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RESEARCH ARTICLE

Conformal Geometric Solitons on the Tangent Bundle of a Statistical Manifold

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ABSTRACT

Let (M_n, ∇, g) denote a statistical manifold equipped with a torsion-free linear connection ∇ and a (pseudo-) Riemannian metric g. The tangent bundle TM of the statistical manifold (M_n, ∇, g) is endowed with a twisted Sasaki metric, denoted as G. The objective of this paper is to explore conformal Ricci, conformal Yamabe, and conformal Ricci-Yamabe solitons on the tangent bundle TM concerning the twisted Sasaki metric G.

Keywords: *Conformal soliton; tangent bundle; twisted Sasaki metric; statistical manifold.*

AMS Subject Classification (2020): *Primary: 53C07, 53B05,53B12; Secondary: 53A45.*

1. Introduction

The theory of statistical manifolds, also known as information geometry, originated in 1945 with the conceptualization of a statistical model as a Riemannian manifold characterized by the Fisher information matrix [25]. The field of information geometry subsequently evolved as an exploration of diverse geometric structures inherent in statistical manifolds, representing models of probability distributions. The introduction of the dual connection, alternatively termed the conjugate connection in affine geometry, was pioneered by Amari in 1985 [6]. A statistical manifold is defined by a statistical model equipped with a Riemannian metric and a pair of dual affine connections, encapsulating these geometric features.

In 1982, Hamilton [19] proposed the concept of Ricci flow and demonstrated its existence. This idea was developed to address Thurston's geometric conjecture, which posits that every closed three-manifold can be geometrically decomposed. Additionally, Hamilton [19] classified all compact manifolds with a positive curvature operator in dimension four. The Ricci flow equation is given by:

$$\frac{\partial g}{\partial t} = -2S.$$

Here, S represents the Ricci tensor, g is the Riemannian metric, and t is time. Hamilton introduced the notion of a Ricci soliton, which is a self-similar solution to the Ricci flow. A Ricci soliton [19, 20] is characterized by moving only through a one-parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by:

$$L_X g + 2S = 2\lambda g.$$

Here L_X is the Lie derivative, S is the Ricci tensor, g is the Riemannian metric, X is a vector field, and λ is a scalar. The Ricci soliton is classified as shrinking, steady, or expanding based on whether λ is positive, zero, or negative, respectively. In [17] the authors studied almost Ricci and Yamabe solitons on tangent bundle.

Fischer formulated the notion of conformal Ricci flow [16]. This variation of the classical Ricci flow equation modifies the unit volume constraint to a scalar curvature constraint. The conformal Ricci flow on a smooth closed connected oriented n-dimensional manifold M is defined by the equation [16]:

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg.$$

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Here, p is a scalar non-dynamical field (time-dependent scalar field), $r(g)$ is the scalar curvature of the manifold, and n is the dimension of the manifold. The constraint $r(g) = -1$ is also imposed. In 2015, Basu and Bhattacharyya [8] introduced the concept of the conformal Ricci soliton equation:

$$L_X g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g \tag{1.1}$$

Here, λ is a constant. This equation serves as a generalization of the Ricci soliton equation and also satisfies the conformal Ricci flow equation.

The concept of the Yamabe flow was originally introduced by Hamilton [19] for the purpose of constructing Yamabe metrics on compact Riemannian manifolds. On a Riemannian or pseudo-Riemannian manifold M, a time-dependent metric $g(., t)$ evolves according to the Yamabe flow if it satisfies the equation:

$$\frac{\partial g}{\partial t} = -rg, g(0) = g_0,$$

where r represents the scalar curvature of the manifold M. A Yamabe soliton [7], corresponding to a self-similar solution of the Yamabe flow, is defined on a Riemannian or pseudo-Riemannian manifold (M, g) as

$$\frac{1}{2} L_X g = (r - \lambda)g. \tag{1.2}$$

Using (1.1) and (1.2), authors defined the concept of a conformal Yamabe soliton in [26]: A Riemannian or pseudo-Riemannian manifold (M, g) of dimension n is said to admit a conformal Yamabe soliton if

$$L_X g + \frac{1}{2}\left[2\lambda - 2r - \left(p + \frac{2}{n}\right)\right] g = 0.$$

The conformal Yamabe soliton is categorized as expanding, steady, or shrinking based on whether λ is positive, zero, or negative, respectively.

In [18], Guler and Crasmareanu introduced a novel geometric flow called the Ricci–Yamabe flow, which is a scalar combination of Ricci and Yamabe flows. This flow, denoted as Ricci–Yamabe flow of the type (α, β), is characterized by the parameters α and β. Specifically, the (α, β)-Ricci–Yamabe flow is defined as follows:

- Ricci flow [20] is represented when $\alpha = 1$ and $\beta = 0$.
- Yamabe flow [19] is captured when $\alpha = 0$ and $\beta = 1$.
- Einstein flow [14] is identified when $\alpha = 1$ and $\beta = -1$.

