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On the Semi-analytical and Hybrid Methods for the Drinfeld-Sokolov-Wilson System Modelling Dispersive Water Waves

Emre Aydın [1,a], İnci Çilingir Süngü [2,b,*]

[1] Department of Mathematics, Ondokuz Mayıs University, Samsun, Türkiye.
[2] Department of Mathematics Education, Ondokuz Mayıs University, Samsun, Türkiye.
*Corresponding author

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ABSTRACT

In this study, modified variational iteration method (MVIM), modified variational iteration Laplace transform method (MVILTM) and modified variational iteration Sumudu transform method (MVISTM) are used to examine the Drinfeld-Sokolov-Wilson (DSW) system. Semi-analytical solutions have been obtained and compared with the analytical solutions. Moreover, it illustrates the effect of wave parameter on the approximate solutions. The exact solutions and semi-analytical solutions of the DSW system are compared with each other. Tables give maximum errors of semi-analytical solutions for various iteration values. The comparison of relative errors for various iteration values and the effect of change of wave constant is visualized by figures. Also, it commented on the effectiveness and usefulness of the methods when applied to the DSW system.

Keywords: Modified Variational İteration Method (MVIM), Modified Variational İteration Laplace Transform Method (MVILTM), Modified Variational İteration Sumudu Transform Method (MVISTM), The Drinfeld-Sokolov-Wilson (DSW) System.

[a] *emre_aydn_55@outlook.com* *https://orcid.org/0000-0001-7480-0965* [b] *incicilingir@gmail.com* *https://orcid.org/0000-0001-7788-181X*

Introduction

It is applied nonlinear systems of partial differential equations in scientific fields such as plasma physics, plasma waves, solid state physics, fluid mechanics, chemical physics. The purpose of this study is to analyze as semi-analytical the solutions of the Drinfeld-Sokolov-Wilson (DSW) System, which is one of the nonlinear partial differential equation systems. Drinfeld and Sokolov [1], Wilson [2], Drinfeld and Sokolov [3] were introduced firstly the DSW model having an important role in fluid dynamics. A generalized form of the DSW system is given by:

$$\begin{cases} u_t + \alpha v v_x = 0 \\ v_t + \beta v_{xxx} + \gamma u v_x + \varepsilon u_x v = 0. \end{cases} \quad (1)$$

Here α, β, γ and ε are real parameters. The coupled of DSW is a system that modeling of dispersive water waves and it is used a model the translation of shallow water waves. Recently, many studies have been conducted on the DSW model. Gao et al. [4] used q-Homotopy analysis transform method, Saifullah et al. [5] used the Laplace transform combined with Adomian decomposition method, Arora and Kumar [6] used Homotopy analysis method, Azizi and Pourgholi [7] used Sine-Cosine wavelets method, Salim et al. [8] used modified Adomian decomposition method, Eskandari and Taghizadeh [9] used the exp-function method and the rational (G'/G)-expansion method, Ali et al. [10] used new iteration method, Taghizadeh and Neirameh [11] investigated the new complex solutions, Raslan and Entesar [12] used Banach contraction method, Lindeberg et al. [13] used finite difference method, Zhang and Zhao [14] used Lie symmetry analysis and Lie-bäcklund symmetries, Singh et al. [15] used homotopy perturbation transform method, Al-Rozbayani and Ali [16] used Sumudu transform with Adomian decomposition method, Alam et al. [17] used $S(\xi)$-expansion method, Usman et al. [18] examined Jacobi elliptic solutions, Shahzad et al. [19] used Φ^6-model expansion method, Iqbal et al. [20] examined multiple solitary wave solutions, Younis et al. [21] used improved finite difference technique via Adominian polynomial, Aydemir [22] used generalized unified method, Hakkaev [23] examined spectral stability of periodic waves.

MVIM is a method that uses variational principles by transforming nonlinear terms into power series [24-26]. Respectively, MVILTM and MVISTM are methods obtained by hybridizing the MVIM with Laplace and Sumudu transform. Laplace and Sumudu transform methods are well-known and useful in many problems in which applied mathematics, physics, science and engineering. Also, these methods are suitable to hybrid the variational methods.

In this study, modified variational iteration method, modified variational iteration Laplace transform method and modified variational iteration Sumudu transform method are used to solve semi-analytically the Drinfeld-Sokolov-Wilson system and compare the results. The maximum and relative errors of proposed methods are evaluated and the results are interpreted. The effect of the wave parameter that emerged in the solution of the DSW system for different iteration values in an example is examined with the help of tables and figures.

Material and Method

In this section, the basic idea of modified variational iteration method (MVIM), modified variational iteration Laplace transform method (MVILTM), modified variational iteration Sumudu transform method (MVISTM) will be introduced. Drinfeld-Sokolov-Wilson (DSW) system can be written to explain the basics of MVIM, MVILTM and MVISTM in the following operator form.

$$\begin{cases} L_t u + Nu = 0 \\ L_t v + R_x v + Nv = 0, \end{cases} \tag{2}$$

where $L_t = \frac{\partial}{\partial t}$, $R_x = \beta \frac{\partial^3}{\partial x^3}$ represent linear differential operators and the notation, $Nu = \alpha v v_x$, $Nv = \gamma u v_x + \varepsilon u_x v$ represent nonlinear differential operator.

Modified Variational Iteration Method

The correction functional for Eq. (2) is given by:

$$\begin{cases} u_{n+1}(x,t) = u_0(x,t) + \int_0^t \lambda[Nu_n(x,s)]\,ds \\ v_{n+1}(x,t) = v_0(x,t) + \int_0^t \lambda[R_x v_n(x,s) + Nv_n(x,s)]\,ds. \end{cases} \tag{3}$$

Here λ is a Lagrange multiplier. For $L_t = \frac{\partial^m}{\partial t^m}$, the Lagrange multiplier is given by [27]:

$$\lambda(x,t) = \frac{(-1)^m}{(m-1)!}(t-x)^{m-1}, m \geq 1. \tag{4}$$

Initial approximate functions u_0, v_0 be taken as $u(x,0)$ and $v(x,0)$ respectively. To avoid computational overhead and unnecessary terms, system in Eq. (3) is rearranged in the form:

$$\begin{cases} u_{n+1} = u_0 + \int_0^t \lambda[G_{n-1}]\,ds + \int_0^t \lambda[G_n - G_{n-1}]\,ds \\ v_{n+1} = v_0 + \int_0^t \lambda[R_x v_{n-1} + J_{n-1}]\,ds + \int_0^t \lambda[R_x(v_n - v_{n-1}) + J_n - J_{n-1}]\,ds. \end{cases} \tag{5}$$

