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Research Article

Analysis of Total Umbilical Fibers in Riemannian Submersions with Quarter Symmetric Non-Metric Connections

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Abstract: In this study, we examine the geometric properties of Riemannian submersions with quarter symmetric non-metric connections. Our work provides a comprehensive analysis of the Weyl projective curvature tensor, the concircular curvature tensor, and the conharmonic curvature tensor. We also investigate how these curvature tensors interact with the total umbilic fibers, particularly focusing on their behavior when such fibers are present in Riemannian submersions. First we introduce basic properties of quarter symmetric non-metric connections. Subsequently, we compute the relevant curvature tensors and analyze their interplay with total umbilic fibers. In this context, we elucidate relationships between various curvature tensors and certain geometric properties, as well as the effects of these relationships on Riemannian submersions.

Keywords: curvature tensors; quarter symmetric non-metric connection; riemannian manifold; riemannian submersion.

Araştırma Makalesi

Quarter Simetrik Non-Metrik Konneksiyonlu Riemann Submersiyonlarda Total Umbilik Liflerin Analizi

Özet: Bu çalışmada quarter simetrik non-metrik konneksiyona sahip Riemann submersiyonların geometrik özellikleri incelenmektedir. Çalışma, Weyl projektif eğrilik tensörü, koncircular eğrilik tensörü ve konharmonic eğrilik tensorlerine ilişkin kapsamlı bir analiz sunmaktadır. Ayrıca bu eğrilik tensörlerinin total um-bilik lifler üzerindeki etkileşimleri de araştırılmaktadır. Ek olarak söz konusu tensörlerin Riemann submersiyonlarında total umbilik liflerinin varlığı duru-munda ortaya çıkan yapısal özellikleri de ele alınmaktadır. İlk olarak, quarter simetrik non-metrik konneksiyonların tanımı ve temel özellikleri incelenmekte, sonrasında bu konneksiyonların Riemann submersiyonları üzerindeki etkileri incelenmektedir. Eğrilik tensörleri hesaplanarak total umbilik liflerle etkileşim-leri analiz edilmektedir. Bu bağlamda, farklı eğrilik tensörlerinin geometrik özel-liklerinin birbiri ile ilişkisi ve Riemann submersiyonları üzerindeki etkileri ortaya konulmaktadır.

Anahtar Kelimeler: eğrilik tensörleri; quarter simetrik non-metrik konneksiyon; riemann manifoldu; riemann submersiyonu.

1. Introduction

Connections play a crucial role in geometry and various scientific disciplines. Quarter symmetric non-metric connection is a notable example of such geometric structures. Friedmann and Schouten [1] introduced the concept of semi-symmetric non-metric connections with in a differentiable manifold. Later, Hayden [2] investigated of metric connections with torsion in Riemannian manifold. Subsequently Yano [3] introduced a new method called semi-symmetric metric connection focusing on a Riemannian manifold containing this connection. Many researchers [4-6] have conducted further studies on semi-symmetric and non-metric connections.

The investigation of Riemannian submersions between Riemannian manifolds was pioneered by O'Neill [7] and Gray [8]. Later; this research was extended to include manifolds with differentiable structures. Riemannian submersion theory finds active application in various fields. Moreover, Riemannian submersions have been the focus on extensive research (see [9-17]).

Akyol and Beyendi investigated Riemannian submersions endowed with a semi-symmetric non-metric connections [18]. Additionally, Sarı conducted a research on semi-invariant Riemannian submersions and investigated the application of semi-symmetric non-metric connections [19]. On the other hand, Demir and Sarı extensively examined the application of Riemannian submersions with semi-symmetric metric connections in their studies [20]. In 1975, Golab introduced the concept of quarter symmetric connection in a differentiable manifold [21]. In 2021, Demir and Sarı introduced Riemannian submersions with quarter symmetric non-metric connections and investigated the geometric properties of these new submersions by examining them in depth, focused on the basic tensor fields of Riemannian submersions related to quarter symmetric non-metric connections and calculated the relevant Riemannian curvatures [22].