A soliton to the Ricci–Yamabe flow is termed a Ricci–Yamabe soliton, provided it evolves solely through a one-parameter group of diffeomorphism and scaling. For an n-dimensional Riemannian manifold (M, g) with $n > 2$, the Riemannian metric g is considered to admit an (α, β)-Ricci–Yamabe soliton, or simply a Ricci–Yamabe soliton denoted as $(g, X, \Lambda, \alpha, \beta)$, if it satisfies the equation:

$$L_X g + 2\alpha S = [2\Lambda - \beta r]\, g, \tag{1.3}$$

where Λ, α, β are real scalars. Utilizing equations (1.3) and (1.1), the concept of a conformal Ricci–Yamabe soliton is introduced as follows [27]: An n-dimensional Riemannian manifold (M, g) with $n > 2$ is considered to possess a conformal Ricci–Yamabe soliton if it satisfies the equation:

$$L_X g + 2\alpha S + \left[2\Lambda - \beta r - \left(p + \frac{2}{n}\right)\right] g.$$

The conformal Ricci–Yamabe soliton is categorized as expanding, steady, or shrinking based on whether Λ is positive, zero, or negative, respectively.

In addition to these, many interesting studies on soliton structures in tangent bundles have been conducted recently, and the different properties of soliton structures have been studied according to various metrics. For example (See [2, 3, 11, 23])

The paper endeavors to provide characterizations of conformal Ricci, conformal Yamabe, and conformal Ricci-Yamabe solitons on the tangent bundle TM over on statistical manifold according to the twisted Sasaki metric G.

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞.

2. The twisted Sasaki metric on the tangent bundle over a statistical manifold

Statistical manifolds, with their broad applications in fields such as information science, information theory, neural networks, and statistical mechanics, represent a geometric model where points correspond to probability distributions [4, 5, 9, 12, 13, 21, 24, 29]. These manifolds are defined by a statistical structure (∇, g) on a differentiable manifold M, where g denotes a (pseudo-) Riemannian metric, and ∇ represents a torsion-free linear connection satisfying the property ∇g is totally symmetric. This structure, termed a statistical manifold, extends the concept of (pseudo-) Riemannian manifolds, providing a broader geometric framework for modeling probabilistic phenomena.

Given an arbitrary linear connection ∇ on a (pseudo-) Riemannian manifold (M, g), a $(0,3)$-tensor field F is introduced as

$$F(X,Y,Z) := (\nabla_{Z}g)(X,Y).$$

This tensor field F is denoted as the cubic form linked with the pair (∇, g) [15]. If we have a symmetric bilinear form ρ defined on a manifold M, we term the pair (∇, ρ) as a Codazzi pair if the covariant derivative $\nabla\rho$ is (totally) symmetric concerning vector fields X, Y, Z [28]:

$$(\nabla_{Z}\rho)(X,Y) = (\nabla_{X}\rho)(Z,Y) = (\nabla_{Y}\rho)(Z,X).$$

Expressed in terms of the cubic form F, this condition can be rephrased as:

$$F(X,Y,Z) = F(Z,Y,X) = F(Z,X,Y),$$

which implies that the condition for (∇, g) to form a Codazzi pair is equivalent to F being entirely symmetric with respect to all of its indices.

Consider an n-dimensional statistical manifold denoted as (M_n, ∇, g). In this exposition, we employ the C^∞-category to thoroughly explicate various concepts, focusing on connected manifolds with a dimension of $n > 1$. To facilitate our analysis, we introduce the tangent bundle of M_n, denoted as TM, with the natural projection defined as $\pi : TM \to M_n$. When utilizing a system of local coordinates $\left(U, x^i\right)$ in M_n, it induces a corresponding system of local coordinates on TM, denoted as $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i\right)$, where $\bar{i}$ ranges from $n+1$ to $2n$. Here, $\left(u^i\right)$ represents the Cartesian coordinates within each tangent space T_pM for all $p \in U$, noting that p is any arbitrary point within U.

Consider the linear connection ∇ on the statistical manifold (M_n, ∇, g). The tangent space of the tangent bundle TM can be decomposed into two distributions: the horizontal distribution determined by ∇ and the vertical distribution defined by $\ker \pi_*$. In this context, the local frame is given by

$$E_i = \frac{\partial}{\partial x^i} - u^s \Gamma^h_{is} \frac{\partial}{\partial u^h};\ i = 1, ..., n,$$

and

$$E_{\bar{i}} = \frac{\partial}{\partial u^i};\ \bar{i} = n+1, ..., 2n.$$

Here, Γ^h_{is} represents the Christoffel symbols of the linear connection ∇. The local frame $\{E_\beta\} = (E_i, E_{\bar{i}})$ is commonly referred to as the adapted frame. Let $A = A^i \frac{\partial}{\partial x^i}$ be a vector field. We can obtain the horizontal and vertical lifts of A with respect to the adapted frame as follows [31]:

$$\begin{aligned} {}^H A &= A^i E_i, \\ {}^V A &= A^i E_{\bar{i}}. \end{aligned}$$