By further simplifying and taking $\lambda(x,t) = -1$, system in Eq. (5) is rearranged in the form:

$$\begin{cases} u_{n+1} = u_n - \int_0^t [G_n - G_{n-1}]\,ds \\ v_{n+1} = v_n - \int_0^t [R_x(v_n - v_{n-1}) + J_n - J_{n-1}]\,ds. \end{cases} \tag{6}$$

Here G_n and J_n can be obtained from the following series expansion of nonlinear terms and are convergent functions. So,
$Nu_n = G_n + o(t^{n+1})$, $Nv_n = J_n + o(t^{n+1})$, $n = 0,1,\dots$, $G_{-1} = J_{-1} = 0$.
System in Eq. (6) is called the modified correction functional. Consequently, the components $u_0, u_1, u_2, u_3, \dots$ and $v_0, v_1, v_2, v_3, \dots$ are identified and the semi-analytical solutions of the DSW system are determined entirely. The convergence occurs without any conditions. That is, $\lim_{n\to\infty} u_n = u(x,t)$ and $\lim_{n\to\infty} v_n = v(x,t)$.

Modified Variational Iteration Laplace Transform Method

Definition 1. For $\forall t \geq 0$, the Laplace transform of the function $f(t)$ defined is given by [28]:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt. \tag{7}$$

Taking the Laplace transform of both sides of the system in Eq. (2) and using the derivative properties of the Laplace transform, the system is rewritten as:

$$\begin{cases} \mathcal{L}_t\{u_t\} + \mathcal{L}_t\{\alpha v v_x\} = 0 \\ \mathcal{L}_t\{v_t\} + \mathcal{L}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\} = 0 \end{cases} \tag{8}$$

$$\Rightarrow \begin{cases} sU(x,s) - u(x,0) + \mathcal{L}_t\{\alpha v v_x\} = 0 \\ sV(x,s) - v(x,0) + \mathcal{L}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\} = 0. \end{cases} \tag{9}$$

By rearranging system in Eq. (9), it can be written as:

$$\begin{cases} U(x,s) = \frac{1}{s}u(x,0) - \frac{1}{s}\mathcal{L}_t\{\alpha v v_x\} \\ V(x,s) = \frac{1}{s}v(x,0) - \frac{1}{s}\mathcal{L}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\}. \end{cases} \tag{10}$$

Taking the inverse Laplace transform of both sides of the Eq. (10), it can be rewritten:

$$\begin{cases} \mathcal{L}_t^{-1}\{U(x,s)\} = \mathcal{L}_t^{-1}\left\{\frac{1}{s}u(x,0)\right\} - \mathcal{L}_t^{-1}\left\{\frac{1}{s}\mathcal{L}_t\{\alpha v v_x\}\right\} \\ \mathcal{L}_t^{-1}\{V(x,s)\} = \mathcal{L}_t^{-1}\left\{\frac{1}{s}v(x,0)\right\} - \mathcal{L}_t^{-1}\left\{\frac{1}{s}\mathcal{L}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\}\right\}. \end{cases} \tag{11}$$

By rearranging system in Eq. (11), it can be written as follows:

$$\begin{cases} u(x,t) = u(x,0) - \mathcal{L}_t^{-1}\left\{\frac{1}{s}\mathcal{L}_t\{\alpha v v_x\}\right\} \\ v(x,t) = v(x,0) - \mathcal{L}_t^{-1}\left\{\frac{1}{s}\mathcal{L}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\}\right\}. \end{cases} \tag{12}$$

Taking first derivative of both sides of the system in Eq. (12) with respect to t, it can be as follows:

$$\begin{cases} u_t + \frac{\partial}{\partial t}\mathcal{L}_t^{-1}\left\{\frac{1}{s}\mathcal{L}_t\{\alpha v v_x\}\right\} = 0 \\ v_t + \frac{\partial}{\partial t}\mathcal{L}_t^{-1}\left\{\frac{1}{s}\mathcal{L}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\}\right\} = 0. \end{cases} \tag{13}$$

The modified correction functional of system in Eq. (13) is given by

$$\begin{cases} u_{n+1} = u_n - \int_0^t [G_n - G_{n-1}]\, ds \\ v_{n+1} = v_n - \int_0^t [J_n - J_{n-1}]\, ds. \end{cases} \tag{14}$$

G_n and J_n can always be obtained from the series expansion of nonlinear terms and are convergent functions. That is,

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{L}_t^{-1}\left\{\frac{1}{s}\mathcal{L}_t\{\alpha v v_x\}\right\} = G_n + \mathrm{o}(t^{n+1}) \\ \frac{\partial}{\partial t}\mathcal{L}_t^{-1}\left\{\frac{1}{s}\mathcal{L}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\}\right\} = J_n + \mathrm{o}(t^{n+1}) \\ u_0 = u(x,0), v_0 = v(x,0), n = 0,1,\dots,\ G_{-1} = J_{-1} = 0. \end{cases} \tag{15}$$

By continuing the iteration process, convergence to the exact solution of the DSW system via modified variational iteration Laplace transform method (MVILTM) occurs without any conditions. That is, $\lim_{n\to\infty} u_n = u(x,t)$ and $\lim_{n\to\infty} v_n = v(x,t)$.