Curvature tensors are essential mathematical tools that are utilized in theoretical physics and mathematics, especially in the study of gravitational theory and geometry. These tensors are used to quantify and characterize gravity's effects as well as the curvature of spaces. The Riemannian curvature tensor is a crucial tool in differential geometry, specifically for defining the curvature of $n-$dimensional spaces, such as Riemannian manifolds. Among curvature tensors, the Riemannian curvature tensor is the most well-known; however, other curvature tensors, such as the Ricci tensor and scalar curvature, also play significant roles in differential geometry. A new class of curvature tensors on Riemannian manifolds, including the concircular curvature tensor, was introduced by Mishra in the work [23]. Building on this work, Pokhariyal and Mishra [24] further developed the Weyl projective curvature tensor for Riemannian manifolds. Afterward, Ojha [25] introduced the $M-$projective curvature tensor. The conditions for the conharmonic curvature tensor were investigated in 1988 by M. Doric et al. [26], particularly in the context of Kaehler hypersurfaces in complex space forms. Furthermore, Ahsan [27] investigated the relativistic implications of the concircular curvature tensor. G. Hall conducted a thorough analysis of projectively linked connections on space-time manifolds in 2018, paying particular attention to the Weyl projective tensor, which is based on Einstein's geodesic postulate [28]. In the field of mathematics, these tensors play a crucial role in categorizing Riemannian and pseudo-Riemannian manifolds, thereby enhancing our comprehension of the diverse geometric structures that can exist. Lately, Akyol and Ayar [29] have been conducting research on New curvature tensors along Riemannian submersions. Their work contributes to the exploration of these tensors within the context of Riemannian submersions.

Building on these foundational ideas we first introduce the foundational concepts of Riemannian submersion, quarter symmetric non-metric connection, and Riemannian submersion with a quarter symmetric non-metric connection, which will be essential for the subsequent sections. Within this framework we conduct detailed calculations for various curvature tensors, including the Weyl projective curvature tensor, concircular and conharmonic curvature tensors. Furthermore, we analyze the behavior of these curvature tensors in the presence of totally umbilic fibers under Riemannian submersion, offering an in-depth examination of their properties in this specific geometric context.

2. Preliminaries

A linear connection $\tilde{\nabla}$ defined on M_1 is termed symmetric if the torsion $\tilde{T}$ of $\tilde{\nabla}$ expressed as

$$\tilde{T}(X_1, X_2) = \tilde{\nabla}_{X_1} X_2 - \tilde{\nabla}_{X_2} X_1 - [X_1, X_2],$$

vanishes for all vector fields X_1 and X_2 in M_1. In case it does not equal zero, it is denoted as non-symmetric [2].

Let (M_1, g_1) be an $n-$dimensional Riemannian manifold, and let

$$\tilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)\varphi X_1 \qquad (2.1)$$

connection be a linear connection on the Riemannian manifold (M_1, g_1). Here, X_1 and X_2 are arbitrary vector fields, η is a $1-$form on M_1 associated with the vector field U_1 on M_1 via $\eta(X_2) = g_1(U_1, X_2)$, ∇ is the Levi-Civita connection, and φ is a $(1,1)-$type tensor field. For all X_1, $X_2 \in \Gamma(TM_1)$, the torsion tensor of the M_1 manifold with the $\tilde{\nabla}$ connection is calculated as

$$\tilde{T}(X_1, X_2) = \eta(X_2)\varphi X_1 - \eta(X_1)\varphi X_2 \qquad (2.2)$$

using the equation (2.1). On the other hand, for $\forall X_1, X_2 \in \Gamma(TM_1)$ and Riemannian metric g_1, the relation

$$(\tilde{\nabla}_{X_1} g_1)(X_2, X_3) = -\eta(X_2) g_1(\varphi X_1, X_3) - \eta(X_3) g_1(X_2, \varphi X_1) \qquad (2.3)$$

is derived from equation (2.1). The linear connection $\tilde{\nabla}$ defined by (2.1) is termed a quarter symmetric non-metric connection for the $\tilde{\nabla}$ connection since it satisfies equations (2.2) and (2.3) [21].

A differentiable transformation $f: M_1 \to M_2$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2), with dimensions m_1 and m_2, respectively, is termed a Riemannian submersion if the following conditions are satisfied:

i.) f has maximal rank.

ii.) The transformation f_{*p} preserves the lengths of horizontal vectors $X_p \in \Gamma(\mathcal{H}_p)$ at each point $p \in M_1$.

