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RESEARCH ARTICLE

On Metrics and Linear Connections on Lines

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ABSTRACT

We discuss linear connections and conformal Riemannian metrics on the real line.

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Introduction

This expository article concerns differential geometric study on 1-manifolds. As is well known connected 1-manifolds are diffeomorphic to either the real line $\mathbb{R}$ or the circle $\mathbb{S}^1$. Thus intrinsic topological study on 1-manifolds is completed. In differential topology, imbeddings of the circle into some spaces have been studied. A *knot* is an imbedding of $\mathbb{S}^1$ into the Cartesian 3-space $\mathbb{R}^3$ (or the 3-sphere $\mathbb{S}^3$). Knot theory has been studied extensively.

On the other hand, from differential geometric viewpoint, we may consider Riemannian 1-manifolds or more generally affine 1-manifolds. However, the notion of curvature does not make sense for Riemannian 1-manifolds. In this sense, no local invariant exists on Riemannian 1-manifolds.

It should be remarked that the curvature functions of planar or spatial curve are *not* intrinsic quantity. Indeed, let $\gamma : M \to \mathbb{E}^n$ be an immersion of a 1-manifold M into the Euclidean n-space (or arbitrary Riemannian n-manifold). Then the curvature function κ is introduced via the *acceleration vector field*

$$\nabla^{\circ}_{\dot{\gamma}}\dot{\gamma} = \kappa \boldsymbol{n}$$

under the affine parametrization. Here ∇° is the Levi-Civita connection of $\mathbb{E}^n$ and $\boldsymbol{n}$ is the principal normal vector field. This formula implies that κ is the mean curvature function of γ.

Since there is no notion of curvature on Riemannian 1-manifolds, we can not develop 1-dimensional Riemannian geometry. In particular we can not introduce the notion of 1-dimensional space form. However we can encounter the following Riemannian 1-manifolds:

$$(\mathbb{R}, \mathrm{d}x^2), \quad \left(\mathbb{R} \cup \{\infty\}, \frac{\mathrm{d}x^2}{(1+x^2)^2}\right), \quad \left(I, \frac{\mathrm{d}x^2}{(1-x^2)^2}\right),$$

where $I = (-1, 1)$. These Riemannian 1-manifolds are regarded as the real part of the following complex 1-dimensional complex space forms:

$$(\mathbb{C}, |\mathrm{d}z|^2), \quad \left(\mathbb{C} \cup \{\infty\}, \frac{|\mathrm{d}z|^2}{(1+|z|^2)^2}\right), \quad \left(\mathbb{D}, \frac{|\mathrm{d}z|^2}{(1-|z|^2)^2}\right),$$

respectively. Here $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the unit disc.

It should be remarked that the distance function induced from the Riemannian metric

$$g_H = \frac{4\mathrm{d}x^2}{(1-x^2)^2}$$

is nothing but the *Hilbert distance* (see Example 1.2). Moreover Riemannian metrics g_c appeared in integrable geometry.

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A conformally immersed surface M in the Euclidean 3-space $\mathbb{E}^3$ is said to be a *surface with harmonic inverse mean curvature* if its reciprocal $1/H$ of the mean curvature function is a harmonic function. The notion of surface with harmonic inverse mean curvature (HIMC surface, in short) was introduced by Bobenko [2] and extended to surfaces in 3-dimensional space forms by Fujioka [5]. A conformally immersed surface in the unit 3-space $\mathbb{S}^3$ [resp. hyperbolic 3-space $\mathbb{H}^3$ of constant curvature -1] is said to be an HIMC surface if its reciprocal $1/H$ of the mean curvature function is a harmonic map into the 1-dimensional Riemannian manifold $(\mathbb{R}, g_1)$ [resp. $((-1,1), g_{-1})$].

The Levi-Civita connections ∇^c of these metrics g_c are given by

$$\nabla^c_X X = \mathit{\Gamma}(x)X, \quad X = \frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathit{\Gamma}(x) = -\frac{2cx}{1+cx^2}.$$

For $c=0$ and $c=1$, the Levi-Civita connection ∇^c of g_c are globally defined on $\mathbb{R}$. The Levi-Civita connection of g_{-1} is defined on the interval $(-1,1)$.

The Gauss-Codazzi equations of HIMC-surfaces can be normalized to certain types of Painlevé equations [3] under isothermic assumption. For more information on HIMC surfaces, we refer to [6, 7, 8, 9].

On the other hand, Nomizu and Sasaki [23] classified globally defined linear connections on the real line $\mathbb{R}$. In this article we discuss relations between the Levi-Civita connections of g_c and the classification due to Nomizu and Sasaki.

This work is motivated by a naive question "*Can we introduce the notion of* 1*-dimensional space form ?*". Obviously the notion of curvature does not make sense for 1-dimensional manifolds. There are several interpretations for 1-dimensional curvatures, see *e.g.*, [5, 19].

As a summary, to develop differential geometry of 1-manifolds, only equipping Riemannian metric (or linear connection) is not sufficient for 1-manifolds. One need to equip additional structures on Riemannian 1-manifolds.

Grigor'yan introduced the notion of weighted manifold [13]. As he exhibited, differential geometry of weighthed manifolds is still valid for dimension 1. A weighted 1-manifold is a Riemannian 1-manifold equipped with a weighted volume element. Crasmareanu [4] pointed out an interesting connection between orthogonal polynomials and weighted 1-manifolds. This fact was observed by Grigor'yan for Hermite polynomials.

On the other hand, Shima introduced the notion of Hessian manifold [25]. A Hessian manifold $M = (M, g, \nabla)$ is a smooth manifold M equipped with a Riemannian metric g and a flat linear connection ∇ such that ∇g is totally symmetric. On a Hessian manifold M, the curvature R of ∇ vanishes. Shima introduced the notion of Hessian curvature tensor field H. Fortunately the notion of Hessian curvature tensor field is still valid for Hessian 1-manifolds and does not automatically vanish. This fact motivates us to study Hessian 1-manifolds. The study of Hessian 1-manifolds has another motivation derived from Information geometry. Statistical 1-manifolds derived from exponential families, *e.g.*, the statistical manifold of *binomial distributions* provides a fundamental example of Hessian 1-manifold (Example 8.2).

In this expository article, we discuss some linear connections and conformal Riemannian metrics on $\mathbb{R}$.

This article is organized as follows. In Section 1 we exhibit some typical examples of Riemannian 1-manifolds. In Section 2, we study imbeddings of Riemannian 1-manifolds exhibited in Section 1 into the lightcone of the Minkowski 3-space. We start our discussion on Riemannian 1-manifolds in Section 3. Section 4 is devoted to the study of affine 1-manifolds. We recall the uniformaization theorem of linear connections on $\mathbb{R}$ due to Nomizu-Sasaki. We give a metrical interpretation of Nomizu-Sasaki's result. In Section 5, we discuss the affine realizations of affine 1-manifolds into the equiaffine plane developed by Nomizu and Sasaki [23]. In Section 6 we recall the notion of harmonic inverse mean curvature surface due to Bobenko [2] and Fujioka [5]. Weighted 1-manifolds will be discussed in Section 7. In Section 8 we study Hessian 1-manifolds. In the final section we discuss statistically harmonic maps between statistical 1-manifolds.

1. Typical examples

We start with exhibiting two typical examples of conformal metrics on open intervals.

First of all we recall the notion of Riemannian metric on open intervals.

1.1. Riemannian metrics

Let $\mathfrak{X}(\mathbb{R}) = \Gamma(T\mathbb{R})$ be the space of all smooth vector fields on the real line $\mathbb{R}$. The space $\mathfrak{X}(\mathbb{R})$ is expressed as

$$\mathfrak{X}(\mathbb{R}) = \{\lambda X \mid \lambda \in C^\infty(\mathbb{R})\}, \quad X = \frac{\mathrm{d}}{\mathrm{d}x}.$$

At a point $x_0 \in \mathbb{R}$, the tangent space $T_{x_0}\mathbb{R}$ is given by

$$T_{x_0}\mathbb{R} = \{aX_{x_0} \mid a \in \mathbb{R}\}$$

which is identified with $\mathbb{R}$ via the correspondence:

$$aX_{x_0} \longmapsto a.$$

A Riemannian metric g on $\mathbb{R}$ is a mapping

$$g : \mathfrak{X}(\mathbb{R}) \times \mathfrak{X}(\mathbb{R}) \to C^\infty(\mathbb{R})$$

satisfying

- $g(\lambda X, \mu X) = \lambda\mu\, g(X, X)$ for any $\lambda, \mu \in C^\infty(\mathbb{R})$ and
- $g(X, X) > 0$.

The Riemannian metric g_0 determined by the condition $g(X, X) = 1$ is expressed as

$$g_0 = \mathrm{d}x^2.$$

The Riemannian metric g_0 is called the *canonical Euclidean metric*. In general, a Riemannian metric g is expressed as

$$g = g(X, X)\,\mathrm{d}x^2.$$

For this reason, we may call Riemannian metric g a *conformal metric* on $\mathbb{R}$.

We may restrict vector fields and Riemannian metrics on the whole line $\mathbb{R}$ to some open intervals.

1.2. The Hilbert distance

On the open interval $I = (-1, 1)$ the *Hilbert distance* d_H is defined by [15, 21]:

$$\mathrm{d}_H(a, b) = |\log[a, b, -1, 1]|\,,$$

where

$$[a, b, x, y] = \frac{|x-a| \cdot |y-b|}{|x-b| \cdot |y-a|}.$$

The Hilbert distance is derived from the Riemannian metric

$$g_H = \frac{4\mathrm{d}x^2}{(1-x^2)^2} = 4g_{-1}$$

on I. Indeed,

$$\int_a^b \frac{2\,\mathrm{d}x}{1-x^2} = \int_a^b \frac{1}{1-x} + \frac{1}{1+x}\,\mathrm{d}x = \left[\log\frac{1+x}{1-x}\right]_a^b = -\log[a, b; 1, -1].$$

1.3. Stereographic projection

Let us consider the unit circle

$$\mathbb{S}^1 = \{(y_1, y_2) \in \mathbb{E}^2 \mid y_1^2 + y_2^2 = 1\}$$

in the Euclidean plane $\mathbb{E}^2$. The stereographic projection π_+ of $U_0 := \mathbb{S}^1 \setminus \{(0, 1)\}$ onto $\mathbb{R}$ with pole $(0, 1)$ is given by

$$\pi_+(y_1, y_2) = \frac{y_1}{1-y_2}$$

with inverse mapping

$$\pi_+^{-1}(x) = \left(\frac{2x}{1+x^2}, \frac{x^2-1}{1+x^2} \right).$$

One can check that the induced metric is given by

$$(\mathrm{d}y_1)^2 + (\mathrm{d}y_2)^2 = \frac{4\mathrm{d}x^2}{(1+x^2)^2} = 4g_1$$

on U_0.

Analogously, the stereographic projection π_- of $U_\infty := \mathbb{S}^1 \smallsetminus \{(0,-1)\}$ onto $\mathbb{R}$ with pole $(0,-1)$ is given by

$$\pi_-(y_1, y_2) = \frac{y_1}{1+y_2}$$

with inverse mapping

$$\pi_-^{-1}(x) = \left(\frac{2x}{1+x^2}, \frac{1-x^2}{1+x^2} \right).$$

The induced metric is given by

$$(\mathrm{d}y_1)^2 + (\mathrm{d}y_2)^2 = \frac{4\mathrm{d}x^2}{(1+x^2)^2}$$

on U_∞.