Modified Variational Iteration Sumudu Transform Method

Definition 2. For set $A = \left\{f(t): \exists K, t_1, t_2 > 0, |f(t)| \le K e^{\frac{|t|}{t_n}}, if\ t \in \{(-1)^n x[0,\infty]\}\right\}$, Sumudu transform of the function $f(t)$ is given by [29]:

$$\mathcal{S}\{f(t)\} = F(\epsilon) = \int_0^\infty e^{-t} f(\epsilon t) dt. \tag{16}$$

Taking the Sumudu transform of both sides of the system in Eq. (2) and using the derivative properties of the Sumudu transform, the system is rewritten as follows:

$$\begin{cases} \mathcal{S}_t\{u_t\} + \mathcal{S}_t\{\alpha v v_x\} = 0 \\ \mathcal{S}_t\{v_t\} + \mathcal{S}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\} = 0 \end{cases} \tag{17}$$

$$\Rightarrow \begin{cases} \frac{1}{\epsilon}U(x,\epsilon) - \frac{1}{\epsilon}u(x,0) + \mathcal{S}_t\{\alpha v v_x\} = 0 \\ \frac{1}{\epsilon}V(x,\epsilon) - \frac{1}{\epsilon}v(x,0) + \mathcal{S}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\} = 0. \end{cases} \tag{18}$$

By rearranging system in Eq. (18), it can be written as follows:

$$\begin{cases} U(x,\epsilon) = u(x,0) - \epsilon\mathcal{S}_t\{\alpha v v_x\} \\ V(x,\epsilon) = v(x,0) - \epsilon\mathcal{S}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\}. \end{cases} \tag{19}$$

Taking the inverse Sumudu transform of both sides of the system in Eq. (19), it can be rewritten.

$$\begin{cases} \mathcal{S}_t^{-1}\{U(x,\epsilon)\} = \mathcal{S}_t^{-1}\{u(x,0)\} - \mathcal{S}_t^{-1}\{\epsilon\mathcal{S}_t\{\alpha v v_x\}\} \\ \mathcal{S}_t^{-1}\{V(x,\epsilon)\} = \mathcal{S}_t^{-1}\{v(x,0)\} - \mathcal{S}_t^{-1}\{\epsilon\mathcal{S}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\}\}. \end{cases} \tag{20}$$

By rearranging system in Eq. (20), it can be written as follows:

$$\begin{cases} u(x,t) = u(x,0) - \mathcal{S}_t^{-1}\{\epsilon \mathcal{S}_t\{\alpha v v_x\}\} \\ v(x,t) = v(x,0) - \mathcal{S}_t^{-1}\{\epsilon \mathcal{S}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\}\}. \end{cases} \quad (21)$$

Taking first derivative of both sides of the system in Eq. (21) with respect to t, it can be as follows:

$$\begin{cases} u_t + \frac{\partial}{\partial t}\mathcal{S}_t^{-1}\{\epsilon \mathcal{S}_t\{\alpha v v_x\}\} = 0 \\ v_t + \frac{\partial}{\partial t}\mathcal{S}_t^{-1}\{\epsilon \mathcal{S}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\}\} = 0. \end{cases} \quad (22)$$

The modified correction functional of system in Eq. (22) is given by

$$\begin{cases} u_{n+1} = u_n - \int_0^t [G_n - G_{n-1}]\, ds \\ v_{n+1} = v_n - \int_0^t [J_n - J_{n-1}]\, ds. \end{cases} \quad (23)$$

G_n and J_n can be found from the following convergent series expansion

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{S}_t^{-1}\{\epsilon \mathcal{S}_t\{\alpha v v_x\}\} = G_n + \mathrm{o}(t^{n+1}) \\ \frac{\partial}{\partial t}\mathcal{S}_t^{-1}\{\epsilon \mathcal{S}_t\{\beta v_{xxx} + \gamma u v_x + \varepsilon u_x v\}\} = J_n + \mathrm{o}(t^{n+1}) \\ u_0 = u(x,0), v_0 = v(x,0), n = 0,1,\dots,\ G_{-1} = J_{-1} = 0. \end{cases} \quad (24)$$

By continuing the iteration process, convergence to the exact solution of the DSW system via modified variational iteration Sumudu transform method (MVISTM) occurs without any conditions. That is, $\lim_{n\to\infty} u_n = u(x,t)$ and $\lim_{n\to\infty} v_n = v(x,t)$

Numerical Experiments and Simulations

In this section, we will examine the following initial value problem for the specific parameters of the Drinfeld-Sokolov-Wilson system in Eq.(1) and get semi-analytical solutions and errors of DSW system with MVIM, MVILTM and MVISTM. Considering the different values of the c wave parameter, the relative errors of the proposed methods for various iteration values will be given with the help of figures.

Example 1. The DSW system with initial value problem for $\alpha = 3, \beta = \gamma = 2, \varepsilon = 1$ in the system in Eq. (1) is given by

$$\begin{cases} u_t + 3 v v_x = 0 \\ v_t + 2 v_{xxx} + 2 u v_x + u_x v = 0 \\ u(x,0) = 0.3\mathrm{sech}^2(\sqrt{0.1}x), v(x,0) = 0.2\mathrm{sech}(\sqrt{0.1}x). \end{cases} \quad (25)$$

For $c = 0.2$, exact solutions of the DSW system given with initial value problem are

$$u(x,t) = \frac{3c}{2} sech^2\left(\sqrt{\frac{c}{2}}(x - ct)\right),\ v(x,t) = c\mathrm{sech}\left(\sqrt{\frac{c}{2}}(x - ct)\right). \quad [30]$$

Semi-analytical solutions founded by MVIM for Eq. (25) can be written as follows.

$$u_0 = \frac{0.3000000000}{\cosh(0.3162277660x)^2}$$

$$u_1 = \frac{0.3000000000}{\cosh(0.3162277660x)^2} + \frac{0.03794733192\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^3}$$

$$u_2 = \frac{0.3000000000}{\cosh(0.3162277660x)^2} + \frac{0.03794733192\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^3} + \frac{0.002400000000t^2}{\cosh(0.3162277660x)^2} - \frac{0.003600000000t^2}{\cosh(0.3162277660x)^4}$$

$$v_0 = \frac{0.2000000000}{\cosh(0.3162277660x)}$$

$$v_1 = \frac{0.2000000000}{\cosh(0.3162277660x)} + \frac{0.01264911064\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^2}$$

$$v_2 = \frac{0.2000000000}{\cosh(0.3162277660x)} + \frac{0.01264911064\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^2} + \frac{0.0003999999990t^2}{\cosh(0.3162277660x)} - \frac{0.0008000000000t^2}{\cosh(0.3162277660x)^3}$$

Semi-analytical solutions founded by MVILTM for Eq. (25) can be written as follows.