On the other hand, for $q \in M_2$, $f^{-1}(q)$ is (m_1-m_2)-dimensional submanifold of M_1. The submanifolds $f^{-1}(q)$ are called the fibers of the submersion. A vector field on M_1 is termed a vertical vector field if it is always tangent to the fibers; if it is orthogonal to the fibers, it is called a horizontal vector field. If the vector field X_1 is a horizontal vector field on the manifold M_1 and X_1 is f-related to the vector field X_1' on the manifold M_2, then X_1 is called the fundamental vector field [30].

In this study, we will denote vertical vector fields and horizontal vector fields with the seymbols $\mathcal{V}$ and $\mathcal{H}$, respectively.

Lemma 2.1 Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, and let $f: M_1 \to M_2$ be Riemannian submersion. In this case, following expressions hold:

i. $g_1(X_1, X_2) = g_2(X_1{'}, X_2{'}) \circ f$.

ii. For the fundamental vector field $h[X_1, X_2]$, $f_* h[X_1, X_2] = [X_1{'}, X_2{'}] \circ f$.

iii. The fundamental vector field $h(\nabla_{X_1} X_2)$ is $f-$related to $(\nabla'_{X_1{'}} X_2{'})$ where ∇ and ∇' are the Levi-Civita connections on M_1 and M_2 respectively.

iv. For any $U_1 \in \Gamma(\mathcal{V})$, $[X_1, U_1] \in \Gamma(\mathcal{V})$,

where X_1 and X_2 are basic vector fields that are f-related to $X_1{'}$ and $X_2{'}$ respectively [31].

The distribution $\mathcal{V}$ corresponds to the foliation of M_1 by setting $\mathcal{V}_p = ker\, f_{*p}$ for any $p \in M_1$. At each point p, $\mathcal{V}_p$ is defined as vertical space, where $\mathcal{V}$ represents the vertical distribution. The sections of $\mathcal{V}$

are determined as a Lie subalgebra, denoted as $\chi^v(M_1)$, of the tangent bundle $\chi(M_1)$. The complementary distribution of $\mathcal{V}$ produced by the Riemannian metric g_1 is denoted by $\mathcal{H}$. Hence, the orthogonal decomposition $T_pM_1 = \mathcal{V}_p \oplus \mathcal{H}_p$ at any $p \in M_1$ is referred to as the horizontal space at p. Given any $E \in \chi(M_1)$, where vE and hE denote the vertical and horizontal components of E respectively [31].

O'Neill tensor fields are determined by a Riemannian submersion $f: M_1 \rightarrow M_2$. The following are the fundamental tensor fields:

$$\mathcal{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF, \tag{2.4}$$

$$\mathcal{A}_E F = v\nabla_{hE} hF + h\nabla_{hE} vF, \tag{2.5}$$

for any $E, F \in \chi(M_1)$, where v and h represent the vertical and horizontal projections, respectively. Moreover

$$\nabla_{U_1} U_2 = \mathcal{T}_{U_1} U_2 + v\nabla_{U_1} U_2, \tag{2.6}$$

$$\nabla_{U_1} X_1 = \mathcal{T}_{U_1} X_1 + h\nabla_{U_1} X_1, \tag{2.7}$$

$$\nabla_{X_1} U_1 = \mathcal{A}_{X_1} U_1 + v\nabla_{X_1} U_1, \tag{2.8}$$

$$\nabla_{X_1} X_2 = \mathcal{A}_{X_1} X_2 + h\nabla_{X_1} X_2, \tag{2.9}$$

where $X_1, X_2 \in \chi^h(M_1)$; $U_1, U_2 \in \chi^v(M_1)$. Furthermore, if X_1 is a basic vector field, then, $h\nabla_{U_1} X_1 = h\nabla_{X_1} U_1 = \mathcal{A}_{X_1} U_1$. We note that $\mathcal{T}_{U_1} U_2 = \mathcal{T}_{U_2} U_1$ [30].

Definition 2.2 Let (M_1, g_1) be a Riemannian manifold with Levi-Civita connection ∇. In this case the Riemannian curvature tensor R is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y, Z are vector fields on M_1.

