s_z^* - \sin\theta^* s_y^* \\ (1-\cos\theta^*)s_x^* s_y^* + \sin\theta^* s_z^* & 1 + (1-\cos\theta^*)(-s_x^{*2} + s_z^{*2}) & (-1+\cos\theta^*)s_y^* s_z^* - \sin\theta^* s_x^* \\ (1-\cos\theta^*)s_x^* s_z^* - \sin\theta^* s_y^* & (-1+\cos\theta^*)s_y^* s_z^* + \sin\theta^* s_x^* & 1 + (1-\cos\theta^*)(-s_x^{*2} + s_y^{*2}) \end{pmatrix}$$

Theorem 3.1. *Let the* **HD**- *hyperbolic angle and the* **HD**- *spacelike vector be* $\theta^* = \overline{\theta}_0 + \varepsilon^*\overline{\theta}_1 \in \mathcal{D}^*$ *and* $S^* = \overline{S}_0 + \varepsilon^*\overline{S}_1 \in S_1^{*2}$, *respectively. Then, a* **HD**- *split rotation with a* **HD**- *spacelike unit axis is expressed by*

$$\mathbf{R}^* = \overline{\mathbf{C}} + \varepsilon^*\overline{\mathbf{D}}$$

Here, the dual part of $\mathbf{R}^*$ *determines a rotation along with the dual spacelike axis* $\overline{S}_0$ *and the dual hyperbolic angle* $\overline{\theta}_0$. *The hyper part of* $\mathbf{R}^*$ *is determined as* $\overline{\mathbf{D}} = \overline{\mathbf{PC}}$ *for a dual semi-skew-symmetric matrix* $\overline{\mathbf{P}}$.

Proof. We can write $g(\mathbf{S}^*) = S^*$ with the isomorphism g. Then, we have $g(\overline{\mathbf{S}}_0) = \overline{S}_0$, $g(\overline{\mathbf{S}}_1) = \overline{S}_1$, and $\mathbf{S}^* = \overline{\mathbf{S}}_0 + \varepsilon^*\overline{\mathbf{S}}_1$. If equations S^* and $\cosh\theta^* = \cosh\overline{\theta}_0 + \varepsilon^*\overline{\theta}_1 \sinh\overline{\theta}_0$ are written in the **HD**- spacelike unit axis rotation matrix, $\mathbf{R}^*$ is written as $\overline{\mathbf{C}} + \varepsilon^*\overline{\mathbf{D}}$ as the sum of two dual matrices. From equation $\mathbf{R}^*\varepsilon(\mathbf{R}^*)^T\varepsilon = I_3$, we get

$$\overline{\mathbf{C}}\varepsilon\overline{\mathbf{C}}^T\varepsilon + \varepsilon^*\left(\overline{\mathbf{C}}\varepsilon\overline{\mathbf{D}}^T\varepsilon + \overline{\mathbf{D}}\varepsilon\overline{\mathbf{C}}^T\varepsilon\right) = I_3 \tag{3.4}$$

where

$$\varepsilon = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From equation (3.4), we can write

$$\overline{\mathbf{C}}\varepsilon\overline{\mathbf{C}}^T\varepsilon = I_3 \quad and \quad \overline{\mathbf{C}}\varepsilon\overline{\mathbf{D}}^T\varepsilon + \overline{\mathbf{D}}\varepsilon\overline{\mathbf{C}}^T\varepsilon = 0. \tag{3.5}$$

Therefore, $\overline{\mathbf{C}}$ is a dual semi-orthogonal matrix. Since $\overline{S}_0$ is a unit dual spacelike vector, by using equation (3.2), we get

$$\overline{\mathbf{C}} = I_3 + \sinh\overline{\theta}_0\overline{\mathbf{S}}_0 + (-1+\cosh\overline{\theta}_0)\overline{\mathbf{S}}_0{}^2 \tag{3.6}$$

It is clear that $\overline{\mathbf{C}}$ determines a dual split rotation with dual spacelike axis $\overline{S}_0$ and dual hyperbolic angle $\overline{\theta}_0$. On the other hand, let us assume that $\overline{\mathbf{D}}\varepsilon(\overline{\mathbf{C}})^T\varepsilon = \overline{\mathbf{P}}$. From equation (3.5), we get $(\overline{\mathbf{P}})^T = -\varepsilon\overline{\mathbf{P}}\varepsilon$. □