$$u_0 = \frac{0.3000000000}{\cosh(0.3162277660x)^2}$$

$$u_1 = \frac{0.3000000000}{\cosh(0.3162277660x)^2} + \frac{0.03794733192\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^3}$$

$$u_2 = \frac{0.3000000000}{\cosh(0.3162277660x)^2} + \frac{0.03794733192\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^3} + \frac{0.002400000000t^2}{\cosh(0.3162277660x)^2} - \frac{0.003600000000t^2}{\cosh(0.3162277660x)^4}$$

$$v_0 = \frac{0.2000000000}{\cosh(0.3162277660x)}$$

$$v_1 = \frac{0.2000000000}{\cosh(0.3162277660x)} + \frac{0.01264911064\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^2}$$

$$v_2 = \frac{0.2000000000}{\cosh(0.3162277660x)} + \frac{0.01264911064\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^2} + \frac{0.0003999999981t^2}{\cosh(0.3162277660x)} - \frac{0.0008000000000t^2}{\cosh(0.3162277660x)^3} + \frac{6.83372784010^{-13}t^2}{\cosh(0.3162277660x)^5}$$

Semi-analytical solutions founded by MVISTM for Eq. (25) can be written as follows.

$$u_0 = \frac{0.3000000000}{\cosh(0.3162277660x)^2}$$

$$u_1 = \frac{0.3000000000}{\cosh(0.3162277660x)^2} + \frac{0.03794733192\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^3}$$

$$u_2 = \frac{0.3000000000}{\cosh(0.3162277660x)^2} + \frac{0.03794733192\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^3} + \frac{0.002400000000t^2}{\cosh(0.3162277660x)^2} - \frac{0.003600000000t^2}{\cosh(0.3162277660x)^4}$$

$$v_0 = \frac{0.2000000000}{\cosh(0.3162277660x)}$$

$$v_1 = \frac{0.2000000000}{\cosh(0.3162277660x)} + \frac{0.01264911064\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^2}$$

$$v_2 = \frac{0.2000000000}{\cosh(0.3162277660x)} + \frac{0.01264911064\sinh(0.3162277660x)t}{\cosh(0.3162277660x)^2} + \frac{0.0003999999981t^2}{\cosh(0.3162277660x)} - \frac{0.0008000000000t^2}{\cosh(0.3162277660x)^3} + \frac{6.83372784010^{-13}t^2}{\cosh(0.3162277660x)^5}$$

Comparison of errors found by MVIM, MVILTM and MVISTM for various iteration values in Eq. (25) are shown in Table 1-6.

Table 1. Comparison of errors for various iteration values in Eq. (25)

MVIM for $u(x,t)$, $t=1$				
x	$N=2$	$N=3$	$N=5$	$N=8$
-6	$6.06160\ 10^{-6}$	$1.13633\ 10^{-7}$	$1.42303\ 10^{-10}$	$4.61369\ 10^{-12}$
-3	$1.10281\ 10^{-5}$	$1.21407\ 10^{-6}$	$7.27618\ 10^{-10}$	$1.02636\ 10^{-11}$
0	$3.19280\ 10^{-6}$	$3.19280\ 10^{-6}$	$7.20005\ 10^{-10}$	$1.31765\ 10^{-12}$
3	$1.35199\ 10^{-5}$	$1.27770\ 10^{-6}$	$8.70581\ 10^{-10}$	$1.00499\ 10^{-11}$
6	$6.28720\ 10^{-6}$	$1.11959\ 10^{-6}$	$1.64544\ 10^{-10}$	$5.41594\ 10^{-12}$

Table 2. Comparison of maximum errors for various iteration values in Eq. (25)

MVIM for $v(x,t)$, $t=1$				
x	$N=2$	$N=3$	$N=5$	$N=8$
-6	$1.16369\ 10^{-6}$	$1.97458\ 10^{-8}$	$5.08659\ 10^{-11}$	$5.27795\ 10^{-12}$
-3	$6.94779\ 10^{-6}$	$2.79873\ 10^{-7}$	$2.71933\ 10^{-10}$	$6.50994\ 10^{-12}$
0	$6.65584\ 10^{-7}$	$6.65584\ 10^{-7}$	$1.081690\ 10^{-9}$	$6.99534\ 10^{-11}$
3	$7.51008\ 10^{-6}$	$2.82411\ 10^{-7}$	$2.96165\ 10^{-10}$	$5.43343\ 10^{-12}$
6	$1.12110\ 10^{-6}$	$2.28447\ 10^{-8}$	$5.27115\ 10^{-11}$	$5.49109\ 10^{-12}$

Table 3. Comparison of maximum errors for various iteration values in Eq. (25)

MVILTM for $u(x,t)$, $t=1$				
x	$N=2$	$N=3$	$N=5$	$N=8$
-6	$6.06161\ 10^{-6}$	$1.13625\ 10^{-7}$	$1.43203\ 10^{-10}$	$6.26648\ 10^{-12}$
-3	$1.10280\ 10^{-5}$	$1.21412\ 10^{-6}$	$6.58228\ 10^{-10}$	$2.01605\ 10^{-11}$
0	$3.19276\ 10^{-6}$	$3.19276\ 10^{-6}$	$7.242262\ 10^{-9}$	$7.61305\ 10^{-11}$
3	$1.35199\ 10^{-5}$	$1.27770\ 10^{-6}$	$8.20719\ 10^{-10}$	$4.33366\ 10^{-12}$
6	$6.28720\ 10^{-6}$	$1.11964\ 10^{-7}$	$1.55581\ 10^{-10}$	$3.83181\ 10^{-12}$

Table 4. Comparison of maximum errors for various iteration values in Eq. (25)

MVILTM for $v(x,t)$, $t=1$				
x	$N=2$	$N=3$	$N=5$	$N=8$
-6	$1.16369\ 10^{-6}$	$1.97432\ 10^{-8}$	5.5985010^{-11}	$1.41204\ 10^{-11}$
-3	$6.94779\ 10^{-6}$	$2.79868\ 10^{-7}$	$2.63163\ 10^{-10}$	$1.72886\ 10^{-10}$
0	$6.65585\ 10^{-7}$	$6.65585\ 10^{-7}$	$1.079563\ 10^{-9}$	$4.736751\ 10^{-9}$
3	$7.51008\ 10^{-6}$	$2.82416\ 10^{-7}$	$3.05784\ 10^{-10}$	$1.39612\ 10^{-10}$
6	$1.12110\ 10^{-6}$	$2.28477\ 10^{-8}$	$4.64987\ 10^{-11}$	$2.48959\ 10^{-12}$