At a point p in M_1, a Riemannian manifold with metric g_1, the sectional curvature K_p defined as follows:

$$K_p = \frac{g_1(R(X_1,X_2)X_2,X_1)}{\|X_1\|^2\|X_2\|^2 - g_1(X_1,X_2)^2}. \tag{2.10}$$

Furthermore, the Ricci curvature, denoted as Ric, is characterized as follows:

$$Ric: C_2^\infty(TM_1) \rightarrow C_0^\infty(TM_1) \text{ by } Ric(X_1, X_2) = \sum_{i=1}^{m} g_1(R(X_1, e_i)e_i, X_2).$$

Thus, the scalar curvature τ is obtained as

$$\tau = \sum_{j=1}^{m} R_i(e_j, e_j) = \sum_{j=1}^{m} \sum_{i=1}^{m} g_1(R(e_i, e_j)e_j, e_i), \tag{2.11}$$

where $\{e_1, e_2, \ldots, e_m\}$ signifies any local orthonormal frame for the tangent bundle [32].

In this work, $S(X_1, X_2)$ is used to signify $Ric(X_1, X_2)$.

Definition 2.3 Let (M_1, g_1) be a Riemannian manifold. An f-adapted local orthonormal frame $\{X_i, U_j\}$ with $1 \leq i \leq n$ and $1 \leq j \leq r$ is defined such that each X_i is horizontal and each U_j is vertical [31].

Lemma 2.4 Given two Riemannian manifolds (M_1, g_1) and (M_2, g_2), let f be a Riemannian submersion between them. Then we have:

$$\sum_{i=1}^{n} g_1(\mathcal{T}_{U_1} X_i, \mathcal{T}_{U_2} X_i) = \sum_{j=1}^{r} g_1(\mathcal{T}_{U_1} U_j, \mathcal{T}_{U_2} U_j), \tag{2.12}$$

$$\sum_{i=1}^{n} g_1(\mathcal{A}_{X_1} X_i, \mathcal{A}_{X_2} X_i) = \sum_{j=1}^{r} g_1(\mathcal{A}_{X_1} U_j, \mathcal{A}_{X_2} U_j), \tag{2.13}$$

$$\sum_{i=1}^{n} g_1(\mathcal{A}_{X_1} X_i, \mathcal{T}_{U_1} X_i) = \sum_{j=1}^{r} g_1(\mathcal{A}_{X_1} U_j, \mathcal{T}_{U_1} U_j), \tag{2.14}$$

where X_1, $X_2 \in \chi^h(M_1)$, U_1, $U_2 \in \chi^v(M_1)$, and $\{X_i, U_j\}$ is an f-adaptable frame on (M_1, g_1) [31].

Definition 2.5 Let the manifold (M_1, g_1) is Riemannian manifold and $\mathcal{V}$ be the local orthonormal frame of the vertical distribution. In this case we introduce horizontal vector field $\mathcal{N}$ on (M_1, g_1) defined as follows [31]:

$$\mathcal{N} = \sum_{j=1}^{r} \mathcal{T}_{U_j} U_j. \tag{2.15}$$

Let us write

$$\tilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)\varphi X_1, \tag{2.16}$$

where X_1, X_2 any vector fields on M_1 and η is a $1-$form, and φ is a (1,1)- type tensor field. Note that we will abbreviate the Riemannian submersions with quarter-symmetric non-metric connections as Q-SNMC.

Let $f: M_1 \to M_2$ be a Riemannian submersion from a Riemannian manifold M_1 to a Riemannian manifold M_2 equipped with Q-SNMC. In this case, expression (2.16) yields

$$\tilde{\mathcal{T}}(E, F) = \tilde{\mathcal{T}}_E F = \mathcal{T}_E F + \eta(vF) h\varphi(vE) + \eta(hF) v\varphi(vE), \tag{2.17}$$

$$\tilde{\mathcal{A}}(E, F) = \tilde{\mathcal{A}}_E F = \mathcal{A}_E F + \eta(hF) v\varphi(hE) + \eta(vF) h\varphi(hE), \tag{2.18}$$

for tensor fields of type (1,2) $\mathcal{T}$ and $\mathcal{A}$ on M_1 with respect to $\tilde{\nabla}$, where $E, F \in \Gamma(TM_1)$ [22]. Consider (M_1, g_1) and (M_2, g_2) as Riemannian manifolds, and let $\tilde{\nabla}$ denote a Q-SNMC. Furthermore, suppose $f: M_1 \to M_2$ is a Riemannian submersion mapping from a Riemannian manifold M_1 to another Riemannian manifold M_2 endowed with a Q-SNMC. In this case following equations are obtained:

$$\tilde{\mathcal{T}}_{U_1} U_2 = \tilde{\mathcal{T}}_{U_2} U_1 + \eta(vU_2) h\varphi(vU_1) - \eta(vU_1) h\varphi(vU_2), \tag{2.19}$$