As usual we add the point at infinity ∞ to $\mathbb{R}$ and extend π to $\mathbb{S}^1$ as $\pi(0,1) = \infty$. Then the 1-manifold $\mathbb{S}^1$ is covered by two charts $\{(U_0, \pi_+), (U_\infty, \pi_-)\}$.

1.4. Projective line

Let us consider the real projective line $\mathbb{P}_1$. The projective line is regarded as the 1-manifold of all lines of $\mathbb{R}^2$ through the origin. Hence $\mathbb{P}_1$ is regarded as the quotient space

$$\mathbb{P}_1 = (\mathbb{R}^2 \smallsetminus \{(0,0)\})/\mathbb{R}^\times = \left\{ [x_1 : x_2] \mid (x_1, x_2) \in \mathbb{R}^2 \smallsetminus \{(0,0)\} \right\},$$

where

$$[x_1 : x_2] = \{(\lambda x_1, \lambda x_2) \mid \lambda \in \mathbb{R}^\times\}, \quad \mathbb{R}^\times = \mathbb{R} \smallsetminus \{0\}.$$

We denote by $\mathrm{pr} : \mathbb{R}^2 \smallsetminus \{(0,0)\} \to \mathbb{P}_1$ the projection. Take

$$\widetilde{U}_+ = \{(x_1, x_2) \mid x_1 \neq 0\}, \quad \widetilde{U}_- = \{(x_1, x_2) \mid x_2 \neq 0\}$$

and set

$$U_+ = \mathrm{pr}(\widetilde{U}_+), \quad U_- = \mathrm{pr}(\widetilde{U}_-).$$

Then $\mathbb{P}_1 = U_+ \cup U_-$. Define smooth maps $\psi_\pm : U_\pm \to \mathbb{R}$ by

$$\psi_+([x_1 : x_2]) = \frac{x_2}{x_1} =: t, \quad \psi_-([x_1 : x_2]) = \frac{x_1}{x_2} =: s.$$

Then, on $U_+ \cap U_-$, we have

$$(\psi_- \circ \psi_+^{-1})(t) = \frac{1}{t}, \quad (\psi_+ \circ \psi_-^{-1})(s) = \frac{1}{s}.$$

Here we recall the fact that $\mathbb{P}_1$ is identified with $\mathbb{R} \cup \{\infty\}$. We identify the line $[x_1 : x_2] \in U_+$ with $t = \psi_+([x_1 : x_2]) \in \mathbb{R}$. Next we identify the line $[0 : 1] \in U_-$ with the point at infinity ∞. Thus we obtain the identification $\mathbb{P}_1 = \mathbb{R} \cup \{\infty\}$. As a result we get the identification $\mathbb{P}_1 = \mathbb{S}^1$.

On the other hand, on the unit circle $\mathbb{S}^1 \subset \mathbb{E}^2$, we introduce an equivalence relation

$$(x_1, x_2) \sim (y_1, y_2) \iff (x_1, y_1) = (x_2, y_2) \text{ or } (x_1, y_1) = (-x_2, -y_2).$$

Then the quotient space is nothing but $\mathbb{P}_1$. Moreover the mapping $f : \mathbb{S}^1/\!\sim \to \mathbb{S}^1$ defined by

$$f((\cos\theta, \sin\theta)) = (\cos(2\theta), \sin(2\theta))$$

is a diffeomorphism. Thus we get again $\mathbb{P}_1 = \mathbb{S}^1$.

1.5. Hyperbola

Let us consider the hyperbola

$$\mathbb{H}^1 = \{(y_1, y_2) \in \mathbb{E}^2_1 \mid y_1^2 - y_2^2 = -1,\ y_2 > 0\}$$

in the Minkowski plane $\mathbb{E}^2_1$. The stereographic projection π of $\mathbb{H}^1$ onto the interval $(-1, 1)$ with pole $(0, -1)$ is given by

$$\pi(y_1, y_2) = \frac{y_1}{1 + y_2}$$

with inverse mapping

$$\pi^{-1}(x) = \left(\frac{2u}{1 - x^2}, \frac{1 + x^2}{1 - x^2}\right).$$

The induced metric of $\mathbb{H}^1$ is computed as

$$(\mathrm{d}y_1)^2 - (\mathrm{d}y_2)^2 = \frac{4\mathrm{d}u^2}{(1 - x^2)^2} = g_H.$$

2. Conics

Let us consider Minkowski 3-space $\mathbb{E}^3_1$ with Minkowski scalar product $\langle \cdot, \cdot \rangle = \mathrm{d}y_1^2 + \mathrm{d}y_2^2 - \mathrm{d}y_3^2$.

The *lightcone* L is given by

$$L = \{(y_1, y_2, y_3) \in \mathbb{E}^3_1 \setminus \{(0, 0, 0)\} \mid y_1^2 + y_2^2 - y_3^2 = 0\}$$

The lightcone is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^\times$. Indeed,

$$\mathbb{S}^1 \times \mathbb{R}^\times \ni (\boldsymbol{x}, t) \longmapsto (|t|\boldsymbol{x}, t) \in \mathbb{E}^3_1$$

gives a diffeomorphism from $\mathbb{S}^1 \times \mathbb{R}^\times$ onto L.

For any $t \in \mathbb{R}^\times$, we define a map $\Phi^t : \mathbb{E}^2 \to \mathbb{E}^3_1$ by

$$\Phi^t(\boldsymbol{x}) = (\boldsymbol{x}, t).$$

Then the image of the circle $\mathbb{S}^1(|t|) \subset \mathbb{E}^2$ of radius $|t|$ under Φ^t is the conic section

$$\Pi_{y_3=t} \cap L = \{(y_1, y_2, t) \in L\}.$$

Note that the plane $y_3 = t$ is a spacelike plane.

The conic sections $L \cap \Pi^{\mathsf{L}}_t$ are parabolas. Here

$$\Pi^{\mathsf{L}}_t = \{(y_1, y_2, y_2 + t) \in \mathbb{E}^3_1\}$$

is a lightlike plane. The conic section $L \cap \Pi^{\mathsf{L}}_t$ is parametrized as

$$L \cap \Pi^{\mathsf{L}}_t = \left\{\left(y_1, -\frac{t}{2} - \frac{y_1^2}{2t}, \frac{t}{2} - \frac{y_1^2}{2t}\right)\right\}.$$

Let us consider the immersion F_t of $\mathbb{R}$ into L by

$$F_t(x) = \left(x, -\frac{t}{2} - \frac{x^2}{2t}, \frac{t}{2} - \frac{x^2}{2t}\right).$$

One can check that $\langle \mathrm{d}F_t, \mathrm{d}F_t \rangle = \mathrm{d}x^2$. Thus Euclidean line is isometrically embedded in L as a parabola.

We define a map $\Psi : \mathbb{E}^2_1 \to \mathbb{E}^3_1$ by

$$\Psi(\boldsymbol{y}) = (1, \boldsymbol{y}).$$

Then the image of $\mathbb{H}^1 \subset \mathbb{E}^2_1$ is the conic section $\Pi^{\mathsf{T}}_{y_1=1} \cap L$. Here $\Pi^{\mathsf{T}}_{y_1=1}$ is a timelike plane defined by $y_1 = 1$. By composing Ψ and $\pi^{-1} : (-1, 1) \to \mathbb{H}^1$, we obtain an isometric imbedding

$$u \longmapsto \left(1, \frac{2u}{1 - u^2}, \frac{1 + u^2}{1 - u^2}\right)$$

of $((-1,1),g_H)$ into the lightcone.

The *conformal circle*, that is, the conformal compactification $\mathcal{M}$ of the Euclidean line $\mathbb{E}^1$ is the projective light cone

$$\{[y_1:y_2:y_3]\in\mathbb{P}_2 \mid y_1^2+y_2^2-y_3^2=0\}\subset\mathbb{P}_2.$$

The conformal transformation group is $\mathrm{O}_1(3)/\mathbb{Z}_2$.

The Euclidean line is conformally imbedded in the conformal circle by

$$x\longmapsto[2x:-1+x^2:1+x^2]=\left[\frac{2x}{1+x^2}:\frac{-1+x^2}{1+x^2}:1\right]$$

Let us identify the Minkowski space $\mathbb{E}^3_1$ with $\mathfrak{sl}_2\mathbb{R}$ via the correspondence

$$y_1\boldsymbol{e}_1+y_2\boldsymbol{e}_2+y_3\boldsymbol{e}_3\longleftrightarrow y_1\boldsymbol{i}+y_2\boldsymbol{j}'+y_3\boldsymbol{k}'=\begin{pmatrix}-y_3 & -y_1+y_2\\ y_1+y_2 & y_3\end{pmatrix}$$

The metric corresponds to the left invariant Lorentz metric on the special linear group $\mathrm{SL}_2\mathbb{R}$ derived from the scalar product

$$\langle X,Y\rangle=\frac{1}{2}\mathrm{tr}(XY).$$

The special linear group $\mathrm{SL}_2\mathbb{R}$ acts isometrically on $\mathbb{E}^3_1$ via the Ad-action:

$$\mathrm{SL}_2\mathbb{R}\times\mathbb{E}^3_1\to\mathbb{E}^3_1;\quad (A,Y)\longmapsto\mathrm{Ad}(A)Y=AYA^{-1}.$$

Hence the map $\mathrm{Ad}:\mathrm{SL}_2\mathbb{R}\to\mathrm{O}_1(3)$ is a Lie group homomorphism. One can see that $\mathrm{SL}_2\mathbb{R}/\mathbb{Z}_2\cong\mathrm{SO}^+_1(3)$. Thus $\mathrm{SL}_2\mathbb{R}$ is the double covering of $\mathrm{SO}^+_1(3)$.

$$\mathrm{Ad}\begin{pmatrix}a&b\\c&d\end{pmatrix}\boldsymbol{i}=\frac{1}{2}(a^2+b^2+c^2+d^2)\boldsymbol{i}+\frac{1}{2}(-a^2-b^2+c^2+d^2)\boldsymbol{j}'-(ac+bd)\boldsymbol{k}',$$
$$\mathrm{Ad}\begin{pmatrix}a&b\\c&d\end{pmatrix}\boldsymbol{j}'=\frac{1}{2}(-a^2+b^2-c^2+d^2)\boldsymbol{i}+\frac{1}{2}(a^2-b^2-c^2+d^2)\boldsymbol{j}'+(ac-bd)\boldsymbol{k}'$$
$$\mathrm{Ad}\begin{pmatrix}a&b\\c&d\end{pmatrix}\boldsymbol{k}'=-(ab+cd)\boldsymbol{i}+(ab-cd)\boldsymbol{j}'+(ad+bc)\boldsymbol{k}'.$$

The lightcone is identified with

$$\{Y\in\mathfrak{sl}_2\mathbb{R}\mid\mathrm{tr}(Y^2)=0\}$$

Hence the isometric action of $\mathrm{SL}_2\mathbb{R}$ induces an action on the projective lightcone as

$$\mathrm{SL}_2\mathbb{R}\times\mathcal{M}\to\mathcal{M};\quad (A,[Y])\longmapsto[AY].$$

The group of all projective transformations preserving the conformal circle is isomorphic to $\mathrm{PSL}_2\mathbb{R}$. Thus the projective transformations coincide with conformal transformations on the conformal circle. In other words, conformal circle is nothing but the projective line.