Theorem 3.2. *Let the **HD**- angle and the **HD**- timelike vector be $\theta^* = \overline{\theta}_0 + \varepsilon^*\overline{\theta}_1 \in \mathcal{D}^*$ and $S^* = \overline{S}_0 + \varepsilon^*\overline{S}_1 \in H_0^{*2}$, respectively. Then, a **HD**- split rotation with a **HD**- timelike unit axis is expressed by*

$$\mathbf{R}^* = \overline{\mathbf{E}} + \varepsilon^*\overline{\mathbf{F}}$$

Here, the dual part of $\mathbf{R}^*$ determines a rotation along with the dual timelike axis $\overline{S}_0$ and the dual angle $\overline{\theta}_0$. The hyper part of $\mathbf{R}^*$ is determined as $\overline{\mathbf{F}} = \overline{\mathbf{P}}\overline{\mathbf{E}}$ for a dual semi-skew-symmetric matrix $\overline{\mathbf{P}}$.

Proof. The proof can be done in a similar way to the proof of theorem 3.1. □

It should be known that $\mathbf{R}^*$ inherits the property that $\mathbf{R}^*(\overline{X}^*) = \overline{Y}^*$ where $\overline{X}^*$ and $\overline{Y}^*$ are any two **HD**- split vectors in $\mathcal{D}_1^{*3}$ such that $\langle\overline{X}^*, \overline{X}^*\rangle_L = \langle\overline{Y}^*, \overline{Y}^*\rangle_L > 0$ or $\langle\overline{X}^*, \overline{X}^*\rangle_L = \langle\overline{Y}^*, \overline{Y}^*\rangle_L < 0$. Consequently, we can give the following.

Proposition 3.1. *The following expressions are given from the E. Study mapping for unit **HD**- split vectors:*

1. *If $\overline{X}^*$ is a **HD**- unit spacelike vector, then the **HD**- split rotation $\mathbf{R}^*(\overline{X}^*) = \overline{Y}^*$ expresses the transforming of the corresponding dual spacelike line of $\overline{X}^*$ to the corresponding dual spacelike line of $\overline{Y}^*$.*
2. *If $\overline{X}^*$ is a **HD**- unit timelike vector, then the **HD**- split rotation $\mathbf{R}^*(\overline{X}^*) = \overline{Y}^*$ expresses the transformation of the corresponding dual time-like line of $\overline{X}^*$ to the corresponding dual timelike line of $\overline{Y}^*$.*

Proof. 1. From Theorem 2.2, if $\overline{X}^*$ is a **HD**- unit spacelike vector, then corresponds to a directed dual spacelike line. Then $\mathbf{R}^*(\overline{X}^*) = \overline{Y}^*$ is a dual spacelike line with the condition $\langle\overline{X}^*, \overline{X}^*\rangle_L = \langle\overline{Y}^*, \overline{Y}^*\rangle_L = 1$.
2. From Theorem 2.2, if $\overline{X}^*$ is a **HD**- unit timelike vector, then corresponds to a directed dual timelike line. Then $\mathbf{R}^*(\overline{X}^*) = \overline{Y}^*$ is a dual timelike line with the condition $\langle\overline{X}^*, \overline{X}^*\rangle_L = \langle\overline{Y}^*, \overline{Y}^*\rangle_L = -1$.

□

Suppose that $\overline{X}^* = \overline{X}_0 + \varepsilon^*\overline{X}_1$ and $\overline{Y}^* = \overline{Y}_0 + \varepsilon^*\overline{Y}_1$ are **HD**- unit spacelike (timelike) vectors satisfying $\mathbf{R}^*(\overline{X}^*) = \overline{Y}^*$ with the **HD**- split rotation $\mathbf{R}^*$ along the **HD**- angle (**HD**- hyperbolic angle) $\theta^* = \overline{\theta}_0 + \varepsilon^*\overline{\theta}_1$ and the **HD**- timelike unit axis (**HD**- spacelike unit axis) $S^* = (s_x^*, s_y^*, s_z^*)$. If $\overline{L}_{\overline{X}^*}$ and $\overline{L}_{\overline{Y}^*}$ are the corresponding dual spacelike (timelike) lines to $\overline{X}^*$ and $\overline{Y}^*$ respectively in $\mathcal{D}_1^3$, then we obtain that $\overline{\theta}_0$ is the dual angle (dual hyperbolic angle) between the dual spacelike (timelike) vectors $\overline{X}_0$ and $\overline{Y}_0$ and $\overline{\theta}_1$ is the closest distance between $\overline{L}_{\overline{X}^*}$ and $\overline{L}_{\overline{Y}^*}$.
Therefore, we can give the following theorems for both cases:

Theorem 3.3. *The **HD**- split rotation $\mathbf{R}^*$ generated by the **HD- angle** $\theta^* = \overline{\theta}_0 + \varepsilon^*\overline{\theta}_1$ and the **HD- timelike unit axis** S^* is determined by a screw motion which is the combination of a dual rotation by **dual angle** $\overline{\theta}_0$ about $\overline{L}_{S^*}$ and a translation by $\overline{\theta}_1$ along $\overline{L}_{S^*}$ that $\overline{L}_{S^*}$ is the **dual timelike line** of the **HD- timelike unit axis** of $\mathbf{R}^*$ in $\mathcal{D}_1^3$.*

Theorem 3.4. *The **HD**- split rotation $\mathbf{R}^*$ generated by the **HD- hyperbolic angle** $\theta^* = \overline{\theta}_0 + \varepsilon^*\overline{\theta}_1$ and the **HD- spacelike unit axis** S^* is determined by a screw motion which is the combination of a dual rotation by **dual hyperbolic angle** $\overline{\theta}_0$ about $\overline{L}_{S^*}$ and a translation by $\overline{\theta}_1$ along $\overline{L}_{S^*}$ that $\overline{L}_{S^*}$ is the **dual spacelike line** of the **HD- spacelike unit axis** of $\mathbf{R}^*$ in $\mathcal{D}_1^3$.*

On the other hand, let us assume that $\overline{X}^* = \overline{X}_0 + \varepsilon^*\overline{X}_1$ is a unit **HD**- spacelike vector. If $\langle \overline{X}_1, \overline{X}_1\rangle_L = -1$, there exists a one to one correspondence between the points $\overline{X}^* = \overline{X}_0 + \varepsilon^*\overline{X}_1$ and any two intersecting perpendicular directed lines in $\mathbb{R}^3_1$ such that one of these lines (corresponding to unit dual spacelike vector $\overline{X}_0$) is spacelike and the other line (corresponding to unit dual timelike vector $\overline{X}_1$) is timelike [1]. If $\langle \overline{X}_1, \overline{X}_1\rangle_L = 1$, there exists a one to one correspondence between the points $\overline{X}^* = \overline{X}_0 + \varepsilon^*\overline{X}_1$ and any two intersecting perpendicular directed spacelike lines in $\mathbb{R}^3_1$ [1]. Assume that $\overline{X}^* = \overline{X}_0 + \varepsilon^*\overline{X}_1$ is a unit **HD**- timelike vector. Then, there exists a one to one correspondence between the points $\overline{X}^* = \overline{X}_0 + \varepsilon^*\overline{X}_1$ and any two intersecting perpendicular directed lines in $\mathbb{R}^3_1$ such that one of these lines (corresponding to unit dual timelike vector $\overline{X}_0$) is timelike and the other line (corresponding to unit dual spacelike vector $\overline{X}_1$) is spacelike [1].

Theorem 3.5. *For **HD**- split rotation $\mathbf{R}^*$, one of the following conditions applies:*

1. *The **HD**- split rotation $\mathbf{R}^*$ converts any two intersecting perpendicular directed spacelike lines to two intersecting perpendicular directed spacelike lines in $\mathbb{R}^3_1$.*
2. *The **HD**- split rotation $\mathbf{R}^*$ converts any two intersecting perpendicular directed spacelike line and timelike line to two intersecting perpendicular directed spacelike and timelike line in $\mathbb{R}^3_1$, respectively.*