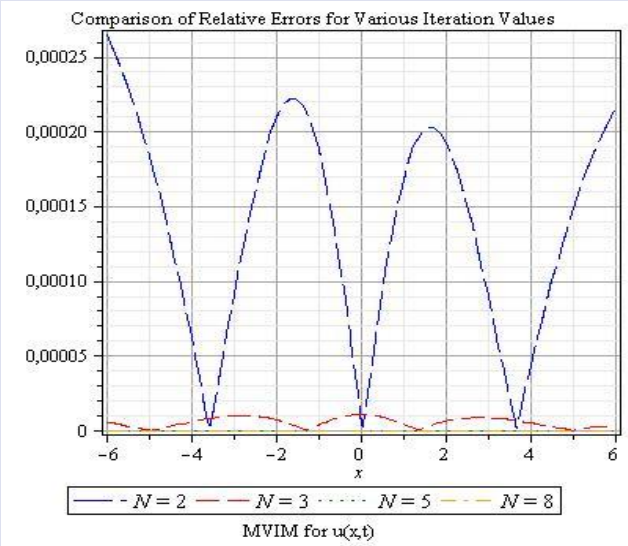
Table 5. Comparison of maximum errors for various iteration values in Eq. (25)

MVISTM for $u(x,t)$, $t=1$				
x	$N=2$	$N=3$	$N=5$	$N=8$
-6	$6.06161\ 10^{-6}$	$1.13625\ 10^{-7}$	$1.43203\ 10^{-10}$	$6.26648\ 10^{-12}$
-3	$1.10280\ 10^{-5}$	$1.21412\ 10^{-6}$	$6.58228\ 10^{-10}$	$2.01605\ 10^{-11}$
0	$3.19276\ 10^{-6}$	$3.19276\ 10^{-6}$	$7.242262\ 10^{-9}$	$7.61305\ 10^{-11}$
3	$1.35199\ 10^{-5}$	$1.27770\ 10^{-6}$	$8.20719\ 10^{-10}$	$4.33366\ 10^{-12}$
6	$6.28720\ 10^{-6}$	$1.11964\ 10^{-7}$	$1.55581\ 10^{-10}$	$3.83181\ 10^{-12}$

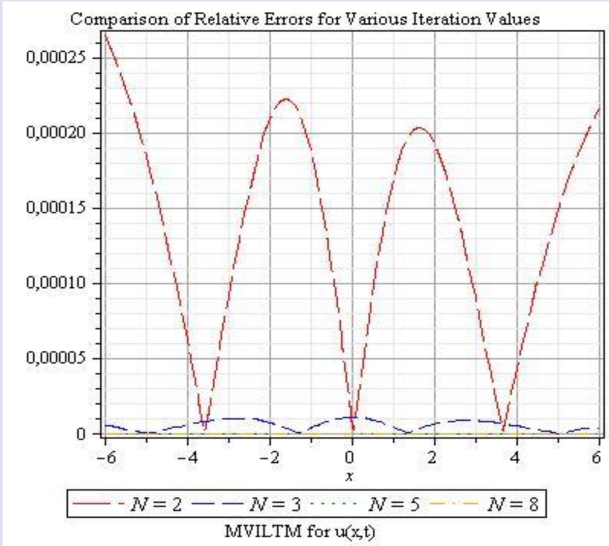
Table 6. Comparison of maximum errors for various iteration values in Eq. (25)

	MVISTM for $v(x,t)$, $t=1$			
x	$N=2$	$N=3$	$N=5$	$N=8$
-6	$1.16369\ 10^{-6}$	$1.97432\ 10^{-8}$	5.5985010^{-11}	$1.41204\ 10^{-11}$
-3	$6.94779\ 10^{-6}$	$2.79868\ 10^{-7}$	$2.63163\ 10^{-10}$	$1.72886\ 10^{-10}$
0	$6.65585\ 10^{-7}$	$6.65585\ 10^{-7}$	$1.079563\ 10^{-9}$	$4.736751\ 10^{-9}$
3	$7.51008\ 10^{-6}$	$2.82416\ 10^{-7}$	$3.05784\ 10^{-10}$	$1.39612\ 10^{-10}$
6	$1.12110\ 10^{-6}$	$2.28477\ 10^{-8}$	$4.64987\ 10^{-11}$	$2.48959\ 10^{-12}$

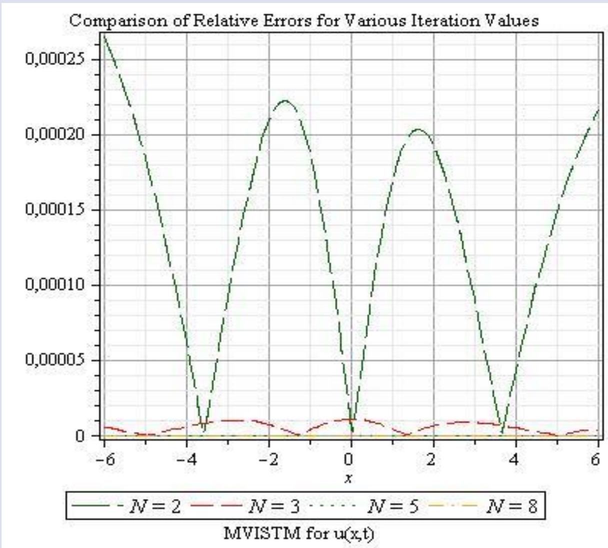
Comparison of relative errors found by MVIM, MVILTM and MVISTM for various iteration values and wave constant c values in Eq. (25) are shown in Figure 1-6.



a) MVIM for u(x,t).