The conic section $\Pi_{y_3=1}\cap L$ is identified with

$$\left\{\begin{pmatrix}-1 & -\cos\theta+\sin\theta\\ \cos\theta+\sin\theta & 1\end{pmatrix}\right\}.$$

3. Conformal metrics on the line

3.1. Linear connection

A linear connection ∇ on $\mathbb{R}$ is determined by the *connection coefficient* $\mathit{\Gamma}\in C^\infty(\mathbb{R})$. Indeed ∇ is a mapping $\mathfrak{X}(\mathbb{R})\times\mathfrak{X}(\mathbb{R})\to\mathfrak{X}(\mathbb{R})$ which is determined by the formula

$$\nabla_X X=\mathit{\Gamma}\,X$$

and the Leipniz rule

$$\nabla_{\lambda X}(\mu X) = \lambda \left(\frac{\mathrm{d}\mu}{\mathrm{d}x} X + \mu \nabla_X X \right), \quad \lambda, \mu \in C^\infty(\mathbb{R})$$

Throughout this article we denote by ∇° the *canonical flat connection* of $\mathbb{R}$, that is

$$\nabla^\circ_X X = 0.$$

Moreover we may restrict linear connections as well as conformal metrics on $\mathbb{R}$ to open submanifolds of $\mathbb{R}$.

3.2. The Levi-Civita connection

Let us take a smooth function $\gamma(x)$ on the real line and consider the Riemannian metric

$$g = e^{2\gamma(x)}\,\mathrm{d}x^2.$$

Obviously g is a global conformal change of the Euclidean metric $g_0 = \mathrm{d}x^2$. The *Levi-Civita connection* ∇^g is a linear connection determined by the connection coefficient

$$\Gamma(x) = \frac{\mathrm{d}\gamma}{\mathrm{d}x}(x).$$

We may restrict γ (and also Γ) on an open submanifold M of $\mathbb{R}$.

Note that under the scaling change $g \longmapsto cg$ for some positive constant c, the Levi-Civita connection is preserved.

Example 3.1 (Hilbert distance)**.** The Levi-Civita connection of $M = (-1, 1)$ equipped with the Hilbert metric g_H is given by

$$\Gamma(x) = \frac{2x}{1 - x^2}.$$

Note that the Levi-Civita connection of the metric $g_{-1} = \mathrm{d}x^2/(1 - x^2)^2$ coincides with that of g_H.

Example 3.2 (Stereographic projection)**.** The Levi-Civita connection of $\mathbb{R}$ equipped with the metric

$$g_S = \frac{4\mathrm{d}x^2}{(1 + x^2)^2}$$

is given by

$$\Gamma(x) = -\frac{2x}{1 + x^2}.$$

The Levi-Civita connection of the metric $g_1 = \mathrm{d}x^2/(1 + x^2)^2$ coincides with that of g_S.

4. Linear connections on the real line

Here we recall Nomizu-Sasaki's work [23] on linear connections on the real line. Let ∇ be a linear connection on the real line with connection coefficient $\Gamma(x)$. Take a smooth map $x : I \to (\mathbb{R}, \nabla)$ defined on an interval I with coordinate t. We consider the pull-backed tangent bundle

$$x^*T\mathbb{R} = \bigcup_{t \in I} T_{x(t)}\mathbb{R}.$$

We denote by ∇^x the linear connection on $x^*T\mathbb{R}$ induced from ∇.

The *velocity* of $x(t)$ is the function

$$\dot{x}(t) = \frac{\mathrm{d}x}{\mathrm{d}t}(t).$$

The *velocity vector field* is

$$x_*T = \dot{x}(t)\frac{\mathrm{d}}{\mathrm{d}x}, \quad T = \frac{\mathrm{d}}{\mathrm{d}t}.$$

The *acceleration* of $x(t)$ is the function
$$\ddot{x}(t) = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t).$$
The *acceleration vector field* $\nabla_{\dot{x}}\dot{x}$ of $x(t)$ is defined by
$$\nabla_{\dot{x}}\dot{x} := \nabla^x_T x_* T = \left(\ddot{x}(t) + \varGamma(x(t))\dot{x}(t)^2\right) X.$$
A smooth map x is said to be a *regular curve* if its velocity vector field does not vanish.

A regular curve $x(t)$ in $(\mathbb{R}, \nabla)$ is said to be a *geodesic* if it satisfies $\nabla_{\dot{x}}\dot{x} = 0$. The ordinary differential equation
$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \varGamma(x(t))\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 0 \tag{4.1}$$
is referred as to the *equation of geodesic* in $(\mathbb{R}, \nabla)$.

Let us perform a parameter change from t to another parameter u. We assume that the orientation preserving property:
$$\frac{\mathrm{d}t}{\mathrm{d}u} > 0.$$
Then one can see that
$$\nabla_{\dot{x}(t)}\dot{x}(t) = \frac{\mathrm{d}u^2}{\mathrm{d}t^2}\frac{\mathrm{d}x}{\mathrm{d}u}X + \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 \nabla_{\dot{x}(u)}\dot{x}(u).$$
This formula shows that the reparametrized curve $x(u) := x(t(u))$ satisfies the equation of geodesic if and only if $u = at + b$ for some constants $a > 0$ and $b \in \mathbb{R}$. Thus, up to orientation preserving affine transformation on $\mathbb{R}$, the parameter t with respect to which the equation of geodesic takes the form (4.1) is unique. Such a parameter is called the *affine parameter* of a geodesic $x = x(t)$.

More generally for a regular curve $x = x(u)$ in $(\mathbb{R}, \nabla)$, if there exists a reparametrization $u = u(t)$ so that the remarametrized curve $x(t) := x(u(t))$ satisfies (4.1), then $x(u)$ is said to be a *pre-geodesic*. One can see that $x(u)$ is a pre-geodesic if and only if
$$\nabla_{\dot{x}(u)}\dot{x}(u) = \Psi(u)\dot{x}(u)X$$
for some function $\Psi(u)$. One can see that
$$t := \int_0^u \left(\exp \int_0^u \Psi(u)\,\mathrm{d}u\right) \mathrm{d}u$$
is an affine parameter for $x(u)$.

A geodesic $x(s)$ in $(\mathbb{R}, \nabla)$ parametrized by an affine parameter s is said to be *complete* if it is defined on the whole line $\mathbb{R}$. A linear connection ∇ is said to be *geodesically complete* if all the geodesics are complete.

Now let $x = x(s)$ be a geodesic parametrized by an affine parameter s. We demand the initial condition
$$x(0) = 0, \quad \dot{x}(0) = 1. \tag{4.2}$$
According to [23], we introduce a function $Q(x)$ by
$$Q(x) = \exp\left(\int_0^x \varGamma(u)\,\mathrm{d}u\right).$$
Then the equation of geodesic is rewritten as
$$\frac{\mathrm{d}}{\mathrm{d}s}\left(Q(x(s))\frac{\mathrm{d}x}{\mathrm{d}s}(s)\right) = 0.$$
Hence
$$a := Q(x(s))\frac{\mathrm{d}x}{\mathrm{d}s}(s)$$
is a conserved quantity of the geodesic. From the initial condition we have $a = 1$. Thus the affine parameter s is determined by
$$s = \int_0^x Q(u)\,\mathrm{d}u.$$
From this result, Nomizu and Sasaki deduced the following theorem:

Theorem 4.1 ([23])**.** *On a* 1*-dimensional manifold* $(\mathbb{R},\nabla)$*, a flat local coordinate* s *around the origin* 0 *is given by*

$$s=\int_0^x Q(u)\,\mathrm{d}u.$$

The inverse function $x=x(s)$ *is a geodesic in* $(\mathbb{R},\nabla)$ *with affine parameter* s*.*

Let us consider the Levi-Civita connection of the Riemannian metric $g=e^{2\lambda(x)}\mathrm{d}x^2$. In this case

$$Q(x)=\exp\int_0^x \mathit{\Gamma}(u)\,\mathrm{d}u=\exp(\gamma(x)-\gamma(0))=\frac{e^{\gamma(x)}}{e^{\gamma(0)}}.$$

Then the flat coordinate s is given by

$$s=\frac{1}{e^{\gamma(0)}}\int_0^x e^{\gamma(u)}\,\mathrm{d}u.$$

Nomizu and Sasaki introduced the notion of affine parametrization of $(\mathbb{R},\nabla)$. An *affine parametrization* of $(\mathbb{R},\nabla)$ is a triplet (I,∇°,x) consisting of an open interval I, natural flat linear connection ∇° and a connection preserving diffeomorphism $x:(I,\nabla^\circ)\to(\mathbb{R},\nabla)$. Compare the notion of affine parametrization with that of *developing map* of affine 1-manifolds ([14, 28]).

Example 4.1 (The canonical flat connection)**.** The canonical flat connection ∇° is determined by $\mathit{\Gamma}=0$. The flat coordinate s around 0 is globally defined and given by $s=x$. Thus the geodesic satisfying the initial condition (4.2) is $x(s)=s$. It should be remarked that ∇° is the Levi-Civita connection of the metric $g_0=\mathrm{d}x^2$.

Example 4.2. The linear connection ∇ with connection coefficient $\mathit{\Gamma}=1$ satisfies $Q(x)=e^x$. The flat coordinate s around 0 is given by

$$s=\int_0^x e^u\,\mathrm{d}u=e^x-1\in(-1,\infty).$$

Thus the geodesic satisfying the initial condition (4.2) is $x(s)=\log(s+1)$ and defined on the interval $(-1,\infty)$. Thus ∇ is not geodesically complete. The geodesic $x:(-1,\infty)\to\mathbb{R}$ is an affine parametrization of $(\mathbb{R},\nabla)$.

The connection ∇ is the Levi-Civita connection of the Riemannian metric $g=e^{2x}\,\mathrm{d}x^2$. The path from x_0 to x_1 is given by

$$x(t)=\log((e^{x_1}-e^{x_0})t+e^{x_0}).$$

The geodesics starting at p with initial velocity v is

$$x(t)=p+\log(1+vs).$$

The Riemannian distance d induced from g is given by

$$\mathrm{d}(x_0,x_1)=|e^{x_1}-e^{x_0}|.$$

Example 4.3 (Hilbert metric)**.** Let us study the Levi-Civita connection of the Hilbert metric g_H on the interval $(-1,1)$ with coordinate u. The Levi-Civita connection ∇^H is given by $\mathit{\Gamma}=2u/(1-u^2)$. Hence $Q(u)=1/(1-u^2)>0$. The flat coordinate around 0 is given by

$$s=\int_0^u\frac{\mathrm{d}u}{1-u^2}=\tanh^{-1}u\in(-\infty,\infty).$$

The geodesic satisfying the initial condition (4.2) is $u(s)=\tanh s$. Hence $u:\mathbb{R}\to(-1,1)$ is an affine parametrization of $((-1,1),\nabla)$.

Next let us consider the diffeomorphism $x:(-1,1)\to\mathbb{R}$ given by

$$x(u)=\log(2\tanh^{-1}u+1),\quad u\in(-1,1)$$

Via this diffeomorphism, the Hilbert metric is transformed as the metric g in Example 4.2. Thus $((-1,1),g_H)$ is isometric to $(\mathbb{R},e^{2x}\,\mathrm{d}x^2)$ exhibited in Example 4.2.

Example 4.4. In [23, Example 3], the linear connection ∇ with connection coefficient

$$\Gamma(x) = -\frac{2x}{1+x^2}$$

is discussed. As we saw in Example 3.2, this connection is nothing but the Levi-Civita connection of the metric $g_S = 4g_1$. One can see that

$$Q(x) = \frac{1}{1+x^2}, \quad s(x) = \int_0^x \frac{\mathrm{d}x}{1+x^2} = \tan^{-1} x.$$

Thus we obtain

$$x = \tan s, \quad s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

The geodesic $x : (-\pi/2, \pi/2) \to \mathbb{R}$ is an affine parametrization of $(\mathbb{R}, \nabla)$.