Example 3.1. Let $\overline{X}^* = (\xi + \varepsilon^*, 1 + \xi\varepsilon^*, -\xi + \xi\varepsilon^*)$ and $\overline{Y}^* = (-\xi - \varepsilon^*, \xi + \xi\varepsilon^*, 1 + \xi\varepsilon)$ be **HD**- split vectors. Since $\langle \overline{X}^*, \overline{X}^*\rangle_L = 1$ and $\langle \overline{Y}^*, \overline{Y}^*\rangle_L = 1$, $\overline{X}^*$ and $\overline{Y}^*$ are **HD**- spacelike vectors. Then, we get

$$\begin{aligned} S^* &= \overline{X}^* \otimes_L \overline{Y}^* \\ &= (-1 - 2\xi\varepsilon^*, -\xi + \xi\varepsilon^* - \varepsilon^*, \xi + \xi\varepsilon^* + \varepsilon^*) \end{aligned} \tag{3.7}$$

where $\overline{S}_0 = (-1, -\xi, \xi)$ and $\overline{S}_1 = (-2\xi, \xi - 1, \xi + 1)$. Since $\langle S^*, S^*\rangle_L = -1$, S^* is a **HD**- timelike vector. By using (3.1), its corresponding semi-skew symmetric matris is explained

$$\mathbf{S}^* = \begin{bmatrix} 0 & \xi + \xi\varepsilon^* + \varepsilon^* & \xi - \xi\varepsilon^* + \varepsilon^* \\ \xi + \xi\varepsilon^* + \varepsilon^* & 0 & 1 + 2\xi\varepsilon^* \\ \xi - \xi\varepsilon^* + \varepsilon^* & -1 - 2\xi\varepsilon & 0 \end{bmatrix}$$

Therefore, we have

$$(\mathbf{S}^*)^2 = \begin{bmatrix} 4\xi\varepsilon^* & -\xi + \xi\varepsilon^* - \varepsilon^* & \xi + \xi\varepsilon^* + \varepsilon^* \\ \xi - \xi\varepsilon^* + \varepsilon^* & -1 - 2\xi\varepsilon^* & 2\xi\varepsilon^* \\ -\xi - \xi\varepsilon^* - \varepsilon^* & 2\xi\varepsilon & -1 - 2\xi\varepsilon^* \end{bmatrix}$$

Since $\overline{X}^*$ and $\overline{Y}^*$ are **HD**- spacelike vectors, by using of **HD**- angle $\theta^* = \overline{\theta}_0 + \varepsilon^*\overline{\theta}_1$, we get

$$\cos\theta^* = 4\xi\varepsilon^*, \quad \sin\theta^* = 1.$$

By according to **HD**- split Euler-Rodrigues equations for timelike axis, we have

$$\begin{aligned} \mathbf{R}^* &= I_3 + \mathbf{S}^* + (1 - 4\xi\varepsilon^*)\mathbf{S}^{*2} \\ \mathbf{R}^* &= \begin{bmatrix} 1 + 4\xi\varepsilon^* & 2\xi\varepsilon^* & 2\xi + 2\varepsilon^* \\ 2\xi + 2\varepsilon^* & 2\xi\varepsilon^* & 1 + 4\xi\varepsilon^* \\ -2\xi\varepsilon^* & -1 & 2\xi\varepsilon^* \end{bmatrix}. \end{aligned}$$

Therefore, we get

$$\overline{\mathbf{E}} = \begin{bmatrix} 1 & 0 & 2\xi \\ 2\xi & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \textit{and} \quad \overline{\mathbf{F}} = \begin{bmatrix} 4\xi & 2\xi & 2 \\ 2 & 2\xi & 4\xi \\ -2\xi & 0 & 2\xi \end{bmatrix}$$

where $\mathbf{R}^* = \overline{\mathbf{E}} + \varepsilon^*\overline{\mathbf{F}}$. It is clear that $\mathbf{R}^*(S^*) = S^*$, $\overline{\mathbf{E}}(\overline{S}_0) = \overline{S}_0$ and the dual rotation angle of $\overline{E}$ is $\overline{\theta}_0 = \frac{\pi}{2}$. On the other hand, $\overline{\mathbf{F}} = \overline{\mathbf{PE}}$ is expressed by the dual semi-skew symmetric matrix

$$\overline{\mathbf{P}} = \begin{bmatrix} 0 & 2 & -2\xi \\ 2 & 0 & -2\xi \\ -2\xi & 2\xi & 0 \end{bmatrix}$$