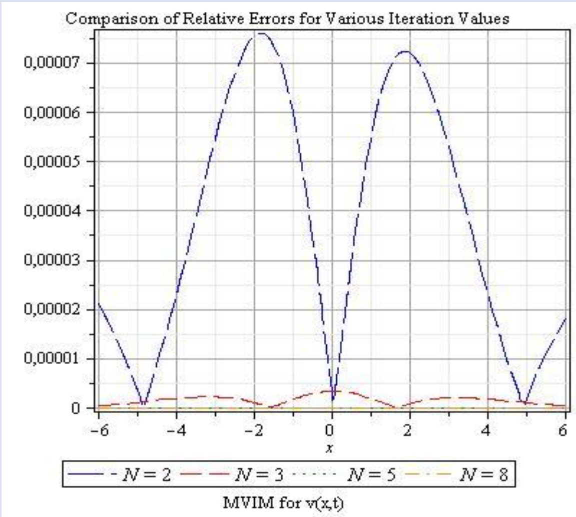


b) MVILTM for u(x,t).

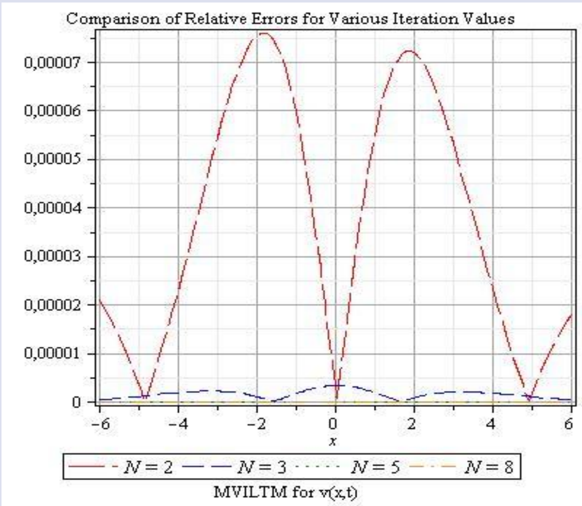


c) MVISTM for $u(x,t)$

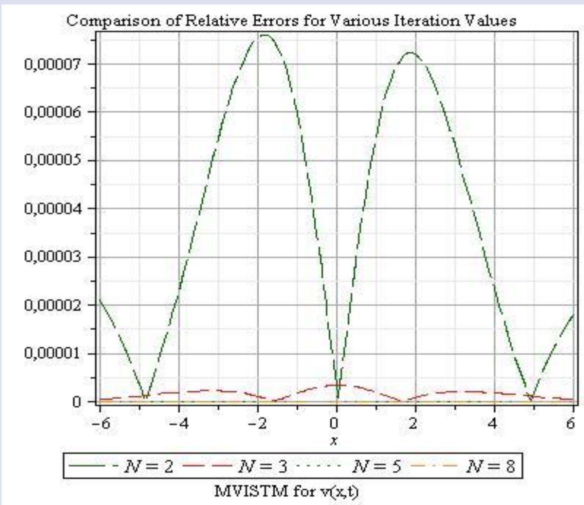
Figure 1. Comparison of relative errors for various iteration values and c=0.2, t=1 in Eq. (25)



a) MVIM for v(x,t).

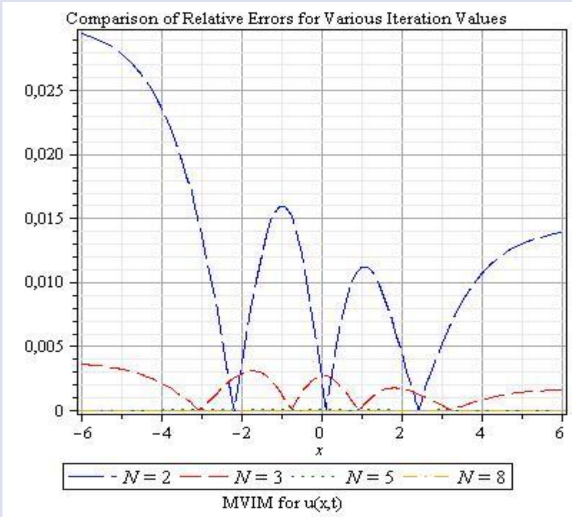


b) MVILTM for v(x,t).

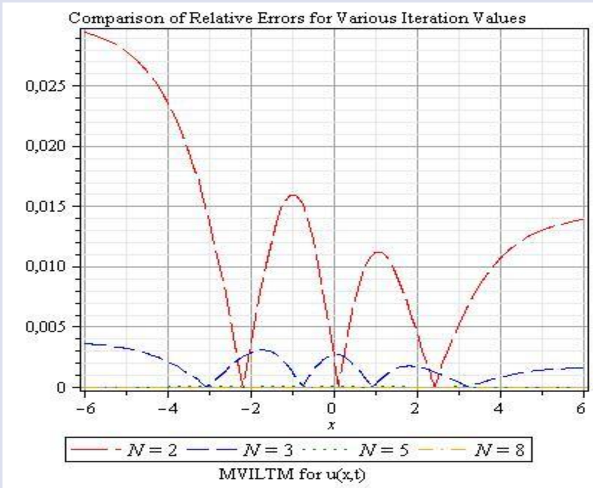


c) MVISTM for v(x, t)

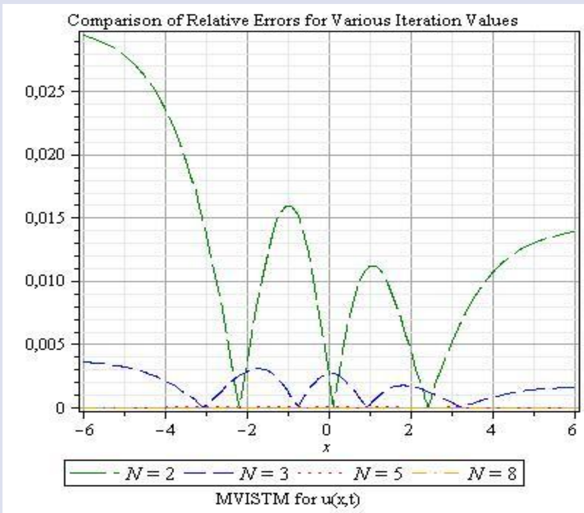
Figure 2. Comparison of relative errors for various iteration values and c=0.2, t=1 in Eq. (25)



a) MVIM for u(x,t).