Example 4.5. Let us consider the metric $g = \mathrm{d}x^2/(1+x^2)$. Then

$$\Gamma(x) = -\frac{x}{1+x^2}, \quad Q(x) = \frac{1}{\sqrt{1+x^2}}.$$

Hence

$$s = \int_0^x \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \sinh^{-1} x \in (-\infty, \infty).$$

Thus the geodesic $x : \mathbb{R} \to \mathbb{R}$ is an affine parametrization of $(\mathbb{R}, \nabla)$.

Nomizu and Sasaki proved the following result (*cf.* [28]).

Theorem 4.2. *Let ∇ be a linear connection on $\mathbb{R}$. Then $(\mathbb{R}, \nabla)$ is obtained from Example 4.1, 4.2 or 4.4 via affine parametrization.*

This classification is rephrased as

Corollary 4.1. *For any linear connection ∇ on $\mathbb{R}$, there exists a global coordinate y so that ∇ is expressed as*

$$\nabla_Y Y = 0, \quad \nabla_Y Y = Y, \quad \textit{or} \quad \nabla_Y Y = -\frac{2y}{1+y^2} Y$$

for $Y = \mathrm{d}/\mathrm{d}y$.

Nomizu-Sasaki's classification is reinterpreted as follows:

Corollary 4.2. *Let ∇ be a linear connection on $\mathbb{R}$. Then $(\mathbb{R}, \nabla)$ is obtained from one of the following spaces via affine parametrizations*:

1. $(\mathbb{R}, \nabla^\circ)$. *The canonical flat connection is the Levi-Civita connection of the Euclidean metric $g_0 = \mathrm{d}x^2$.*
2. $(\mathbb{R}, \nabla^S)$, *where ∇^S is the Levi-Civita connection of the metric $g_S = 4\mathrm{d}x^2/(1+x^2)^2$.*
3. $((-1,1), \nabla^H)$, *where ∇^H is the Levi-Civita connection of the Hilbert metric $g_H = 4\mathrm{d}x^2/(1-x^2)^2$.*

Thus globally defined linear connections are exhausted by Levi-Civita connections of the metrics g_0 and $g_{\pm 1}$. In other words, those linear connections are conformally realizable as in $\mathbb{E}^1$, $\mathbb{S}^1 \setminus \{\infty\}$ or a one-sheet of $\mathbb{H}^1$.

Concerning on linear connections on the circle $\mathbb{S}^1$, Nomizu and Sasaki proved the following result (compare with Kuiper's theorem [30]. See also [14, 27, 28]).

Theorem 4.3. *For any linear connection ∇ on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, there exists a diffeomorphim $\phi : \mathbb{S}^1 \to \mathbb{R}/\mathbb{Z}$ or $\phi : \mathbb{S}^1 \to \mathbb{R}^+/G_a$ such that $\nabla = \phi^*\nabla^\circ$. Here G_a is a group of transformations on $\mathbb{R}^+$ defined by $x \longmapsto ax$ with $a \neq 1$. The linear connection of the former case is complete but the latter one is not. The quotient $\mathbb{R}^+/G_a$ is identified with $\mathbb{R}/\mathbb{Z}_a$. The discrete subgroup $\mathbb{Z}_a$ is the group of translations $x \longmapsto x + \log a$. The induced connection on $\mathbb{R}$ coincides with the one in Example 4.2.*

Nomizu and Sasaki pointed out that the connection $\nabla = \phi^*\nabla^\circ$ on $\mathbb{S}^1$ induced from $\mathbb{R}^+/G_a$ is non-metrical. Because the connection $\nabla = \phi^*\nabla^\circ$ is not complete on $\mathbb{S}^1$.

Question. *We know that* $((-1,\infty),\nabla)$ *in Example 4.2 is derived from the Levi-Civita connection of* $(\mathbb{S}^1\setminus\{\infty\}, 2\mathrm{d}u^2/(1+u^2)^2)$. *How to understand/interpret the non-metrizability of* $\nabla=\phi^*\nabla^\circ$ *?*

Remark 4.1 (Schwarzian). Let $x=x(t)$ be a regular curve in a Riemannian n-manifold (M,g). The *Schwarzian derivative* of $x(t)$ in the sense of Kobayashi-Wada [19] is defined by

$$s^2x := (\nabla_{\dot{x}}\nabla_{\dot{x}}\dot{\gamma})\dot{x}^{-1} - \frac{3}{2}((\nabla_{\dot{x}}\dot{x})\dot{x}^{-1})^2 - \frac{\mathrm{s}}{2n(n-1)}\dot{x}^2,$$

where s is the scalar curvature of M. Here we used the Clifford multiplication. In case $n=1$, the term $\mathrm{s}/n(n-1)$ is indefinite. Kobayashi and Wada gave the following interpretation:

- $M=\mathbb{E}^1$: $\mathrm{s}/n(n-1)=0$.
- $M=\mathbb{S}^1(r)\subset\mathbb{E}^2$: $\mathrm{s}/n(n-1)=r^{-2}$.

5. Equiaffine realizations

Let $(\mathbb{R}^2, D, \mathrm{d}y_1\wedge\mathrm{d}y_2)$ be the *equiaffine plane*, that is, the Cartesian plane equipped with canonical flat connection D and the area element $\mathrm{d}y_1\wedge\mathrm{d}y_2$ parallel with respect to D.

Let I be an interval equipped with a linear connection ∇. An immersion $f: I\to(\mathbb{R}^2, D, \mathrm{d}y_1\wedge\mathrm{d}y_2)$ into the equiaffine plane is said to be an *equiaffine immersion* if there exists a vector field ξ along f transversal to f. Then the *Gauss formula* holds:

$$D^f_X f_*X = f_*(\nabla_X X) + h(X,X)\xi,$$

where $X=\mathrm{d}/\mathrm{d}x$ as before. Moreover D^f is the connection on the pull-backed bundle $f^*T\mathbb{R}^2$ induced from D.

Assume that f is non-degenerate, *i.e.*, $\det(\dot{f}(x),\ddot{f}(t))\neq 0$. Then the *equiaffine parameter* s is defined by

$$\mathsf{s}(x) := \int_0^x \det(\dot{f}(x),\ddot{f}(t))^{1/3}\,\mathrm{d}x.$$

The equiffine frame $\mathcal{F}(\mathsf{s})=(e_1(\mathsf{s}),e_2(\mathsf{s}))$ is an $\mathrm{SL}_2\mathbb{R}$-valued function defined by

$$e_1(\mathsf{s}) := f_*\frac{\mathrm{d}}{\mathrm{d}\mathsf{s}}, \quad e_2(\mathsf{s}) := \frac{\mathrm{d}}{\mathrm{d}\mathsf{s}}e_1(\mathsf{s}).$$

The equiaffine Frenet formula is

$$\frac{\mathrm{d}}{\mathrm{d}\mathsf{s}}\mathcal{F}(\mathsf{s}) = \mathcal{F}(\mathsf{s})\begin{pmatrix} 0 & -k(\mathsf{s}) \\ 1 & 0 \end{pmatrix}.$$

The function $k(s)$ is called the *equiaffine curvature*. The Gauss formula becomes

$$D^f_X e_1 = e_2, \quad h(X,X)=1.$$

Definition 5.1. Let I be an open interval equipped with a linear connection ∇. If there exists an affine immersion $f: I\to(\mathbb{R}^2, D)$ with transversal vector field ξ so that the induced connection coincides with ∇, then (I,∇) is said to be *realizable* in $(\mathbb{R}^2, D)$.

Example 5.1. The immersion $f(x)=(x,x^2/2)$ of $(\mathbb{R},\nabla^\circ)$ into $(\mathbb{R}^2,D)$ is realizable with transversal vector field $\xi=(0,1)$.

Example 5.2. The immersion $f(x)=(-2/x^3, 3x^2/5)$ of $\mathbb{R}^+$ into $(\mathbb{R}^2,D)$ is an equiaffine curve with equiaffine parameter x. One can see that

$$e_1(x)=\left(\frac{1}{x^2},\frac{1}{4x^3}\right), \quad e_2(x)=\left(-\frac{2}{x^3},\frac{3x^2}{5}\right),$$

and $k(x)=-6/x^2$. The induced connection is flat.

Example 5.3. Consider the immersion $f(x) = (x^{-p}, (1-x)^{-p})$ of $(0,1)$ into $(\mathbb{R}^2, D)$. Then we have

$$\dot{f}(x) = \left(-px^{-p-1}, p(1-x)^{-p-1}\right), \quad \ddot{f}(x) = \left(p(p+1)x^{-p-2}, p(p+1)(1-x)^{-p-2}\right).$$

Thus we obtain

$$\det(\dot{f}(x), \ddot{f}(x)) = -p^2(p+1)\{x(1-x)\}^{-p-2}.$$

This shows that f is non-degenerate when $p \neq 0, -1$.

$$\frac{\mathrm{d}s}{\mathrm{d}x} = -(p^2(p+1))^{1/3}\{x(1-x)\}^{-(p+2)/3}.$$

In case $0 < p < 1$, s varies on a bounded open interval.

6. HIMC surfaces in space forms

6.1. Harmonic maps

A smooth map $\varphi : (N, \bar{g}, \mathrm{d}v_{\bar{g}}) \to (M, g)$ of an oriented Riemannian manifold $(N, \bar{g}, \mathrm{d}v_{\bar{g}})$ to a Riemannian manifold (M, g) is said to be a *harmonic map* if it is a critical point of the Dirichlet energy functional:

$$E(\varphi) = \int_N \frac{1}{2} g(\mathrm{d}\varphi, \mathrm{d}\varphi)\, \mathrm{d}v_{\bar{g}}.$$

The Euler-Lagrange equation of this variational problem is

$$\tau(\varphi) = \mathrm{tr}_{\bar{g}}(\nabla \mathrm{d}\varphi) = 0.$$

Here $\nabla \mathrm{d}\varphi$ is the *second fundamental form* of φ defined by

$$(\nabla \mathrm{d}\varphi)(W; V) = \nabla^{\varphi}_V \varphi_* W - \mathrm{d}\varphi(\nabla^{\bar{g}}_V W), \quad V, W \in \Gamma(TN),$$

where $\nabla^{\bar{g}}$ is the Levi-Civita connection of $\bar{g}$ and ∇^{φ} is the linear connection on the pull-backed bundle $\varphi^* TM$ induced from the Levi-Civita connection ∇^g of g. The operator $\mathrm{tr}_{\bar{g}}$ is the metrical trace with respect to $\bar{g}$

In case, $\dim N = 2$, the Dirichlet energy is conformal invariant. Thus the harmonicity makes sense for maps from Riemann surfaces into Riemannian manifolds.

Let M be an open interval equipped with a Riemannian metric $e^{2\gamma(x)}\, \mathrm{d}x^2$. Then for a smooth map $x : \Sigma \to M$ from a Riemann surface Σ into M, its tension field $\tau(x)$ is computed as

$$\tau(x) = \frac{4}{E}\left(\frac{\partial^2 x}{\partial z \partial \bar{z}} + \frac{\mathrm{d}\gamma}{\mathrm{d}x}(x)\left|\frac{\partial x}{\partial z}\right|^2\right)\frac{\partial}{\partial x}.$$

Here z is a local complex coordinate and we use a Riemannian metric $E(z, \bar{z})\, \mathrm{d}z\, \mathrm{d}\bar{z}$ in the conformal class of Σ. Hence we obtain

Proposition 6.1. *A smooth map $x : (\Sigma, E(z, \bar{z})\, \mathrm{d}z\, \mathrm{d}\bar{z}) \to (M, e^{2\gamma(x)} \mathrm{d}x^2)$ is a harmonic map if and only if it satisfies*

$$\frac{\partial^2 x}{\partial z \partial \bar{z}} + \frac{\mathrm{d}\gamma}{\mathrm{d}x}\left|\frac{\partial x}{\partial z}\right|^2 = 0.$$

From this characterization one may generalize the notion of harmonic maps in the following manner.