We can write

i. $\overline{X}_0 = (\xi, 1, -\xi)$ and $\overline{X}_1 = (1, \xi, \xi)$ for $\overline{X}^* = \overline{X}_0 + \varepsilon^* \overline{X}_1$,
ii. $\overline{Y}_0 = (-\xi, \xi, 1)$ and $\overline{Y}_1 = (-1, \xi, \xi)$ for $\overline{Y}^* = \overline{Y}_0 + \varepsilon^* \overline{Y}_1$,
iii. $\overline{S}_0 = (-1, -\xi, \xi)$ and $\overline{S}_1 = (-2\xi, \xi - 1, \xi + 1)$ for $S^* = \overline{S}_0 + \varepsilon^* \overline{S}_1$

and we have

$$\langle \overline{X}_0, \overline{X}_0 \rangle_L = 1, \quad \langle \overline{Y}_0, \overline{Y}_0 \rangle_L = 1, \quad \langle \overline{S}_0, \overline{S}_0 \rangle_L = -1$$

Therefore, $\overline{X}_0$ and $\overline{Y}_0$ are unit dual spacelike vectors and $\overline{S}_0$ is unit dual timelike vector. On the other hand, $\langle \overline{X}_0, \overline{X}_1 \rangle_L = 0$, $\langle \overline{Y}_0, \overline{Y}_1 \rangle_L = 0$ and $\langle \overline{S}_0, \overline{S}_1 \rangle_L = 0$, we get $\overline{X}^* \in S_1^{*2}$, $\overline{Y}^* \in S_1^{*2}$ and $S^* \in H_0^{*2}$. Each point of the Lorentz unit **HD**- sphere corresponds to a directed spacelike line, and each point of the hyperbolic unit **HD**- sphere corresponds to a directional timelike line. We can express lines as follows:

$$\begin{aligned} \overline{X}^* &\longleftrightarrow \overline{L}_{\overline{X}^*} = (-\xi, -\xi, -1) + \overline{t}_x(\xi, 1, -\xi) \\ \overline{Y}^* &\longleftrightarrow \overline{L}_{\overline{Y}^*} = (\xi, -1, \xi) + \overline{t}_y(-\xi, \xi, 1) \\ S^* &\longleftrightarrow \overline{L}_{S^*} = (0, \xi + 1, -\xi + 1) + \overline{t}_s(-1, -\xi, \xi) \end{aligned}$$

where $\overline{t}_x, \overline{t}_y, \overline{t}_z \in \mathcal{D}$. Geometrically, it rotates the direction lines $\mathbf{R}^*$, the dual spacelike line $\overline{L}_{\overline{X}^*}$ and dual spacelike line $\overline{L}_{\overline{Y}^*}$ along the dual timelike line $\overline{L}_{S^*}$ at an angle $\overline{\theta}_0 = \dfrac{\pi}{2}$ and applies a translation $\overline{\theta}_1 = 4\xi$. Since $\overline{X}_1 \in H_0^{*2}$, $\overline{X}^* \in S_1^{*2}$, $\overline{Y}_1 \in H_0^{*2}$ and $\overline{Y}^* \in S_1^{*2}$, the points of $\overline{X}^*$ and $\overline{Y}^*$ correspond exactly to any two intersecting perpendicular lines. In $\mathbb{R}_1^3$, one of these lines is spacelike and the other is timelike.

$$\overline{X}^* \longleftrightarrow \begin{cases} d_{\overline{X}_0} = & (1, 0, -1) + \lambda_0(0, 1, 0) \\ d_{\overline{X}_1} = & (0, -1, 1) + \lambda_1(1, 0, 0) \end{cases}$$

$$\overline{Y}^* \longleftrightarrow \begin{cases} d_{\overline{Y}_0} = & (1, -1, 0) + \mu_0(0, 0, 1) \\ d_{\overline{Y}_1} = & (0, 1, -1) + \mu_1(-1, 0, 0) \end{cases}$$