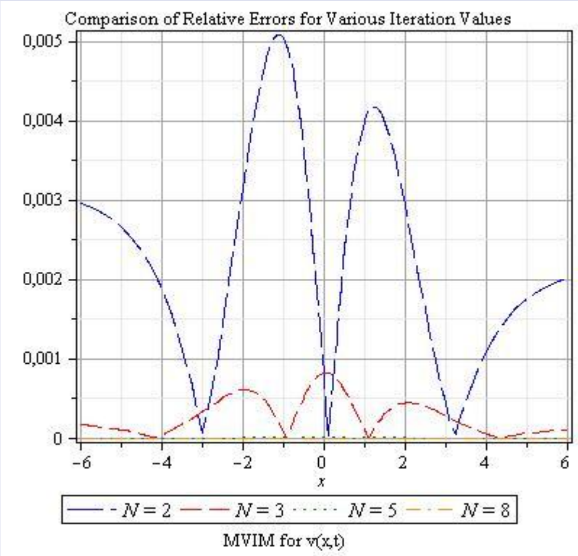


b) MVILTM for u(x,t).

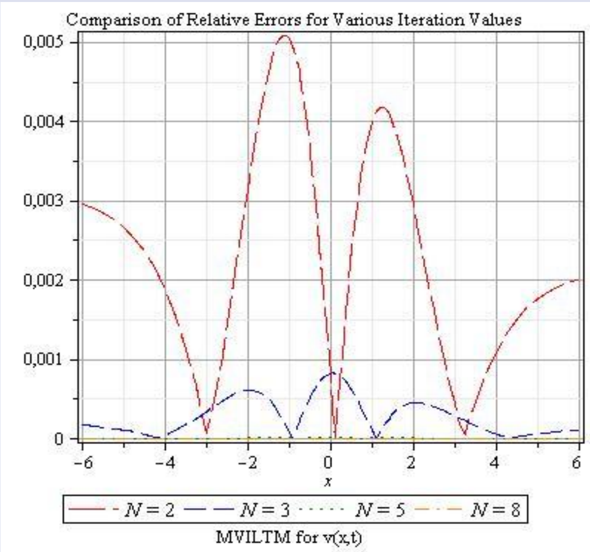


c) MVISTM for u(x, t)

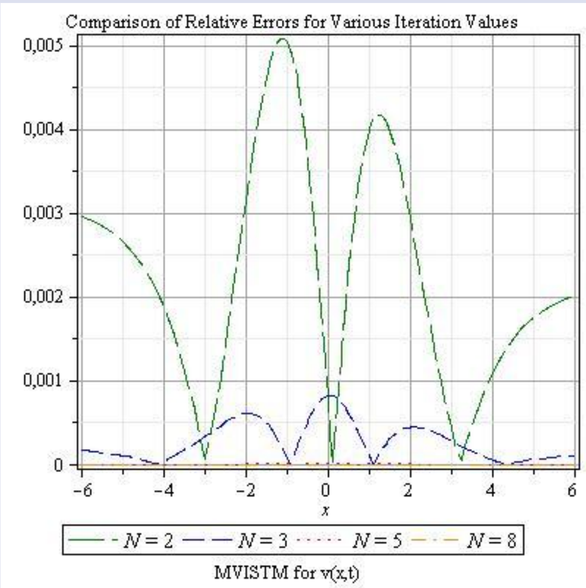
Figure 3. Comparison of relative errors for various iteration values and c=0.5, t=1 in Eq. (25)



a) MVIM for v(x,t).

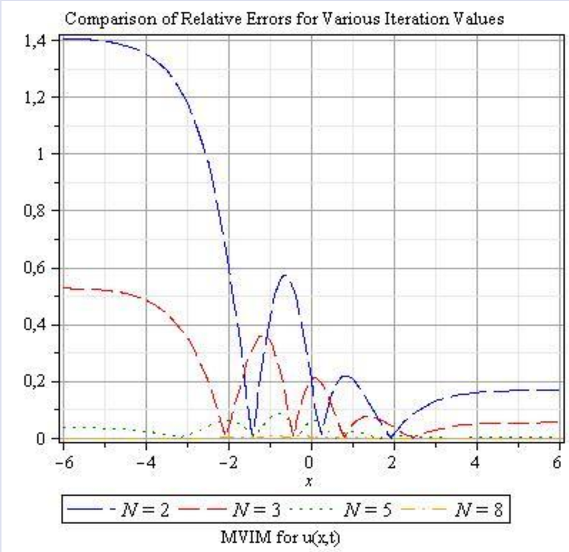


b) MVILTM for v(x,t).

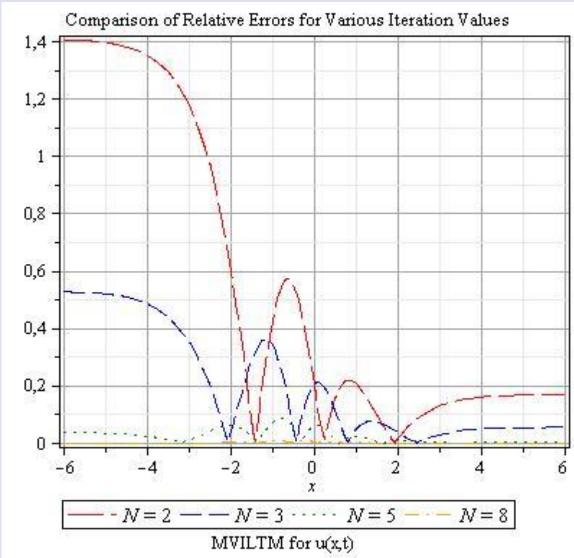


c) MVISTM for v(x, t)

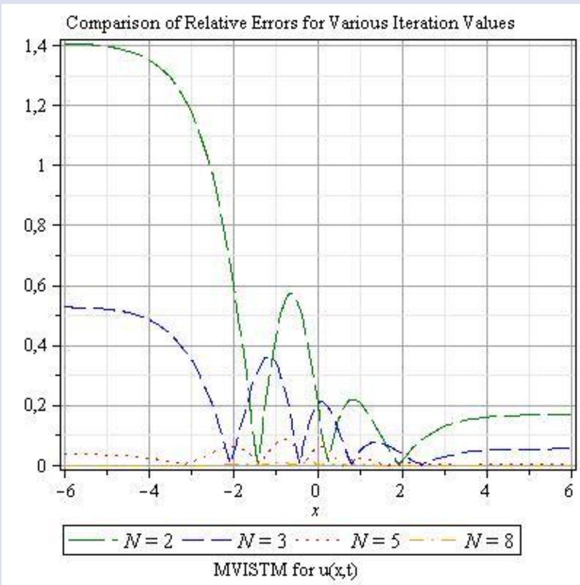
Figure 4. Comparison of relative errors for various iteration values and c=0.5, t=1 in Eq. (25)



a) MVIM for u(x,t).