Within TM, the local $1-$form system $\left(dx^i, \delta u^i\right)$ forms the dual frame of the adapted frame $\{E_\beta\}$, where:

$$\delta u^i = {}^H\left(dx^i\right) = du^i + u^s \Gamma^i_{hs} dx^h.$$

Lifting from the Riemannian manifold (M_n, g) to its tangent bundle TM, various Riemannian or pseudo-Riemannian metrics have been formulated. These metrics, also known as g-natural metrics, are created by naturally extending the Riemannian metric g to the tangent bundle TM [10]. In [1], the authors developed a comprehensive family of Riemannian g-natural metrics based on six arbitrary functions that define the norm of a vector $u \in TM$. The exploration of natural metrics on tangent bundles arises from the imperative to comprehend the geometric and physical properties of entities in motion on a Riemannian manifold. These metrics offer a means to extend the geometric attributes of the base manifold to the tangent bundle, proving indispensable in diverse fields such as physics, differential geometry, and mechanics. Now, let us introduce the twisted Sasaki metric on the tangent bundle of a statistical manifold.

Definition 2.1. [22] Let (M_n, ∇, g) be a statistical manifold equipped with a torsion-free linear connection ∇ and a (pseudo-) Riemannian metric g and $a, b \in \mathbb{R}$. On the tangent bundle TM, the twisted Sasaki metric G is defined by

$$\begin{aligned} i)\ G(^H X, ^H Y) &= ag(X, Y), \\ ii)\ G(^V X, ^H Y) &= 0, \\ iii)\ G(^V X, ^V Y) &= bg(X, Y) \end{aligned}$$

for all vector fields X, Y on (M_n, ∇, g).

For the linear connection $\tilde{\nabla}$ of the twisted Sasaki metric G, we give the following proposition.

Proposition 2.1. *[22] Let (M_n, ∇, g) be a statistical manifold equipped with a torsion-free linear connection ∇ and a (pseudo-) Riemannian metric g and (TM, G) be its tangent bundle with the twisted Sasaki metric G. The local expression for the Levi-Civita connection $\tilde{\nabla}$ associated with the twisted Sasaki metric G on TM can be stated as follows:*

$$\begin{aligned} \tilde{\nabla}_{E_i} E_j &= \left(\Gamma_{ij}^k\right) E_k + \left(\frac{1}{2} y^s R_{jis}{}^k\right) E_{\bar{k}}, \\ \tilde{\nabla}_{E_i} E_{\bar{j}} &= \left(\frac{b}{2\alpha} y^s R_{sji}{}^k\right) E_k + \left(\Gamma_{ij}^k + \frac{1}{2} g^{km} \left(\nabla_i g_{mj}\right)\right) E_{\bar{k}}, \\ \tilde{\nabla}_{E_{\bar{i}}} E_j &= \left(\frac{b}{2\alpha} y^s R_{sij}{}^k\right) E_k + \left(\frac{1}{2} g^{km} \left(\nabla_j g_{mi}\right)\right) E_{\bar{k}}, \\ \tilde{\nabla}_{E_{\bar{i}}} E_{\bar{j}} &= \left(-\frac{b}{2a} g^{kh} \left(\nabla_h g_{ij}\right)\right) E_k, \end{aligned}$$

where R is the Riemannian curvature tensor of ∇.

Next, we give the Ricci curvature tensor and scalar curvature.

Proposition 2.2. *[22] Let (M_n, ∇, g) be a statistical manifold equipped with a torsion-free linear connection ∇ and a (pseudo-) Riemannian metric g and (TM, G) be its tangent bundle with the twisted Sasaki metric G. Then the corresponding Ricci curvature tensor is given locally by ($\tilde{R}_{IJ} = \tilde{R}_{MIJ}{}^M$ is Ricci curvature tensor.):*

$$\begin{aligned} \tilde{R}_{ij} &= R_{ij} + \frac{b}{4a} y^s y^p \left[R_{mis}{}^h R_{phj}{}^m + R_{msi}{}^h R_{jhp}{}^m\right] - \frac{1}{4} \left(\nabla_i g^{ml}\right) \left(\nabla_j g_{ml}\right) \\ &\quad - \frac{1}{2} g^{ml} \left(\nabla_i \nabla_j g_{ml}\right), \\ \tilde{R}_{\bar{i}j} &= \frac{b}{2a} y^s \nabla_m R_{sij}{}^m + \frac{b}{4a} y^s \left[R_{sij}{}^h A_{hm}^m + R_{jsm}{}^h A_{ih}^m\right], \\ \tilde{R}_{i\bar{j}} &= \frac{b}{2a} y^s \nabla_m R_{sji}{}^m + \frac{b}{4a} y^s \left[R_{sji}{}^h A_{hm}^m + R_{ism}{}^h A_{jh}^m\right], \\ \tilde{R}_{\bar{i}\bar{j}} &= \frac{b}{4a} \left[2A_{hi}^m A_{mj}^h - A_{hm}^m A_{ij}^h\right] - \frac{b}{2a} \left(\nabla_m A_{ij}^m\right) - \frac{b^2}{4a^2} y^s y^p R_{sih}{}^m R_{pjm}{}^h, \end{aligned}$$

where R is Riemannian curvature tensor of the linear connection ∇ and $A_{ij}^k = g^{kl} \left(\nabla_i g_{jl}\right)$.