Example 3.2. Let $\overline{X}^* = (1, \varepsilon^*, \xi\varepsilon^*)$ and $\overline{Y}^* = (\sqrt{2} + \varepsilon^*, 1 + \sqrt{2}\varepsilon^*, \xi\varepsilon^*)$ be **HD**- split vectors. Since $\langle \overline{X}^*, \overline{X}^* \rangle_L = -1$ and $\langle \overline{Y}^*, \overline{Y}^* \rangle_L = -1$, $\overline{X}^*$ and $\overline{Y}^*$ are **HD**- timelike vectors. Therefore, we have

$$\begin{aligned} S^* &= \overline{X}^* \otimes_L \overline{Y}^* \\ &= (\xi\varepsilon^*, \sqrt{2}\xi\varepsilon^* - \xi\varepsilon^*, 1) \end{aligned} \tag{3.8}$$

where $\overline{S}_0 = (0, 0, 1)$ and $\overline{S}_1 = (\xi, \sqrt{2}\xi - \xi, 0)$. Since $\langle S^*, S^* \rangle_L = 1$, S^* is a **HD**- spacelike vector. By using (3.1), its corresponding semi-skew symmetric matris is explained

$$\mathbf{S}^* = \begin{bmatrix} 0 & 1 & -\sqrt{2}\xi\varepsilon^* + \xi\varepsilon^* \\ 1 & 0 & -\xi\varepsilon^* \\ -\sqrt{2}\xi\varepsilon^* + \xi\varepsilon^* & \xi\varepsilon^* & 0 \end{bmatrix}$$

Then, we get

$$(\mathbf{S}^*)^2 = \begin{bmatrix} 1 & 0 & -\xi\varepsilon^* \\ 0 & 1 & -\sqrt{2}\xi\varepsilon^* + \xi\varepsilon^* \\ \xi\varepsilon^* & -\sqrt{2}\xi\varepsilon^* + \xi\varepsilon^* & 0 \end{bmatrix}$$

Since $\overline{X}^*$ and $\overline{Y}^*$ are **HD**- timelike vectors, by using of **HD**- hyperbolic angle $\theta^* = \overline{\theta}_0 + \varepsilon^* \overline{\theta}_1$, we get

$$\cosh \theta^* = \sqrt{2}, \quad \sinh \theta^* = 1.$$

From **HD**- split Euler-Rodrigues equations for spacelike axis, we have

$$\begin{aligned} \mathbf{R}^* &= I_3 + \mathbf{S}^* + (-1 + \sqrt{2})\mathbf{S}^{*2} \\ \mathbf{R}^* &= \begin{bmatrix} \sqrt{2} & 1 & (2 - 2\sqrt{2})\xi\varepsilon^* \\ 1 & \sqrt{2} & (2\sqrt{2} - 4)\xi\varepsilon^* \\ 0 & (2\sqrt{2} - 2)\xi\varepsilon^* & 1 \end{bmatrix}. \end{aligned}$$

Then, we get

$$\overline{\mathbf{C}} = \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad and \quad \overline{\mathbf{D}} = \begin{bmatrix} 0 & 0 & (2-2\sqrt{2})\xi \\ 0 & 0 & (2\sqrt{2}-4)\xi \\ 0 & (2\sqrt{2}-2)\xi & 0 \end{bmatrix}$$

where $\mathbf{R}^* = \overline{\mathbf{C}} + \varepsilon^* \overline{\mathbf{D}}$. Then we have $\mathbf{R}^*(S^*) = S^*$, $\overline{\mathbf{C}}(\overline{S}_0) = \overline{S}_0$ and the dual rotation angle of $\overline{C}$ is $\overline{\theta}_0 = \operatorname{arccosh}(\sqrt{2})$. Then, $\overline{\mathbf{D}} = \overline{\mathbf{PC}}$ is expressed by the dual semi-skew symmetric matrix

$$\overline{\mathbf{P}} = \begin{bmatrix} 0 & 0 & (2-2\sqrt{2})\xi \\ 0 & 0 & (2\sqrt{2}-4)\xi \\ (2-2\sqrt{2})\xi & (4-2\sqrt{2})\xi & 0 \end{bmatrix}$$