Definition 6.1. Let Σ be a Riemann surface and D be a linear connection on an interval $M \subset \mathbb{R}$ with connection coefficient Γ. Then $x : \Sigma \to (M, D)$ is said to be *affine harmonic* with respect to D if it satisfies

$$\frac{\partial^2 x}{\partial z \partial \bar{z}} + \Gamma(x)\left|\frac{\partial x}{\partial z}\right|^2 = 0.$$

6.2. HIMC surfaces

According to Fujioka [5], let us define a 1-dimensional Riemannian manifold $\mathcal{M}^1(c)$ with $c = 0$ or $c = \pm 1$ in the following manner (see Introduction).

- $c = 0$: $\mathcal{M}^1(0) = \mathbb{R}$ and $g_0 = \mathrm{d}x^2$.
- $c = 1$: $\mathcal{M}^1(1) = \mathbb{R} \cup \{\infty\}$ and

$$g_1 = \frac{\mathrm{d}x^2}{(1+x^2)^2}.$$

- $c = -1$: $\mathcal{M}^1(-1) = (-1, 1)$ and

$$g_{-1} = \frac{\mathrm{d}x^2}{(1-x^2)^2}.$$

Let us consider harmonic maps from Riemann surfaces into $\mathcal{M}^1(c)$. The harmonic map equation for $\varphi : \Sigma \to \mathcal{M}^1(c)$ is given by

$$\frac{\partial^2 x}{\partial z \partial \bar{z}} - \frac{2cx}{1+cx^2} \left| \frac{\partial x}{\partial z} \right|^2 = 0.$$

The harmonic map equation can be solved explicitly.

Proposition 6.2. *Let $\varphi : \Sigma \to \mathcal{M}^1(c)$ be a harmonic map. Then there exists a holomorphic function $f(z)$ on Σ such that*

$$x(z, \bar{z}) = \begin{cases} f + \bar{f} \ \text{if} \ c = 0 \\ \frac{f+\bar{f}}{1-c|f|^2} \ \text{or} \ \frac{1-c|f|^2}{f+\bar{f}}. \end{cases}$$

Definition 6.2 ([2, 5])**.** A conformally immersed surface Σ of a Riemannian space form $\mathcal{M}^3(c)$ of constant curvature $c = 0, \pm 1$ is said to be a *surface of harmonic inverse mean curvature* if its mean curvature function H does not vanish and $1/H$ is a harmonic map into $\mathcal{M}^1(c)$.

7. Orthogonal polynomials

7.1. Weighted Laplacian

Let $(M, g, \mathrm{d}v_g)$ be an oriented Riemannian m-manifold with volume element

$$\mathrm{d}v_g = \sqrt{\det(g_{ij})}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^m.$$

Take a positive smooth function Υ and set $\mathrm{d}\mu = \Upsilon\, \mathrm{d}v_g$. According to Grigor'yan [13], a Riemannian manifold (M, g) equipped with a volume element $\mathrm{d}\mu$ is called a *weighted manifold*. The positive smooth function Υ is called the *density function* of $\mathrm{d}\mu$. The *weighted divergence operator* div_μ is defined by

$$\mathrm{div}_\mu V = \frac{1}{\Upsilon} \mathrm{div}(\Upsilon V), \quad V \in \Gamma(TM).$$

The *weighted Laplacian* $\boldsymbol{\Delta}$ of a weighted manifold $(M, g, \mathrm{d}\mu)$ is introduced as

$$\boldsymbol{\Delta}_\mu = -\mathrm{div}_\mu \circ \mathrm{grad}_g,$$

where grad_g is the gradient operator with respect to g.

The following variant of Green's formula holds for the weighted divergence operator and the weighted Laplacian:

$$\int_M (\mathrm{div}_\mu V) u\, \mathrm{d}\mu = -\int_M g(V, \mathrm{grad}\, u)\, \mathrm{d}\mu = -\int_M V(\boldsymbol{\Delta}_\mu u)\, \mathrm{d}\mu$$

for any smooth function u on M with compact support and any vector field V on M with compact support.

In local coordinate fashion, $\boldsymbol{\Delta}_\mu$ is expressed as

$$\boldsymbol{\Delta}_\mu = -\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} - \sum_{i,j=1}^{n} \left(\frac{1}{\rho} \frac{\partial \rho}{\partial x^i} g^{ij} + \frac{\partial g^{ij}}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

Let us return our attention to the real line $\mathbb{R}$ equipped with a conformal metric $g = e^{2\gamma(x)}\mathrm{d}x^2$. Take a density function $\Upsilon(x)$ and set $\mathrm{d}\mu = \Upsilon\,\mathrm{d}x$. Then the weighted Laplacian is given by

$$-\boldsymbol{\Delta}_\mu = e^{-2\gamma(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(e^{-2\gamma(x)}\frac{\dot{\Upsilon}(x)}{\Upsilon(x)} - \Gamma(x)\right)\frac{\mathrm{d}}{\mathrm{d}x}. \tag{7.1}$$

In particular, for the flat metric $g_0 = \mathrm{d}x^2$, we have

$$-\boldsymbol{\Delta}_\mu = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{\dot{\Upsilon}(x)}{\Upsilon(x)}\right)\frac{\mathrm{d}}{\mathrm{d}x}.$$

7.2. The Rodrigues formula

Let I be an interval and consider the function

$$\Xi(x) = \begin{cases} (x-a)(b-x), & I=[a,b], \quad ,a,b\in\mathbb{R} \\ x-a, & I=[a,+\infty), \quad a\in\mathbb{R}, \\ b-x, & I=[-\infty,b], \quad b\in\mathbb{R} \\ 1, & I=(-\infty,+\infty). \end{cases}$$

Take a positive continuous function $w(x)$ satisfying

$$\left|\int_a^b w(x)\,\mathrm{d}x\right| < \infty.$$

Such a function $w(x)$ is called a *weight*. Let us introduce a sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials by the so-called *Rodrigues formula*:

$$p_n(x) = \frac{C_n}{w(x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(w(x)\Xi(x)^n\right), \quad n=0,1,2,\ldots.$$

Here C_n are normalizing constants.

Lemma 7.1. *If we choose $w(x)$ as*

- $w(x) = (x-a)^\alpha(b-x)^\beta$ *with* $\alpha,\beta > -1$ *if* $I=[a,b]$, a, $b\in\mathbb{R}$,
- $w(x) = (x-a)^\nu e^{-x}$ *with* $\nu > -1$ *if* $I=[a,+\infty]$ *with* $a\in\mathbb{R}$, *or*
- $w(x) = e^{-x^2}$ *if* $I=(-\infty,+\infty)$.

Then the polynomials

$$f_n(x) = \frac{1}{w(x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(w(x)\Xi(x)^n\right), \quad n=0,1,2,\ldots$$

are orthogonal with respect to the inner product

$$\langle F|G\rangle = \int_a^b F(x)G(x)w(x)\,\mathrm{d}x.$$

Moreover every f_n is a solution to the ordinary differential equation:

$$\Xi(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + f_1(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x) = \Lambda_n\,u(x), \tag{7.2}$$

where

$$f_1(x) = \alpha_1 x + c_0, \quad \Xi(x) = \frac{X_0}{2}x^2 + c_1 x + c_2, \quad \lambda_n = n\alpha_1 + \frac{n(n-1)}{2}X_0.$$

Example 7.1 (Legendre polynomials)**.** On the interval $[-1,1]$, we choose

$$w(x) = 1, \quad C_n = \frac{(-1)^n}{2^n n!}.$$

Then the resulting polynomials are orthogonal and called the *Legendre polynomials* (and denoted by $P_n(x)$).

Example 7.2 (Chebyshev polynomials)**.** On the interval $[-1, 1]$, we choose

$$w(x) = \frac{1}{\sqrt{1-x^2}}, \quad C_n = \frac{(-1)^n 2^n n!}{(2n)!}.$$

Then the resulting polynomials are orthogonal and called the *Chebyshev polynomials* (and denoted by $T_n(x)$).

Example 7.3 (Gegenbauer polynomials)**.** On the interval $[-1, 1]$, we choose

$$w(x) = (1-x^2)^{\nu-\frac{1}{2}}, \quad C_n = \frac{(-1)^n (2\nu)_n}{2^n\, n!(\nu+\frac{1}{2})_n}, \quad \nu > -\frac{1}{2}.$$

Then the resulting polynomials are orthogonal and called the *Gegenbauer polynomials* (and denoted by $C_n^{\nu}(x)$).

Example 7.4 (Jacobi polynomials)**.** On the interval $[-1, 1]$, we choose

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad C_n = \frac{(-1)^n}{2^n\, n!}, \quad \alpha, \beta > -1.$$

Then the resulting polynomials are orthogonal and called the *Jacobi polynomials* (and denoted by $P_n^{(\alpha,\beta)}(x)$).

Example 7.5 (Laguerre polynomials)**.** On the interval $[0, +\infty)$, we choose

$$w(x) = e^{-x}, \quad C_n = 1.$$

Then the resulting polynomials are orthogonal and called the *Laguerre polynomials* (and denoted by $L_n(x)$).

Example 7.6 (Sonine polynomials)**.** On the interval $[0, +\infty)$, we choose

$$w(x) = e^{-x}x^{\mu}, \quad C_n = \frac{1}{n!}, \quad \mu > -1.$$

Then the resulting polynomials are orthogonal and called the *Sonine polynomials* (and denoted by $S_n^{\mu}(x)$).

Example 7.7 (Hermite polynomials)**.** On the interval $(-\infty, +\infty)$, we choose

$$w(x) = e^{-x^2}, \quad C_n = (-1)^n, \quad \mu > -1.$$

Then the resulting polynomials are orthogonal and called the *Hermite polynomials* (and denoted by $H_n(x)$).

Grigor'yan [13] pointed out the following interesting fact.

Proposition 7.1. *On the weighted manifold* $(\mathbb{R}, \mathrm{d}x^2, e^{-x^2}\mathrm{d}x)$*, Hermite polynomials are eigenfunctions of the weighted Laplacian. More precisely we have*

$$\mathbf{\Delta}_{\mu} H_n = 2nH_n, \quad n = 0, 1, 2, \dots.$$

Now let us consider orthogonal polynomials $\{f_n\}$ determined by Lemma 7.1. Comparing the ODE (7.2) and the eigenvalue problem:

$$\mathbf{\Delta}_{\mu} f = \lambda f,$$

we notice the following fact pointed out by Crasmareanu.

Proposition 7.2 ([4])**.** *On the weighted manifold* $(I, \mathrm{d}x^2/\Xi, \mathrm{d}\mu)$ *with density function* $\Upsilon(x) = w(x)\Xi(x)$*, each polynomial* $f_n(x)$ *as well as* $p_n(x)$ *are eigenfunctions of the weighted Laplacian. More precisely*

$$\mathbf{\Delta}_{\mu} f_n = \lambda_n f_n, \quad \mathbf{\Delta}_{\mu} p_n = \lambda_n p_n.$$

Here eigenvalues λ_n *are non-negative integers.*

7.3. *Conformal metrics and orthogonal polynomials*

Proposition 7.2 motives us to study conformal metrics $g = \mathrm{d}x^2/\Xi$ derived from orthogonal polynomials.