Proposition 2.3. *[22] Let (M_n, ∇, g) be a statistical manifold equipped with a torsion-free linear connection ∇ and a (pseudo-) Riemannian metric g and (TM, G) be its tangent bundle with the twisted Sasaki metric G. Then the corresponding scalar curvature $\tilde{r}$ is locally given by*

$$\tilde{r} = \frac{1}{a} r + \frac{b}{4a^2} \|R\| + \frac{1}{4a} g^{ij} \left[A_{hi}^m A_{mj}^h - A_{hm}^m A_{ij}^h\right] - \frac{1}{2a} g^{ij} \left[\nabla_i A_{jm}^m + \nabla_m A_{ij}^m\right].$$

where $A_{ij}^k = g^{kl} \left(\nabla_i g_{jl}\right)$ and r, R are the scalar curvature and Riemannian curvature tensors of the torsion-free linear connection ∇, respectively. Also in here $\|R\| = y^s y^p R_{silh} R_p{}^{ilh}$.

3. Main results

Let $L_{\widetilde{V}}$ denotes the Lie derivative with respect to the vector field $\widetilde{V}$. A vector field $\widetilde{V}$ with components $(v^h, v^{\bar{h}})$ is considered fibre-preserving if and only if v^h depends solely on the variables (x^h). Consequently, each fibre-preserving vector field $\widetilde{V}$ on TM induces a vector field $V = v^h \frac{\partial}{\partial x^h}$ on M. We will start by stating the following lemma, which will be used later.

Lemma 3.1. *[30] Consider a statistical manifold (M_n, g, ∇) equipped with a torsion-free linear connection ∇ and a (pseudo-) Riemannian metric g and its tangent bundle TM. Let $\widetilde{V}$ be a fibre-preserving vector field on TM with the components $(v^h, v^{\bar{h}})$. Then, the Lie derivates of the adapted frame and the dual basis are given as follows:*

$$\begin{aligned}
i)\ \ L_{\widetilde{V}} E_i &= -(E_i v^h) E_h + \left[y^b v^c R_{icb}{}^h - v^{\bar{b}} \tilde{\Gamma}^h_{bi} - (E_i v^{\bar{h}}) \right] E_{\bar{h}}, \\
ii)\ \ L_{\widetilde{V}} E_{\bar{i}} &= \left[v^b \tilde{\Gamma}^h_{bi} - (E_{\bar{i}} v^{\bar{h}}) \right] E_{\bar{h}}, \\
iii)\ \ L_{\widetilde{V}} dx^i &= (E_h v^i) dx^h, \\
iv)\ \ L_{\widetilde{V}} \delta y^i &= -\left[y^c v^b R_{bhc}{}^i + v^{\bar{b}} \tilde{\Gamma}^i_{bh} + (E_h v^{\bar{i}}) \right] dx^h - \left[v^b \tilde{\Gamma}^i_{bh} - (E_{\bar{h}} v^{\bar{i}}) \right] \delta y^h.
\end{aligned}$$

Through the Lemma given above, we will give the following lemma that we will use later.

Lemma 3.2. *[22] In the context of a statistical manifold (M_n, g, ∇) and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, the Lie derivative of twisted Sasaki metric G with respect to the fibre-preserving vector field $\tilde{X}$ is given as follows:*

$$\begin{aligned}
L_{\widetilde{V}} G &= a \left[L_V g_{ij} + 2 \left(E_i v^h \right) g_{hj} \right] dx^i dx^j \\
&\quad -2b g_{hj} \left[y^s v^b R_{bis}{}^h + v^{\bar{b}} \Gamma^h_{bi} + \left(E_i v^{\bar{h}} \right) \right] dx^i \delta y^j \\
&\quad +b \left[L_V g_{ij} - 2 v^b \Gamma^h_{bi} g_{hj} + 2 g_{hj} \left(E_{\bar{i}} v^{\bar{h}} \right) \right] \delta y^i \delta y^j
\end{aligned}$$

where $L_V g_{ij}$ denotes the components of the Lie derivative of $L_V g$ and $\widetilde{V} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$. $V = v^h \frac{\partial}{\partial x^h}$ is a vector field on M_n.

3.1. *Conformal Ricci soliton on the tangent bundle over statistical manifold according to the twisted Sasaki metric*

Consider a smooth manifold M_n, ($n \geq 2$). A conformal Ricci soliton on M_n is a triple (g, V, λ) that satisfies the equation:

$$L_V g + 2R = \left[2\lambda - \left(p + \frac{2}{n} \right) \right] g,$$

where R is the Ricci tensor, p is a scalar non-dynamical field(time dependent scalar field), λ is constant, n is the dimension of the manifold. The conformal Ricci soliton is categorized as either shrinking, steady, or expanding based on whether λ is positive, zero, or negative, respectively. A conformal Ricci soliton on the tangent bundle TM with the twisted Sasaki metric G over a statistical manifold (M_n, g, ∇) is defined as [8]:

$$L_{\widetilde{V}} G + 2\widetilde{R} = \left[2\lambda - \left(p + \frac{1}{n} \right) \right] G, \tag{3.1}$$

where $\tilde{R}$ is the Ricci tensor of G, $\widetilde{V}$ is a vector field on TM and λ is a smooth function on TM.