We can write

i. $\overline{X}_0 = (1,0,0)$ and $\overline{X}_1 = (0,1,\xi)$ for $\overline{X}^* = \overline{X}_0 + \varepsilon^* \overline{X}_1$,
ii. $\overline{Y}_0 = (\sqrt{2},1,0)$ and $\overline{Y}_1 = (1,\sqrt{2},\xi)$ for $\overline{Y}^* = \overline{Y}_0 + \varepsilon^* \overline{Y}_1$,
iii. $\overline{S}_0 = (0,0,1)$ and $\overline{S}_1 = (\xi,(\sqrt{2}-1)\xi,0)$ for $S^* = \overline{S}_0 + \varepsilon^* \overline{S}_1$

and we get

$$\langle \overline{X}_0, \overline{X}_0 \rangle_L = -1, \quad \langle \overline{Y}_0, \overline{Y}_0 \rangle_L = -1, \quad \langle \overline{S}_0, \overline{S}_0 \rangle_L = 1$$

Therefore, $\overline{X}_0$ and $\overline{Y}_0$ are unit dual timelike vectors and $\overline{S}_0$ is unit dual spacelike vector. Since $\langle \overline{X}_0, \overline{X}_1 \rangle_L = 0$, $\langle \overline{Y}_0, \overline{Y}_1 \rangle_L = 0$ and $\langle \overline{S}_0, \overline{S}_1 \rangle_L = 0$, we get $\overline{X}^* \in H_0^{*2}$, $\overline{Y}^* \in H_0^{*2}$ and $S^* \in S_1^{*2}$. Each point of the Lorentz unit **HD**-sphere corresponds to a directed spacelike line, and each point of the hyperbolic unit **HD**- sphere corresponds to a directional timelike line. We can express lines as follows:

$$\begin{aligned} \overline{X}^* \quad &\longleftrightarrow \quad \overline{L}_{\overline{X}^*} = (0,-\xi,1) + \overline{t}_x(1,0,0) \\ \overline{Y}^* \quad &\longleftrightarrow \quad \overline{L}_{\overline{Y}^*} = (-\xi,-\sqrt{2}\xi,1) + \overline{t}_y(\sqrt{2},1,0) \\ S^* \quad &\longleftrightarrow \quad \overline{L}_{S^*} = ((\sqrt{2}-1)\xi,\xi,0) + \overline{t}_s(0,0,1) \end{aligned}$$

where $\overline{t}_x, \overline{t}_y, \overline{t}_z \in \mathcal{D}$. Geometrically, it rotates the direction lines $\mathbf{R}^*$, the dual timelike line $\overline{L}_{\overline{X}^*}$ and dual timelike line $\overline{L}_{\overline{Y}^*}$ along the dual spacelike line $\overline{L}_{S^*}$ at an angle $\overline{\theta}_0 = \operatorname{arccosh}(\sqrt{2})$ and applies a translation $\overline{\theta}_1 = 0$. Since $\overline{X}_1 \in S_1^{*2}$, $\overline{X}^* \in H_0^{*2}$, $\overline{Y}_1 \in S_1^{*2}$ and $\overline{Y}^* \in H_0^{*2}$, the points of $\overline{X}^*$ and $\overline{Y}^*$ correspond exactly to any two intersecting perpendicular lines. In $\mathbb{R}_1^3$, one of these lines is timelike and the other is spacelike.

$$\overline{X}^* \longleftrightarrow \begin{cases} d_{\overline{X}_0} = & (0,0,0) + \lambda_0(1,0,0) \\ d_{\overline{X}_1} = & (-1,0,0) + \lambda_1(0,1,0) \end{cases}$$

$$\overline{Y}^* \longleftrightarrow \begin{cases} d_{\overline{Y}_0} = & (0,0,0) + \mu_0(\sqrt{2},1,0) \\ d_{\overline{Y}_1} = & (-\sqrt{2},-1,0) + \mu_1(1,\sqrt{2},0) \end{cases}$$

4. Construction of the HD-split Euler's Theorem

In this section, the fixed point theorem given by Euler will be constructed using the **HD**- split rotation matrix and will be called **HD**- split Euler's theorem.