For instance the conformal metric on $(-1, 1)$ derived from Legendre, Chebyshev, Gegenbauer or Jacobi polynomials is

$$g = \frac{\mathrm{d}x^2}{1-x^2}, \quad \Gamma(x) = \frac{x}{1-x^2}.$$

Then we have

$$Q(x) = \frac{1}{\sqrt{1-x^2}}, \quad s(x) = \sin^{-1} x.$$

Hence the geodesic starting at p with initial velocity v is given by

$$x(s) = \sin\left(\frac{vs}{\cos(\sin^{-1} p)} + \sin^{-1} p\right).$$

For any points x and y, the geodesic segment from x to y is given by

$$\sin\left((\sin^{-1} y - \sin^{-1} x)s + \sin^{-1} x\right).$$

The Riemannian distance is given by

$$\mathrm{d}(x, y) = |\sin^{-1} x - \sin^{-1} y|.$$

The injectivity radius at p is

$$\frac{2\pi - \sin^{-1} p}{\sqrt{1-(\sin^{-1} p)^2}}.$$

The Riemannian manifold $([-1, 1], \mathrm{d}x^2/(1-x^2))$ has the diameter π.

7.4. *Conformal metric* $\mathrm{d}x^2/x$ *on* $\mathbb{R}^+$

Next we study the Riemannian metric

$$g = \frac{\mathrm{d}x^2}{x}$$

on $I = (0, +\infty)$. The connection coefficient is

$$\Gamma(x) = -\frac{1}{2x}.$$

For any points x and y, the geodesic segment from x to y is given by

$$(y - x)\sqrt{s} + x.$$

The Riemannian distance is given by

$$\mathrm{d}(x, y) = 2|\sqrt{x} - \sqrt{y}|.$$

8. Hessian metrics

8.1. *Statistical structures*

Let M be a manifold equipped with a pair (g, ∇) consisting of a Riemannian metric g and a torsion free linear connection ∇. Then (M, g, ∇) is said to be a *statistical manifold* if $C = \nabla g$ is a section of $T^*M \odot T^*M \odot T^*M$. The section C is called the *cubic form* of a statistical manifold (M, g, ∇). One can associate a tensor field K to C by

$$C(U, V, W) = g(K(U)V, W), \quad U, V, W \in \Gamma(TM).$$

Then we have

$$\nabla = \nabla^g - \frac{1}{2}K.$$

The *conjugate connection* ∇^* of ∇ with respect to g is defined by

$$\nabla^* = \nabla^g + \frac{1}{2}K.$$

The conjugate connection is characterized the formula:

$$U\, g(V, W) = g(\nabla_U V, W) + g(V, \nabla^*_U W).$$

Let (M, g, C) be a Riemannian manifold equipped with a section C of $T^*M \odot T^*M \odot T^*M$. Then by introducing a linear connection $\nabla = \nabla^g - K/2$, then we obtain a statistical manifold (M, g, ∇). Thus we may regard (M, g, C) as a statistical manifold.

A statistical manifold (M, g, ∇) is said to be of *trace free* if $\mathrm{tr}_g K = 0$.

Remark 8.1. Properly convex $\mathbb{R}P^n$-structures can be characterized by statistical structures of negative constant curvature. See [29, 31].

Definition 8.1. A statistical manifold (M, g, ∇) is said to be a *Hessian manifold* if the metric g is locally expressed as the Hessian $\mathrm{Hess}^{\nabla}\varPhi$ of some locally defined smooth function $\varPhi$ with respect to ∇. The local function $\varPhi$ is called a *Hesse potential* of g with respect to ∇.

A Hessian manifold of dimension greater than 1 is characterized as a statistical manifold with vanishing curvature $R = R^{\nabla}$ of ∇.

On a Hessian manifold(M, g, ∇), the *Hessian curvature tensor field* H is introduced as [11, 25]:

$$H(U, V)W = \frac{1}{2}(\nabla_U K)(V, W), \quad K = -2(\nabla - \nabla^g).$$

Here use the sign convention of [12]. A Hessian manifold is said to be of *constant Hessian sectional curvature* c if

$$H(U, V)W = -\frac{c}{2}(g(U, V)W + g(W, U)V).$$

Shima [25] proved that a Hessian manifold M is of constant Hessian sectional curvature c if and only if its tangent bundle is of constant holomorphic sectional curvature $-c$.

8.2. Statistical 1-manifolds

Let I be an open interval equipped with a conformal metric $g = e^{2\gamma(x)}\, \mathrm{d}x^2$. Take any linear connection ∇ with connection coefficient $\varGamma(x)$:

$$\nabla_X X = \varGamma(x)\, X, \quad X = \frac{\mathrm{d}}{\mathrm{d}x}.$$

As we saw before, if ∇ is the Levi-Civita connection ∇^g of g, then

$$\varGamma(x) = \frac{\mathrm{d}\gamma}{\mathrm{d}x}(x).$$

To distinguish the connection coefficient of ∇ and that of the Levi-Civita connection ∇^g of g, hereafter we use the following notation.

$$\varGamma(x) = \text{connection coefficient of } \nabla,$$
$${}^g\varGamma(x) = \text{connection coefficient of the Levi-Civita connection } \nabla^g.$$

We have

$$C = \nabla g = 2e^{2\gamma(x)}(\dot\gamma(x) - \varGamma(x))\, \mathrm{d}x^3, \quad \mathrm{d}x^3 = \mathrm{d}x \odot \mathrm{d}x \odot \mathrm{d}x.$$

Hence (I, g, ∇) is always statistical. The operator K is given by

$$K(X)X = -2(\varGamma(x) - \dot\gamma(x))X.$$

Hence (I, g, ∇) is of torsion free if and only if $\nabla = \nabla^g$. It should be remarked that $R = 0$ for any statistical 1-manifold (I, g, ∇).

8.3. Hessian 1-manifolds

Let (I, g, ∇) be a statistical 1-manifold. For any positive smooth function f on I, its *Hessian* $\mathrm{Hess}^{\nabla} f$ with respect to ∇ is given by

$$\mathrm{Hess}^{\nabla} f = \left(\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) - \Gamma(x)\,\frac{\mathrm{d}f}{\mathrm{d}x}(x)\right)\,\mathrm{d}x^2.$$

A 1-manifold (I, g, ∇) is said to be a *Hessian 1-manifold* if the metric g is (locally) expressed as $g = \mathrm{Hess}^{\nabla} \Phi$. In such a case Φ is called a *Hesse potential* of g with respect to ∇. For a 1-manifold (I, ∇) equipped with a linear connection ∇. Then a conformal metric $g = e^{2\gamma(x)}dx^2$ is a Hessian with respect to ∇ for some potential Φ if and only if there exists a solution Φ to the following *Hesse potential equation*:

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2}(x) - \Gamma(x)\,\frac{\mathrm{d}\Phi}{\mathrm{d}x}(x) = \exp(2\gamma(x)). \tag{8.1}$$

For prescribed functions $\Gamma(x)$ and $\gamma(x)$. Let us consider the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu(x) - \Gamma(x)\,\mu(x) = \exp(2\gamma(x)). \tag{8.2}$$

Obviously, the derivative $\mu(x) = \dot{\Phi}(x)$ of the Hesse potential $\Phi(x)$ is a solution to (8.2). The general solution of (8.2) is given by (see [1]):

$$\mu(x) = C\exp\left(\int_{x_0}^{x}\Gamma(u)\,\mathrm{d}u\right) + \int_{x_0}^{x} e^{2\gamma(u)}\exp\left(\int_{u}^{x}\Gamma(v)\,\mathrm{d}v\right)\,\mathrm{d}u.$$

On a Hessian 1-manifold (I, g, ∇) with metric $g = e^{2\gamma(x)}\mathrm{d}x^2$, the Hessian curvature tensor field is given by

$$H(X, X)X = \left(\ddot{\gamma}(x) - 2\Gamma(x)\dot{\gamma}(x) - \dot{\Gamma}(x) + 2\Gamma(x)^2\right)X.$$

Thus the notion of Hessian sectional curvature is valid on (I, g, ∇). The *Hessian sectional curvature* on (I, g, ∇) is defined as the smooth function

$$\mathcal{H} = e^{-2\gamma(x)}\left(\ddot{\gamma}(x) - 2\Gamma(x)\dot{\gamma}(x) - \dot{\Gamma}(x) + 2\Gamma(x)^2\right)$$

on (I, g, ∇). Note that when $\nabla = \nabla^g$, we have $H = 0$.

Example 8.1. On a statistical 1-manifold $(\mathbb{R}^+, \mathrm{d}x^2/x^2, \nabla^{\circ})$, we can see that

$$g = \frac{\mathrm{d}x^2}{x^2} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(-\log x)\,\mathrm{d}x^2.$$

Thus $(\mathbb{R}^+, \mathrm{d}x^2/x^2, \nabla^{\circ})$ is Hessian. The Hessian sectional curvature is constant 1. The tangent bundle of this statistical manifold is the half plane

$$T\mathbb{R}^+(x) = \mathbb{R}^+(x) \times \mathbb{R}(y) = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$$

equipped with the Poincaré metric of constant curvature -1.

Example 8.2 (Binomial distribution)**.** Let us take a sample space $\Omega = \{0, 1, 2, \ldots, n\}$. The probability density function of the binomial distribution $B(n, x)$ is given by

$$p(k; x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}, \quad k \in \Omega,\ x \in I = (0, 1).$$

The set of all binomial distributions on Ω is denoted by $\mathcal{B}(n)$. The Fisher metric g of $\mathcal{B}(n)$ is given by

$$g = \frac{n\,\mathrm{d}x^2}{x(1-x)}.$$

The Levi-Civita connection ∇^g is described as

$$\nabla^g_X X = {}^g\Gamma(x)X, \quad {}^g\Gamma(x) = \frac{2x-1}{2x(1-x)}, \quad X = \frac{\mathrm{d}}{\mathrm{d}x}.$$

It is known that $(\mathbb{R}, g)$ is isometric to

$$\mathbb{S}^1_+(2\sqrt{n}) = \{(y_1, y_2) \in \mathbb{E}^2 \mid y_1^2 + y_2^2 = 4n,\ y_1, y_2 > 0\}$$

equipped with the Riemannian metric induced from $\mathbb{E}^2$.

On $\mathcal{B}(n)$ we equip a linear connection $\nabla = \nabla^{\mathsf{e}}$ called the e-connection (exponential connection) by

$$\nabla_X X = \Gamma(x) X, \quad \Gamma(x) = \frac{2x-1}{x(1-x)}.$$

Thus we have

$$\nabla_X X = 2\nabla^g_X X.$$

Note that ∇ is the Levi-Civita connection of the Riemannian metric

$$g^{\mathsf{e}} = \frac{n\, dx^2}{x^2(1-x)^2}.$$

The tensor field $K = -2(\nabla - \nabla^g)$ is given by

$$K(X)X = -2\nabla^g_X X = -\nabla_X X.$$

The α-connection $\nabla^{(\alpha)} = \nabla^g - \alpha K/2$ is given by $\nabla^{(\alpha)} = (1+\alpha)\nabla^g$. In particular the mixture connection (m-connection) ∇^{m} is determined by $\nabla^{\mathsf{m}}_X X = 0$. Note that ∇^{m} is the conjugate connection of ∇^{e}

Introducing a new coordinate θ by

$$\theta = \log \frac{x}{1-x},$$

and set

$$\Phi(\theta) = n\, \log(1 + e^x).$$

Then θ is an affine coordinate of ∇ and Φ is a Hesse potential of g with respect to ∇. The probability density function is rewritten as

$$p(k;\theta) = \exp(C(k) + F(k)\theta - \Phi(\theta)),$$

where

$$C(k) = \log \begin{pmatrix} n \\ k \end{pmatrix}, \quad F(k) = k.$$

Thus $\mathcal{B}(n)$ is an exponential family (see *c.f.*, [26, Example 6.2]). One can see that $\mathcal{B}(n)$ is a Hessian 1-manifold of constant Hessian sectional curvature $-1/n$ ([26, Example 2.2,2.8, Proposition 3.9]). Note that $\mathcal{B}(n)$ is rewritten as

$$(\mathbb{R}(\theta), g, \nabla^{\circ}), \quad g = \frac{n\, d\theta^2}{(1+e^{\theta})^2}.$$

Here we prove the following important result.