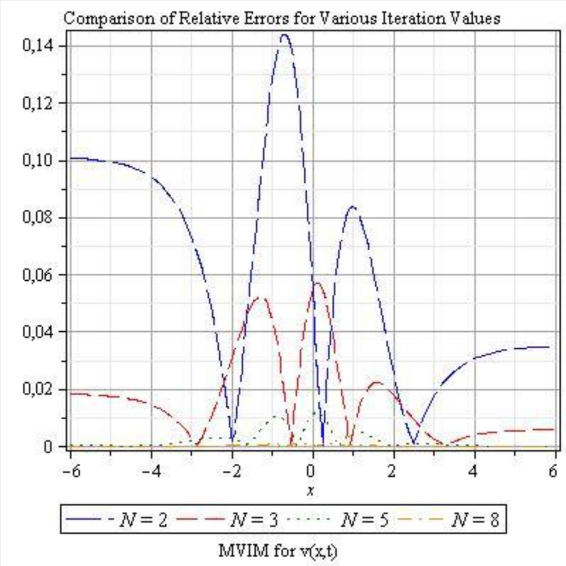


b) MVILTM for u(x,t).

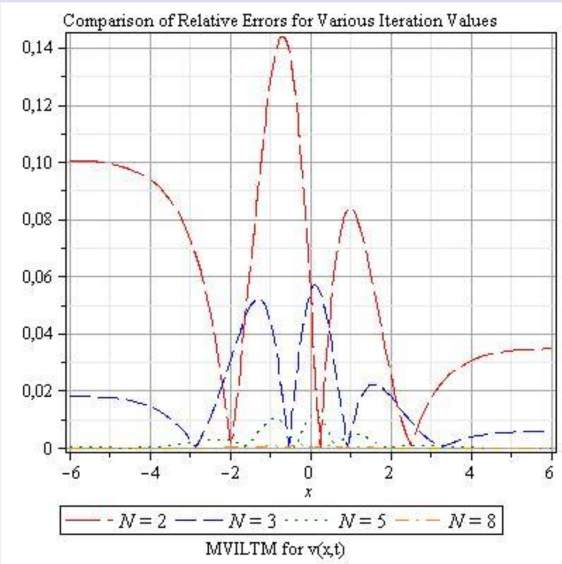


c) MVISTM for u(x, t)

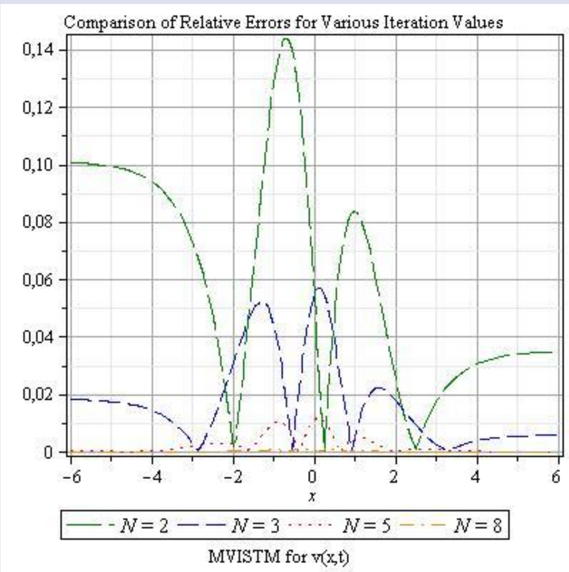
Figure 5. Comparison of relative errors for various iteration values and c=1, t=1 in Eq. (25)



a) MVIM for v(x,t).



b) MVILTM for v(x,t).



c) MVILTM for v$(\boldsymbol{x}, \boldsymbol{t})$

Figure *6*. Comparison of relative errors for various iteration values and c=1, t=1 in Eq. (25)

Conclusion

In this study, MVIM, MVILTM and MVISTM are used to utilize the Drinfeld-Sokolov-Wilson system semi-analytically. Semi-analytical solutions of the DSW system via MVIM, MVILTM and MVISTM are determined. The comparison of maximum errors of MVIM for various iteration values is given in Table 1-6. Also, the c value is a parameter that occurs in the solution of the DSW system and is called the wave parameter r. The comparison of relative errors for various iteration values and wave parameters (c values) are visualized in Figure 1-6.

Via MVIM, MVILTM and MVISTM used to solve the DSW system, the semi-analytical results of proposed methods are obtained in a short time and using less memory. It has been observed that MVIM, MVILTM, MVSTM proposed for the DSW system not only eliminate the complexity and intense processing load of the exact solution but also provide practicality, time saving and effectiveness. Therefore, they are very useful methods in terms of both time and computational cost. It is highly recommended as an alternative to exact solution methods. MVILTM and MVISTM are highly effective, fast, more practical and reliable methods to solve the DSW system as an alternative to MVIM. Also, the maximum errors of proposed methods for c=0.2 in the DSW system are around $10^{-9} - 10^{-11}$. It has been observed that the maximum and relative errors increase as the value of the c parameter increases. Therefore, it has been determined that the c parameter affects convergence. As a result, it has been signed that MVILTM and MVISTM used to solve the DSW system are equivalent methods and have the same effectiveness. It has been shown that MVIM gives more accurate results than MVILTM and MVISTM in solving the DSW problem.

Consequently, proposed semi-analytical solutions can be used instead of exact solutions of the DSW system that models the translation of shallow water waves and the dispersive water waves. These obtained semi-analytical solutions can be more useful and functional in explaining the physical aspects of various models that originated from engineering and science. The parameters in the Drinfeld-Sokolov-Wilson system can be also examined according to spatial. Finally, this study can be taken further by bringing the DSW system to fractional form with the help of fractional derivative definitions.

This study was partially presented orally at "The 8th International Conference on Computational Mathematics and Engineering Sciences / 17 – 19 May. 2024, Sanlıurfa – Turkiye".

Conflict of interest

There are no conflicts of interest in this work.

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