Theorem 3.1. *In the context of a statistical manifold (M_n, g, ∇) and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, the quadruple $(TM, G, {}^C V, \lambda)$ is a conformal Ricci soliton if and only if the following conditions are satisfied:*

$$\begin{aligned}
i)\ \ & \lambda = \frac{1}{2n} g^{ij} (L_V g_{ij}) + \frac{1}{n} (E_i v^i) - \frac{1}{4na} g^{ij} \left(\nabla_i g^{ml} \right) \left(\nabla_j g_{ml} \right) \\
& \quad - \frac{1}{2na} g^{ij} g^{ml} \left(\nabla_i \nabla_j g_{ml} \right) - \frac{b}{2na^2} \| R \| + \frac{r}{na} + \frac{p}{2} + \frac{1}{2n}, \\
ii)\ \ & \frac{1}{a} g^{hm} \left(\nabla_m R_{sijh} \right) + \frac{1}{2a} \left(\nabla_l g^{ml} \right) R_{jis}{}^h = 2 g_{hj} \left[v^l R_{lis}{}^h + \nabla_i \left(\nabla_s v^h \right) \right] \\
iii)\ \ & r = \frac{b}{4a} \| R \| - \frac{n-3}{4a} g^{ij} \left(\nabla_m g^{ml} \right) \left(\nabla_i g_{lj} \right).
\end{aligned}$$

Here, the potential vector field ${}^{C}V = \left(v^h, y^s\nabla_s v^h\right)$ *is the complete lift of a vector field* $V = v^h\frac{\partial}{\partial x^h}$ *on* M_n *to* TM *and* $A^k_{ij} = g^{kl}\left(\nabla_i g_{jl}\right)$ *on* M_n.

Proof. Considering equation (3.1), we derive:

$$L_{\tilde{V}}G_{ij} + 2\tilde{R}_{ij} = \left[2\lambda - \left(p + \frac{1}{n}\right)\right]G_{ij}$$

which leads to:

$$\begin{aligned}\left[2\lambda - \left(p + \frac{1}{n}\right)\right]ag_{ij} &= a\left[L_V g_{ij} + 2\left(E_i v^h\right)g_{hj}\right] + \frac{b}{2a}y^s y^p\left[R_{mis}{}^{h}R_{phj}{}^{m} + R_{msi}{}^{h}R_{jhp}{}^{m}\right]\\ &\quad -\frac{1}{2}\left(\nabla_i g^{ml}\right)\left(\nabla_j g_{ml}\right) - g^{ml}\left(\nabla_i\nabla_j g_{ml}\right) + 2R_{ij}.\end{aligned}$$

By contracting with g^{ij} in the last equation, we have the following equation:

$$\begin{aligned}2na\lambda - pna - 2a &= ag^{ij}L_V g_{ij} + 2a\left(E_i v^i\right) - \frac{1}{2}g^{ij}\left(\nabla_i g^{ml}\right)\left(\nabla_j g_{ml}\right)\\ &\quad -g^{ij}g^{ml}\left(\nabla_i\nabla_j g_{ml}\right) + 2r - \frac{b}{a}\left\|R\right\|.\end{aligned}$$

Multiplying both sides by $\frac{1}{2na}$ in the above equation, we derive:

$$\begin{aligned}\lambda &= \frac{1}{2n}g^{ij}(L_V g_{ij}) + \frac{1}{n}(E_i v^i) - \frac{1}{4na}g^{ij}\left(\nabla_i g^{ml}\right)\left(\nabla_j g_{ml}\right)\\ &\quad -\frac{1}{2na}g^{ij}g^{ml}\left(\nabla_i\nabla_j g_{ml}\right) - \frac{b}{2na^2}\left\|R\right\| + \frac{r}{na} + \frac{p}{2} + \frac{1}{2n}.\end{aligned} \tag{3.2}$$

Similarly, from equation (3.1), we also infer:

$$L_{\tilde{X}}G_{\bar{i}j} + 2\tilde{R}_{\bar{i}j} = \left[2\lambda - \left(p + \frac{1}{n}\right)\right]G_{\bar{i}j}$$

$$\begin{aligned}0 &= -2g_{hj}\left[y^s v^b R_{bis}{}^{h} + v^{\bar{b}}\Gamma^h_{bi} + \left(E_i v^{\bar{h}}\right)\right] + \frac{1}{a}y^s\nabla_m R_{sij}{}^{m}\\ &\quad +\frac{1}{2a}y^s\left[R_{sij}{}^{h}A^m_{hm} + R_{jsm}{}^{h}A^m_{ih}\right].\end{aligned}$$