Theorem 8.1. *Every statistical* 1*-manifold is Hessian.*

Proof. Let (I, g, ∇) be a statistical 1-manifold. Take an affine parameter s of ∇. Represent g as $g = e^{2\gamma(s)} ds^2$. Then

$$\Phi(s) = \int_{s_0}^{s} \left(\int_{s_0}^{v} e^{2\gamma(u)} du \right) dv \tag{8.3}$$

is a Hesse potential. □

Molitor studied Hessian 1-manifolds of constant Hessian sectional curvature.

Proposition 8.1 ([22])**.** *Let (M, g, ∇) be a Hessian* 1*-manifold of constant Hessian sectional curvature c, then there exists an affine parameter x with respect to ∇ such that g is locally expressed in the following form*:

1. *If $c = 0$, then $g = a\, e^{bx}\, dx^2$ for some positive constants a and b.*

2. *If $c > 0$, then*

$$g = \frac{a^2 \mathrm{d}x^2}{c\,\cos^2(ax+b)}, \quad \frac{a^2 \mathrm{d}x^2}{c\,\sinh^2(ax+b)} \quad \text{or} \quad \frac{\mathrm{d}x^2}{c\,(x+b)^2}$$

for some positive constant a and constant b.

3. *If $c < 0$, then*

$$g = \frac{a^2 \mathrm{d}x^2}{(-c)\,\cosh^2(ax+b)}$$

for some positive constant a and constant b.

To obtain explicit examples of Hessian 1-manifolds, one need to carry out the integration (8.3). Instead of integration procedure, Bercu, Corcodel and Postolache [1] gave some examples of Hessian 1-manifolds by using special functions, especially orthogonal polynomials.

Example 8.3 (Bessel functions)**.** Let us consider *Bessel equation*:

$$x^2\ddot{y}(x) + x\dot{y}(x) + (x^2 - \alpha^2)y(x) = 0,$$

where α is a constant. The *Bessel function*

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\alpha+n+1)} \left(\frac{x}{2}\right)^{2n+\alpha}$$

is a real analytic function defined on the whole line and satisfies the Bessel equation. Here $\Gamma(x)$ is the Gamma function. One can confirm that

$$\mathit{\Gamma}(x) = -\frac{1}{x}, \quad g = -\frac{x^2-\alpha^2}{x^2} J_\alpha(x)\,\mathrm{d}x^2$$

on an interval I on which g is positive definite. Then (I, g, ∇) is a Hessian 1-manifold with Hesse potential $f(x) = J_\alpha(x)$.

Example 8.4 (Hermite polynomials)**.** The Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}$$

are solutions to Hermite's differential equation:

$$\ddot{y}(x) - 2x\dot{y}(x) + 2ny(x) = 0, \quad n = 0, 1, 2, \ldots.$$

Then we obtain a Hessian structure

$$\mathit{\Gamma}(x) = 2x, \quad g = -2nH_n(x)\,\mathrm{d}x^2$$

on an open interval on which g is positive definite.

Example 8.5 (Legendre polynomials)**.** The Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$$

are solutions to the ODE

$$(1-x^2)\ddot{y}(x) - 2xy'(x) + n(n+1)y(x) = 0, \quad n = 0, 1, 2, \ldots.$$

Then

$$\mathit{\Gamma}(x) = \frac{2x}{1-x^2}, \quad g = -\frac{n(n+1)}{1-x^2} P_n(x)\,\mathrm{d}x^2$$

gives a Hessian structure on an open interval on which g is positive definite.

Example 8.6 (Laguerre polynomials)**.** Let us consider the Laguerre equation

$$x^2\ddot{y}(x) + (1-x)y'(x) + ny(x) = 0, \quad n = 0, 1, 2, \ldots.$$

The Laguerre polynomials

$$L_n(x) = e^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x}x^n)$$

are solutions to the Laguerre equation. One can confirm that

$$\mathit{\Gamma}(x) = -\frac{1-x}{x}, \quad g = -\frac{n}{x}L_n(x)\,\mathrm{d}x^2$$

gives a Hessian structure on an open interval on which g is positive definite.

Example 8.7 (The sinc function)**.** The sinc function

$$\mathrm{sinc}\, x = \frac{\sin x}{x}$$

is a solution to

$$x\ddot{y}(x) + 2\dot{y}(x) + xy(x) = 0.$$

More generally $y(x) = \lambda \mathrm{sinc}\,(\lambda x)$ is a solution to

$$x\ddot{y}(x) + 2\dot{y}(x) + \lambda^2 xy(x) = 0.$$

Here λ is a positive constant. The function $\mathrm{sinc}(\pi x)$ is often called the *normalized sinc function* and used in digital processing and information theory. By using sinc function we may construct a Hessian structure

$$\mathit{\Gamma}(x) = -\frac{2}{x}, \quad g = -\mathrm{sinc}\, x\,\mathrm{d}x^2.$$

Example 8.8 (Chebyshev polynomials)**.** The Chebyshev polynomials $T_n(x)$ are solutions to

$$(1-x^2)\ddot{y}(x) - xy'(x) + n^2 y(x) = 0, \quad n = 0, 1, 2, \ldots.$$

One can confirm that

$$\mathit{\Gamma}(x) = \frac{x}{1-x^2}, \quad g = -\frac{n^2}{1-x^2}T_n(x)\,\mathrm{d}x^2$$

gives a Hessian structure on an open interval on which g is positive definite.

Problem 1. Compute the Hessian sectional curvatures of Hessian 1-manifolds derived from orthogonal polynomials.

8.4. Product manifolds

Bercu, Corcodel and Postolache [1] studied product manifolds of the form

$$(\mathbb{R}(x), e^{2\gamma(x)}\,\mathrm{d}x^2) \times (\mathbb{R}(y), \mathrm{d}y^2).$$

The product manifold is interpreted as the Cartesian plane $\mathbb{R}^2(x, y)$ equipped with the Riemannian metric

$$g = e^{2\gamma(x)}\mathrm{d}x^2 + \mathrm{d}y^2.$$

Take a smooth function $f(x, y)$ of the form

$$f(x, y) = \phi(x) + \psi(y).$$

Let us consider the Hessian $\mathrm{Hess}^g f$ with respect to the Levi-Civita connection of the product metric g. Bercu, Corcodel and Postolache studied the problem when the Hessian metric $\mathrm{Hess}^g f$ induces the Levi-Civita connection of g. Concerning on this problem, they obtained the following result.

Theorem 8.2 ([1])**.** *Let us set*

$$\phi(x) = \int_{x_0}^{x}\left(k + \int_{x_0}^{t} e^{\gamma(t)}\,\mathrm{d}t\right) e^{\gamma(t)}\,\mathrm{d}t,$$

where $k = C\,e^{-\gamma(x_0)}$ and C is an arbitrary constant. Then

$$f(x, y) = \phi(x) + \frac{y^2}{2} + ay + b, \quad a, b \in \mathbb{R}$$

produces a Hessian metric $\mathrm{Hess}^g f$ whose Levi-Civita connection coincides with that of g.

9. Statistically harmonic maps and statistically biharmonic maps

9.1. Statistically harmonic maps

Here we recall the following notion from our work [17]:

Definition 9.1 ([17])**.** Let (M, g, ∇) be a statistical manifold and $\varphi : M \to M$ a smooth map. Then f is said to be *statistically harmonic* if its *statistical tension field*

$$\tau_g^{\nabla}(\varphi) = \mathrm{tr}_g(\nabla^{\mathsf{S}}\mathrm{d}\varphi)$$

vanishes. Here the *statistical second fundamental form* $\nabla^{\mathsf{S}}\mathrm{d}\varphi$ of φ is defined by

$$(\nabla^{\mathsf{S}}\mathrm{d}\varphi)(W; V) = \nabla_V^{*\varphi}\varphi_* W - \varphi_*(\nabla_V W),$$

where $\nabla^{*\varphi}$ is the connection on $\varphi^* TM$ induced from the conjugate connection ∇^* of ∇.

In case $\nabla = \nabla^g$, the statistical-harmonicity is equivalent to the usual harmonicity.

Problem 2. Classify statistically harmonic automorphisms on statistical Lie groups, *e.g.*, on the statistical Lie group of normal distributions. For harmonic inner automorphisms of compact semi-simple Lie groups, see [24].

Now let us deduce the statistically harmonic map equation for a smooth map

$$y : (I, e^{2\gamma(x)}\mathrm{d}x^2, \nabla) \to (I, e^{2\gamma(y)}\mathrm{d}y^2, \nabla^*).$$

We can take a unit vector field

$$E = e^{-\gamma(x)}X, \quad X = \frac{\mathrm{d}}{\mathrm{d}x}$$

on the domain of $y = y(x)$. Since $\nabla_X X = \mathit{\Gamma} X$, one can see that

$$\nabla_E E = e^{-2\gamma(x)}(\mathit{\Gamma}(x) - \dot{\gamma}(x))X = -\frac{1}{2}\mathrm{tr}_g K.$$

Next, we get

$$y_* X = \dot{y}(x)\, Y, \quad Y = \frac{\mathrm{d}}{\mathrm{d}y}.$$

From this formula, we get

$$y_*(\nabla_X X) = \mathit{\Gamma}(y(x))\dot{y}(x)\, Y.$$

On the other hand, we have

$$\nabla_X^{*y} y_* X = (\ddot{y}(x) + \mathit{\Gamma}^*(y(x))\dot{y}(x)^2)\, Y,$$

where $\mathit{\Gamma}^*$ is the connection coefficient of the conjugate connection ∇^*. Hence

$$\nabla_E^{*y} y_* E = e^{-2\gamma(x)}(\ddot{y}(x) + \mathit{\Gamma}^*(y(x))\dot{y}(x)^2)\, Y.$$

Thus we obtain the formula:

$$\tau_g^{\nabla}(y) = e^{-2\gamma(x)}\left(\ddot{y}(x) + \{2\dot{\gamma}(x) - \mathit{\Gamma}(y(x))\}\dot{y}(x)^2 - \mathit{\Gamma}(y(x))\dot{y}(x)\right) Y.$$

Here we used the formula $\mathit{\Gamma}^* = 2\dot{\gamma} - \mathit{\Gamma}$.

Proposition 9.1. *A smooth map* $y : (I, g, \nabla) \to (I, g, \nabla^*)$ *is statistically harmonic if and only if* $y = y(x)$ *satisfies*

$$\ddot{y}(x) + \{2\dot{\gamma}(x) - \mathit{\Gamma}(y(x))\}\dot{y}(x)^2 - \mathit{\Gamma}(y(x))\dot{y}(x) = 0. \tag{9.1}$$

It should be remarked that even if $\nabla = \nabla^g$, the ordinary differential equation can *not* be the geodesic equation unless $\dot{\gamma} = \mathit{\Gamma} = 0$. The geodesic equation

$$\ddot{y}(x) + \{2\dot{\gamma}(x) - \mathit{\Gamma}(y(x))\}\dot{y}(x)^2 = 0 \tag{9.2}$$

of D^* is derived from the setting

$$y : (I, \mathrm{d}x^2, \nabla^g) \to (I, \nabla^*).$$

Analogously, the geodesic equation

$$\ddot{y}(x) + \mathit{\Gamma}(y(x))\,\dot{y}(x)^2 = 0 \tag{9.3}$$

of ∇ is derived from the setting

$$y : (I, \mathrm{d}x^2, \nabla^g) \to (I, \nabla).$$

The geodesic equation (9.3) does not depend on the Riemannian metrics on the target 1-manifold.