Theorem 4.1. *Let $\mathbf{R}^* \in SO_1^*(3)$ be a **HD**- split rotation matrix. Then, there exists a non-zero **HD**- split vector $S^* \in \mathcal{D}_1^{*3}$ where $\mathbf{R}^* S^* = S^*$.*

Proof. Let $\mathbf{R}^*$ be the **HD**- split rotation matrix where $(\mathbf{R}^*)^T \varepsilon \mathbf{R}^* \varepsilon = \mathbf{R}^* \varepsilon (\mathbf{R}^*)^T \varepsilon = I_3, det(\mathbf{R}^*) = 1$. $\mathbf{R}^*$ can be written by the sum of a semi-symmetric matrix and a semi-skew symmetric matrix as follows:

$$\mathbf{R}^* = \frac{\mathbf{R}^* + \varepsilon(\mathbf{R}^*)^T \varepsilon}{2} + \frac{\mathbf{R}^* - \varepsilon(\mathbf{R}^*)^T \varepsilon}{2}. \tag{4.1}$$

Let the semi-skew symmetric part of $\mathbf{R}^*$ be $\mathbf{S}^*$, then we can write

$$\mathbf{R}^*\mathbf{S}^*\varepsilon(\mathbf{R}^*)^T\varepsilon = \mathbf{S}^* \tag{4.2}$$

By using equation (3.1), there exists a corresponding **HD**- split vector $S^* \in \mathcal{D}_1^{*3}$ where $g(\mathbf{S}^*) = S^*$. Therefore, we get $\mathbf{S}^* = g_{S^*}^{-1}$ and $g_{S^*}^{-1}(\beta^*) = S^* \otimes_L \beta^*$ for **HD**- split vector β^*. Then we get

$$\mathbf{R}^*(S^* \otimes_L \beta^*) = \mathbf{R}^*S^* \otimes_L \mathbf{R}^*\beta^*. \tag{4.3}$$

From the equation (4.3) it is clear that

$$\mathbf{R}^*g_{S^*}^{-1} = g_{\mathbf{R}^*S^*}^{-1}\mathbf{R}^* \tag{4.4}$$

If we apply $\varepsilon(\mathbf{R}^*)^T\varepsilon$ to both sides of the equation (4.4), then we have

$$\mathbf{R}^*g_{S^*}^{-1}\varepsilon(\mathbf{R}^*)^T\varepsilon = g_{\mathbf{R}^*S^*}^{-1}.$$

Therefore, we have

$$g_{\mathbf{R}^*S^*}^{-1} = \mathbf{S}^*$$

and

$$\mathbf{R}^*S^* = g(\mathbf{S}^*) = S^*.$$

□

Corollary 4.1. *If $\mathbf{R}^*$ is semi-symmetric (i.e $(\mathbf{R}^*)^T = \varepsilon\mathbf{R}^*\varepsilon$), then $\mathbf{S}^* = 0$ and $(\mathbf{R}^*)^2 = I_3$. It is clear that $\mathbf{R}^*(I_3 + \mathbf{R}^*) = I_3 + \mathbf{R}^*$. Consequently, each column of $I_3 + \mathbf{R}^*$ remains constant and those that are non-zero become constant vectors.*

Example 4.1. Let the **HD**- split matrix $\mathbf{R}^*$ be as follows:

$$\mathbf{R}^* = \begin{bmatrix} 1+4\xi\varepsilon^* & 2\xi\varepsilon^* & 2\xi+2\varepsilon^* \\ 2\xi+2\varepsilon^* & 2\xi\varepsilon^* & 1+4\xi\varepsilon^* \\ -2\xi\varepsilon^* & -1 & 2\xi\varepsilon^* \end{bmatrix} \in SO_1^*(3). \tag{4.5}$$

From the equation $\mathbf{S}^* = \dfrac{\mathbf{R}^* - \varepsilon(\mathbf{R}^*)^T\varepsilon}{2}$, the semi-skew symmetric part of $\mathbf{R}^*$ is found as follows:

$$\mathbf{S}^* = \begin{bmatrix} 0 & \xi+\xi\varepsilon^*+\varepsilon^* & \xi-\xi\varepsilon^*+\varepsilon^* \\ \xi+\xi\varepsilon^*+\varepsilon^* & 0 & 1+2\xi\varepsilon^* \\ \xi-\xi\varepsilon^*+\varepsilon^* & -1-2\xi\varepsilon & 0 \end{bmatrix}.$$

The vector corresponding to $\mathbf{S}^*$ is

$$S^* = (-1-2\xi\varepsilon^*, -\xi+\xi\varepsilon^*-\varepsilon^*, \xi+\xi\varepsilon^*+\varepsilon^*). \tag{4.6}$$

It can be seen from the equations (4.5) and (4.6) that $\mathbf{R}^*S^* = S^*$.

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