If the equation $v^{\bar{h}} = y^s\nabla_s v^h$ is used in this last equation, we get

$$\frac{1}{a}g^{hm}\left(\nabla_m R_{sijh}\right) + \frac{1}{2a}\left(\nabla_l g^{ml}\right)R_{jis}{}^{h} = 2g_{hj}\left[v^l R_{lis}{}^{h} + \nabla_i\left(\nabla_s v^h\right)\right].$$

Again, from equation (3.1) we obtain

$$L_{\tilde{V}}G_{\overline{ij}} + 2\tilde{R}_{\overline{ij}} = \left[2\lambda - \left(p + \frac{1}{n}\right)\right]G_{\overline{ij}},$$

leading to:

$$\begin{aligned}\left[2\lambda - \left(p + \frac{1}{n}\right)\right]bg_{ij} &= b\left[L_V g_{ij} - 2v^l\Gamma^h_{li}g_{hj} + 2g_{hj}\left(\partial_{\bar{i}}\left(y^s\nabla_s v^h\right)\right)\right] - \frac{b}{a}\left(\nabla_m A^m_{ij}\right)\\ &\quad +\frac{b}{2a}\left[2A^m_{hi}A^h_{mj} - A^m_{hm}A^h_{ij}\right] - \frac{b^2}{2a^2}y^s y^p R_{sih}{}^{m}R_{pjm}{}^{h}.\end{aligned}$$

Contracting with $\frac{1}{2n}g^{ij}$ in the above equation, we arrive to

$$\begin{aligned}\lambda &= \frac{1}{2n}g^{ij}L_V g_{ij} - \frac{1}{n}v^l\Gamma^i_{li} + \frac{1}{n}\left(\nabla_i v^i\right) + \frac{1}{4na}g^{ij}\left[2A^m_{hi}A^h_{mj} - A^m_{hm}A^h_{ij}\right]\\ &\quad -\frac{1}{2na}g^{ij}\left(\nabla_m A^m_{ij}\right) - \frac{b}{4na^2}\left\|R\right\| + \frac{p}{2} + \frac{1}{2n}.\end{aligned} \tag{3.3}$$

From equations of (3.2) and (3.3) we have

$$r = \frac{b}{4a}\left\|R\right\| - \frac{n-3}{4a}g^{ij}\left(\nabla_m g^{ml}\right)\left(\nabla_i g_{lj}\right).$$

Thus, the proof is concluded. □

3.2. Conformal Yamabe soliton on tangent bundle over statistical manifold according to the twisted Sasaki metric

A Riemannian or pseudo-Riemannian manifold (M, g) of dimension n is considered to admit a conformal Yamabe soliton if it satisfies the equation:

$$L_V g + \left[2\lambda - 2r - \left(p + \frac{2}{n}\right)\right] g = 0,$$

where $L_V g$ represents the Lie derivative of the metric g along the vector field V, r is the scalar curvature and λ is a constant, p is a scalar non-dynamical field (time dependent scalar field), n is the dimension of the manifold. The conformal Yamabe soliton is classified as shrinking, steady, or expanding, depending on whether λ is positive, zero, or negative, respectively. Furthermore, the conformal Yamabe soliton defined on the tangent bundle TM with the twisted Sasaki metric G over a statistical manifold (M_n, g, ∇) can be expressed as:

$$L_{\widetilde{V}} G + \left[2\lambda - 2\tilde{r} - \left(p + \frac{1}{n}\right)\right] G = 0. \tag{3.4}$$

Here, $\tilde{r}$ denotes the scalar curvature of G, $\widetilde{V}$ is a vector field on TM and λ is a smooth function defined on TM.

Theorem 3.2. *In the context of a statistical manifold (M_n, g, ∇) and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, the quadruple $(TM, G, {}^C V, \lambda)$ is a conformal Yamabe soliton if and only if the following conditions are satisfied:*

$$\begin{aligned} i) \quad & \lambda = \frac{-1}{2n} g^{ij}(L_V g_{ij}) - \frac{1}{n}(E_i v^i) + \tilde{r} + \frac{p}{2} + \frac{1}{n}, \\ ii) \quad & \nabla_i \left(\nabla_s v^i\right) = v^l R_{ls}. \end{aligned}$$

Here, the potential vector field ${}^C V = \left(v^h, y^s \nabla_s v^h\right)$ is the complete lift of a vector field $V = v^h \frac{\partial}{\partial x^h}$ on M_n to TM and $A_{ij}^k = g^{kl}\left(\nabla_i g_{jl}\right)$ on M_n.