Problem 3. Construct explicit examples of statistical harmonic maps on 1-dimensional statistical manifolds by using orthogonal polynomials.

Remark 9.1. One may consider the following conditions for smooth maps of a statistical manifold M into itself:

- $\tau_g^{+,0}(\varphi) = \mathrm{tr}_g(\nabla^{+,0}\mathrm{d}\varphi) = 0$, where

$$(\nabla^{+,0}\mathrm{d}\varphi)(W; V) = \nabla^{\varphi}_V \varphi_* W - \varphi_*(\nabla^g_V W),$$

  and ∇^{φ} is the connection on φ^*TM induced from ∇.
- $\tau_g^{0,+}(\varphi) = \mathrm{tr}_g(\nabla^{0,+}\mathrm{d}\varphi) = 0$, where

$$(\nabla^{0,+}\mathrm{d}\varphi)(W; V) = \nabla^{g;\varphi}_V \varphi_* W - \varphi_*(\nabla_V W),$$

  and $\nabla^{g;\varphi}$ is the connection on φ^*TM induced from the Levi-Civita connection ∇^g of g.

Obviously for the identity map id,

$$\tau_g^{+,0}(\mathrm{id}) = 0 \iff \tau_g^{0,+}(\mathrm{id}) = 0 \iff \tau_g^{\nabla}(\mathrm{id}) = 0 \iff \mathrm{tr}_g K = 0.$$

9.2. Statistically biharmonic maps

Let us return once to general situation. Let (M, g, ∇) be a statistical manifold and $\varphi : M \to M$ a smooth map. When we choose $\varphi = \mathrm{id}$ the identity map. In case $\nabla = \nabla^g$, id is automatically harmonic. The stability of identity maps was studied extensively in 1970's and 1980's. On the other hand, we know the following fact.

Proposition 9.2 ([17]). *On a statistical manifold (M, g, ∇), the identity map is statistically harmonic when and only when (M, g, ∇) is of trace free.*

As a result, the identity map of a 1-dimensional statistical manifold (I, g, ∇) can not be statistically harmonic if $\nabla \neq \nabla^g$. Indeed, if $y = x$, then (9.1) becomes

$$\tau_g^{\nabla}(x) = 2e^{-2\gamma(x)}(\dot{\gamma}(x) - \mathit{\Gamma}(x))X = 0.$$

This formula means that $\tau(x)$ measures how ∇ is far from ∇^g. In other words, the trace free condition is characterized by the statistical-harmonicity of the identity map.

For a smooth map $\varphi : M \to M$ from an oriented statistical manifold $(M, g, \nabla, \mathrm{d}v_g)$ into itself, one can consider the functional (called the *bienergy*):

$$E_2(\varphi) = \int_M \frac{1}{2} g(\tau_g^{\nabla}(\varphi), \tau_g^{\nabla}(\varphi))\,\mathrm{d}v_g.$$

A smooth map φ is said to be *statistically biharmonic* if it is a critical point of the bienergy.

As we mentioned above, the trace free condition of (M, g, ∇) is equivalent to the statistical harmonicity of the identity map. Here we propose the following problem:

Problem 4. When is the identity map of a statistical manifold statistically biharmonic ?

Remark 9.2. The notion of statistical biharmonicity in this article is more restrictive than that of [12].

Let (M_1, g_1, ∇^1) and (M_2, g_2, ∇^2) be statistical manifolds. Assume that M_1 is oriented by an volume element $\mathrm{d}v_{g_1}$. For a smooth map $\varphi : M_1 \to M_2$, set

$$\tau_1(\varphi) = \mathrm{tr}_{g_1}(\nabla^{2,1,\varphi}\mathrm{d}\varphi),$$

where

$$(\nabla^{2,1,\varphi}\mathrm{d}\varphi)(Y;X)=\nabla^{2,\varphi}_X\varphi_*Y-\varphi_*(\nabla^1_XY),$$

where $\nabla^{2,\varphi}$ is the connection on φ^*TM_2 induced from ∇^2. One can see that $\tau_1(\varphi)$ depends on the statistical structures (g_1,∇^1) on M_1 and the connection ∇^2. It does *not* depend on the metric g_2. The bienergy functional proposed in [12] is

$$E_2(\varphi)=\int_{M_1}\frac{1}{2}g_2(\tau_1(\varphi),\tau_1(\varphi))\,\mathrm{d}v_{g_1}.$$

By computing the Euler-Lagrange equations of E_2 with respect to compactly supported variations, they deduced the Euler-Lagrange equation $\tau_2(\varphi)=0$, where

$$\tau_2(\varphi)=\boldsymbol{\Delta}^{\varphi}\tau_1(\varphi)-\frac{1}{2}\mathrm{div}_{g_1}(\mathrm{tr}_{g_1}K_1)\,\tau_1(\varphi)-\mathrm{tr}_{g_1}L_2(\mathrm{d}\varphi,\tau_1(\varphi))\mathrm{d}\varphi+\frac{1}{2}K_2(\tau_1(\varphi))\tau_1(\varphi).$$

Here

$$\nabla^1-\nabla^{g_1}=-\frac{1}{2}K_1,\quad \nabla^2-\nabla^{g_2}=-\frac{1}{2}K_2,$$

$$g_2(L_2(Z,W)X,Y)=g_2(R^{\nabla^2}(X,Y)Z,W).$$

The operator $\boldsymbol{\Delta}^{\varphi}$ is the Laplace-Beltrami operator of the vector bundle $(\varphi^*TM_2,\nabla^{2,\varphi},\varphi^*g_2)$.

A statistically biharmonic map in the sense of Furuhata-Ueno [12] is a smooth map satisfying $\tau_2(\varphi)=0$.

If we choose

$$M_1=M_2=M,\quad g_1=g_2=g,\quad \nabla^1=\nabla,\quad \nabla^2=\nabla^*,$$

then the statistically biharmonicicity of φ in the sense of [12] coincides with ours.

Problem 5. Complexify all the stories in this article.

A. The moduli problem

As we saw before, the statistical manifold $\mathcal{B}(n)$ of the binomial distributions is one of the typical example of Hessian 1-manifold. On the other hand the statistical manifold $\mathcal{N}$ of the normal distributions is the most well known example of Hessian 2-manifold.

Kito [18] studied the moduli problems of Hessian structures on the Euclidean n-space $\mathbb{E}^n$ and the hyperbolic n-space $\mathbb{H}^n$ of constant curvature -1 with $n>1$. More precisely he studied the set

$$\mathcal{H}(M,g)=\{C\in\Gamma(T^*M\odot T^*M\odot T^*M)\mid (M,g,C)\text{ is Hessian }\}$$

for $M=\mathbb{E}^n$ and $M=\mathbb{H}^n$. Here we interpret a Hessian structure on a manifold M as a pair (g,C) consisting of a Riemannian metric g and a symmetric covariant tensor field C of degree 3. Kito [18] proved the following results.

Theorem A.1. *The set $\mathcal{H}(\mathbb{E}^n)$ has at least the freedom of n functions on $\mathbb{R}$. In particular, the set $\mathcal{H}(\mathbb{T}^n)$ of Hessian structure of the flat torus has at least the freedom of n periodic functions. $\mathbb{T}^n$.*

Theorem A.2. *The set $\mathcal{H}(\mathbb{H}^n)$ has at least the freedom of $(n-1)$ functions on $\mathbb{R}$.*

In a local situation Kito obtained the following result.

Theorem A.3. *The set $\mathcal{H}(\mathbb{E}^2,\mathbf{0})$ of Hessian structures of a neighborhood of the origin has the freedom of three local functions.*

On the other hand, in our previous work [10] we studied left invariant statistical structures on the statistical manifold $\mathcal{N}$ of normal distributions. The set

$$\mathcal{N}=\{N(x,y^2)\mid x,y\in\mathbb{R},\ y>0\}$$

of all normal distributions $N(x,y^2)$ (of mean x and variance y) is identified with the upper half plane

$$\{(x,y)\in\mathbb{R}^2\mid y>0\}.$$

The *Fisher metric*

$$g = \frac{dx^2 + 2dy^2}{y^2}$$

and e-connection (exponential connection)

$$\nabla^{\mathsf{e}}_{\partial x}\partial_x = 0, \quad \nabla^{\mathsf{e}}_{\partial x}\partial_y = \nabla^{\mathsf{e}}_{\partial y}\partial_x = -\frac{2}{y}\,\partial_x, \quad \nabla^{\mathsf{e}}_{\partial y}\partial_y = -\frac{3}{y}\partial_y$$

gives a Hessian structure (g, ∇^{E}). Moreover the m-connection (mixture connection)

$$\nabla^{\mathsf{m}}_{\partial x}\partial_x = \frac{1}{y}\,\partial_y, \quad \nabla^{\mathsf{m}}_{\partial x}\partial_y = \nabla^{\mathsf{m}}_{\partial y}\partial_x = 0, \quad \nabla^{\mathsf{m}}_{\partial y}\partial_y = \frac{1}{y}\partial_y$$

also defines a Hessian structure (g, ∇^{m}). The triplet $(g, \nabla^{\mathsf{e}}, \nabla^{\mathsf{m}})$ is referred as to a *dually flat structure*. More generally we know the one-parameter family of statistical structures $\{(g, \nabla^{(\alpha)})\}_{\alpha\in\mathbb{R}}$ on $\mathcal{N}$. The connection $\nabla^{(\alpha)}$ defined by

$$\nabla^{(\alpha)}_{\partial x}\partial_x = \frac{1-\alpha}{2y}\,\partial_y, \quad \nabla^{(\alpha)}_{\partial x}\partial_y = \nabla^{(\alpha)}_{\partial y}\partial_x = -\frac{1+\alpha}{y}\partial_x, \quad \nabla^{(\alpha)}_{\partial y}\partial_y = -\frac{1+2\alpha}{y}\partial_y$$

is called the *Amari-Chentsov α-connection*. Note that

$$\nabla^{(1)} = \nabla^{\mathsf{e}}, \quad \nabla^{(-1)} = \nabla^{\mathsf{m}}, \quad \nabla^{(0)} = \nabla^{g} \text{ (Levi-Civita connection of } g).$$

The statistical manifold $(\mathcal{N}, g, \nabla^{(\alpha)})$ is identified with the Lie group

$$\left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R},\ y > 0 \right\}.$$

The statistical structures are left invariant. By suitable modification, Kito's result is rephrased for $\mathcal{N}$ as follows:

Corollary A.1. *The set $\mathcal{H}(\mathcal{N}, g)$ has at least the freedom of one functions on $\mathbb{R}$.*

On the other hand the α-connections are characterized in our work [10] as follows:

Theorem A.4 ([10])**.** *The only left invariant connections on the Lie group of normal distributions compatible to the Fisher metric which are conjugate symmetric are Amari-Chentsov α-connections. In particular the only left invariant connections on the Lie group of normal distributions which together with Fischer metric define Hessian structures are* e*-connection and* m*-connection.*

Motivated by Kito's work [18] and our previous work, here we propose the following problem:

Problem 6. Classify all the left invariant linear connections on the Lie group of normal distributions which is compatible to the Fisher metric g.

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Competing interests

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