Proof. From equation (3.4), we have

$$L_{\tilde{X}} G_{ij} + \left[2\lambda - 2\tilde{r} - \left(p + \frac{1}{n}\right)\right] G_{ij} = 0,$$

which yields:

$$L_V g_{ij} + 2\left(E_i v^h\right) g_{hj} + \left[2\lambda - 2\tilde{r} - \left(p + \frac{1}{n}\right)\right] g_{ij} = 0.$$

Transvecting with g^{ij} in the last equation, we obtain:

$$\lambda = \frac{-1}{2n} g^{ij}(L_V g_{ij}) - \frac{1}{n}(E_i v^i) + \tilde{r} + \frac{p}{2} + \frac{1}{2n}.$$

Additionally, from equation (3.4), we have:

$$L_{\tilde{X}} G_{\bar{i}j} + \left[2\lambda - 2\tilde{r} - \left(p + \frac{1}{n}\right)\right] G_{\bar{i}j} = 0,$$

which implies:

$$g_{hj}\left[y^s v^l R_{lis}^{\ \ \ h} + \Gamma_{li}^h \left(y^s \nabla_s v^l\right) + \left(\partial_i - y^p \Gamma_{pi}^l \partial_{\bar{l}}\right)\left(y^s \nabla_s v^h\right)\right] = 0$$
$$y^s g_{hj}\left[v^l R_{lis}^{\ \ \ h} + \nabla_i \left(\nabla_s v^h\right)\right] = 0.$$

Contracting with g^{ij} in the last equation, we have:

$$\nabla_i \left(\nabla_s v^i\right) = v^l R_{ls}.$$

Thus, the proof is complete. □

3.3. Conformal Ricci-Yamabe soliton on tangent bundle over statistical manifold according to the twisted Sasaki metric

Güler and Crasmareanu [18] established the notion of Ricci-Yamabe flow on a Riemannian manifold (M_n, g), $(n \geq 2)$, by investigating a scalar combination of the Ricci flow and Yamabe flow represented as:

$$\frac{\partial g}{\partial t}(t) + 2\alpha R(t) + \beta r(t) g(t) = 0.$$

Here, g denotes a Riemannian metric, R is the Ricci tensor, r is the scalar curvature tensor and $\alpha, \beta \in \mathbb{R}$.

A Riemannian or pseudo-Riemannian manifold (M, g) of dimension n is is deemed to exhibit a conformal Ricci-Yamabe soliton if:

$$L_V g + 2\alpha R + \left[2\lambda - \beta r - \left(p + \frac{2}{n}\right)\right] g = 0,$$

where r denotes the scalar curvature, R is Ricci tensor and λ, α, β are real scalars, p is a scalar non-dynamical field (time dependent scalar field) and n signifies the dimension of the manifold. The conformal Ricci-Yamabe soliton is is classified as shrinking, steady, or expanding based on the sign of λ (positive, zero, or negative, respectively). The conformal Ricci-Yamabe soliton is said to be gradient if the soliton vector field $\widetilde{V}$ is the gradient of a C^∞ function f on M, then the equation (3.5) is called conformal gradient Ricci-Yamabe soliton. The conformal Yamabe soliton on the tangent bundle TM with the twisted Sasaki metric G over a statistical manifold (M_n, g, ∇) is expressed as:

$$L_{\widetilde{V}} G + 2\alpha \tilde{R} + \left[2\lambda - \beta \tilde{r} - \left(p + \frac{1}{n}\right)\right] G = 0 \tag{3.5}$$

where $\tilde{R}$ is the Ricci tensor, $\tilde{r}$ is the scalar curvature of G, $\widetilde{V}$ is a vector field on TM and λ is a smooth function on TM.

Theorem 3.3. *In the context of a statistical manifold (M_n, g, ∇) and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, the quadruple $(TM, G, {}^C V, \lambda)$ is a conformal Ricci-Yamabe soliton if and only if the following conditions are satisfied:*

$$\begin{aligned} i) \quad & \lambda = -\frac{1}{2n} g^{ij} L_V g_{ij} - \frac{1}{n}\left(E_i v^i\right) + \frac{\beta}{2}\tilde{r} + \frac{p}{2} + \frac{1}{2n}, \\ ii) \quad & \nabla_i \left(\nabla_s v^i\right) - v^l R_{ls} = \frac{\alpha}{2a} g^{mh} g^{ij} \left(\nabla_h R_{sijm}\right), \\ iii) \quad & r = \frac{1}{4}\left(\nabla^j g^{ml}\right)\left(\nabla_j g_{ml}\right) + \frac{1}{2} g^{ml} \nabla^j \left(\nabla_j g_{ml}\right) - \frac{b}{2a}\|R\|. \end{aligned}$$

Here, the potential vector field ${}^C V = \left(v^h, y^s \nabla_s v^h\right)$ is the complete lift of a vector field $V = v^h \frac{\partial}{\partial x^h}$ on M_n to TM and $A^k_{ij} = g^{kl}\left(\nabla_i g_{jl}\right)$ on M_n.

Proof. From equation (3.5), we deduce:

$$L_{\widetilde{V}} G_{ij} + 2\alpha \tilde{R}_{ij} + \left[2\lambda - \beta \tilde{r} - \left(p + \frac{1}{n}\right)\right] G_{ij} = 0$$

which leads to:

$$\begin{aligned} 0 \quad = \quad & a\left[L_V g_{ij} + 2\left(E_i v^h\right) g_{hj}\right] + 2\alpha \Bigg[R_{ij} + \frac{b}{4\alpha} y^s y^p \left(R_{mis}{}^h R_{phj}{}^m + R_{msi}{}^h R_{jhp}{}^m\right) \\ & -\frac{1}{4}\left(\nabla_i g^{ml}\right)\left(\nabla_j g_{ml}\right) - \frac{1}{2} g^{ml}\left(\nabla_i \nabla_j g_{ml}\right)\Bigg] + \left[2\lambda - \beta \tilde{r} - \left(p + \frac{1}{n}\right)\right] a g_{ij}. \end{aligned}$$

Contracting with $\frac{1}{2na} g^{ij}$ in the last equation, we get

$$\begin{aligned} \lambda \quad = \quad & -\frac{1}{2n} g^{ij} L_V g_{ij} - \frac{1}{n}\left(E_i v^i\right) - \frac{\alpha}{na} r + \frac{\alpha}{4na}\left(\nabla^j g^{ml}\right)\left(\nabla_j g_{ml}\right) \\ & -\frac{b\alpha}{2na^2}\|R\| + \frac{\alpha}{2na} g^{ml}\left(\nabla^j \left(\nabla_j g_{ml}\right)\right) + \frac{\beta}{2}\tilde{r} + \frac{p}{2} + \frac{1}{2n}. \end{aligned}$$

Also from equation (3.5), we have

$$L_{\widetilde{V}} G_{\bar{i}j} + 2\alpha \tilde{R}_{\bar{i}j} + \left[2\lambda - \beta\tilde{r} - \left(p + \frac{1}{n}\right)\right] G_{\bar{i}j} = 0$$

from which we can express:

$$\begin{aligned} 0 \quad = \quad & \frac{\alpha}{a} y^s \nabla_m R_{sij}{}^m - 2g_{hj}\left[y^s v^l R_{lis}{}^h + v^{\bar{l}}\Gamma_{li}^h + \left(E_i v^{\bar{h}}\right)\right] \\ & + \frac{\alpha}{2a} y^s \left[R_{sij}{}^h A_{hm}^m + R_{jsm}{}^h A_{ih}^m\right]. \end{aligned} \tag{3.6}$$

If we use the equality $v^{\bar{h}} = y^s \nabla_s v^h$, we infer:

$$0 = -2g_{hj}\left(v^l R_{lis}{}^h + \nabla_i\left(\nabla_s v^h\right)\right) + \frac{\alpha}{a}\nabla_m R_{sij}{}^m + \frac{\alpha}{2a}\left(R_{sij}{}^h A_{hm}^m + R_{jsm}{}^h A_{ih}^m\right).$$

This is equivalent to the equation:

$$0 = -2g_{hj}\left(v^l R_{lis}{}^h + \nabla_i\left(\nabla_s v^h\right)\right) + \frac{\alpha}{a} g^{mh}\left(\nabla_h R_{sijm}\right) + \frac{\alpha}{2a}\left(\nabla_l g^{ml}\right) R_{jism}.$$

Contracting with g^{ij} in the last equation, we get

$$\nabla_i\left(\nabla_s v^i\right) - v^l R_{ls} = \frac{\alpha}{2a} g^{mh} g^{ij}\left(\nabla_h R_{sijm}\right).$$

Similarly, from equation (3.5), we obtain

$$L_{\widetilde{X}} G_{\overline{ij}} + 2\alpha \tilde{R}_{\overline{ij}} + \left[2\lambda - \beta\tilde{r} - \left(p + \frac{1}{n}\right)\right] G_{\overline{ij}} = 0$$

from which we obtain

$$\left[L_V g_{ij} - 2v^l \Gamma_{li}^h g_{hj} + 2g_{hj}(E_{\bar{i}} v^{\bar{h}})\right] + \left[2\lambda - \beta\tilde{r} - \left(p + \frac{1}{n}\right)\right] g_{ij} = 0.$$

Contracting with $\frac{1}{2n} g^{ij}$ in the last equation, we deduce:

$$\lambda = -\frac{1}{2n} g^{ij} L_V g_{ij} + \frac{1}{n} v^l \Gamma_{li}^i - \frac{1}{n}\left(E_{\bar{i}} v^{\bar{i}}\right) + \frac{\beta}{2}\tilde{r} + \frac{p}{2} + \frac{1}{2n}.$$

If we use the equality $v^{\bar{h}} = y^s \nabla_s v^h$, we get:

$$\lambda = -\frac{1}{2n} g^{ij} L_V g_{ij} - \frac{1}{n}\left(E_i v^i\right) + \frac{\beta}{2}\tilde{r} + \frac{p}{2} + \frac{1}{2n} \tag{3.7}$$

With help of (3.6) and (3.7), we have:

$$r = \frac{1}{4}\left(\nabla^j g^{ml}\right)\left(\nabla_j g_{ml}\right) + \frac{1}{2} g^{ml} \nabla^j\left(\nabla_j g_{ml}\right) - \frac{b}{2a}\|R\|.$$

So the proof is completed. □

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