$$\tilde{\mathcal{T}}_{U_1} X_1 = \mathcal{T}_{U_1} X_1 + \eta(hX_1) v\varphi(vU_1), \tag{2.20}$$

$$\tilde{\mathcal{A}}_{X_1} X_2 = -\tilde{\mathcal{A}}_{X_2} X_1 + \eta(hX_2) v\varphi(hX_1) + \eta(hX_1) v\varphi(hX_2), \tag{2.21}$$

$$\tilde{\mathcal{A}}_{X_1} U_1 = \mathcal{A}_{X_1} U_1 + \eta(vU_1) h\varphi(hX_1), \tag{2.22}$$

where $U_1, U_2 \in \Gamma(\mathcal{V})$, $X_1, X_2 \in \Gamma(\mathcal{H})$ [22].

On the other hand, the following equations are derived using equation (2.16):

$$\tilde{\nabla}_{U_1} U_2 = \tilde{\mathcal{T}}_{U_1} U_2 + \hat{\nabla}_{U_1} U_2 - \eta(U_2)\varphi(U_1), \tag{2.23}$$

$$\tilde{\nabla}_{U_1} X_1 = \mathcal{T}_{U_1} X_1 + h\tilde{\nabla}_{U_1} X_1 + \eta(X_1) v\varphi(U_1), \tag{2.24}$$

$$\tilde{\nabla}_{X_1} U_1 = \mathcal{A}_{X_1} U_1 + v\tilde{\nabla}_{X_1} U_1 + \eta(U_1) h\varphi(X_1), \tag{2.25}$$

$$\tilde{\nabla}_{X_1} X_2 = \mathcal{A}_{X_1} X_2 + h\tilde{\nabla}_{X_1} X_2 + \eta(X_2) v\varphi(X_1), \tag{2.26}$$

where $U_1, U_2 \in \Gamma(\mathcal{V})$, $X_1, X_2 \in \Gamma(\mathcal{H})$, and $\hat{\nabla}_{U_1} U_2 = v\tilde{\nabla}_{U_1} U_2$.

Theorem 2.6 Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, with $f: (M_1, g_1) \to (M_2, g_2)$ being a Riemannian submersion. Let $\tilde{R}$, R' and $\hat{R}$ be the Riemannian curvature tensors of M_1, M_2, and the fiber $(f^{-1}(x), \hat{g}_x)$ with respect to a Q-SNMC, respectively. In this case, the following equations are obtained:

$$\begin{aligned} g_1\big(\tilde{R}(U_1, U_2)U_3, U_4\big) &= g_1(\hat{R}(U_1, U_2)U_3, U_4) - g_1(\eta(\hat{\nabla}_{U_2} U_3)\varphi(U_1), U_4) + g_1(\mathcal{T}_{U_1}\tilde{\mathcal{T}}_{U_2} U_3, U_4) \\ &+ g_1\big(\eta(\tilde{\mathcal{T}}_{U_2} U_3) v\varphi(U_1), U_4\big) - g_1\big(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2), U_4\big) + g_1(\eta(\hat{\nabla}_{U_1} U_3)\varphi(U_2), U_4) \\ &- g_1\big(\mathcal{T}_{U_2}\tilde{\mathcal{T}}_{U_1} U_3, U_4\big) - g_1\big(\eta(\tilde{\mathcal{T}}_{U_1} U_3) v\varphi(U_2), U_4\big) + g_1\big(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1), U_4\big) \end{aligned}$$

$$+g_1(\eta(U_3)\varphi([U_1,U_2]),U_4),$$

(2.27)

$$g_1\big(\tilde{R}(U_1,U_2)U_3,X_1\big)=g_1\big(\tilde{\mathcal{T}}_{U_1}\hat{\nabla}_{U_2}U_3,X_1\big)-g_1\big(\eta\big(\hat{\nabla}_{U_2}U_3\big)\varphi(U_1),X_1\big)+g_1(h\tilde{\nabla}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,X_1)$$
$$-g_1\big(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),X_1\big)-g_1\big(\tilde{\mathcal{T}}_{U_2}\hat{\nabla}_{U_1}U_3,X_1\big)+g_1\big(\eta\big(\hat{\nabla}_{U_1}U_3\big)\varphi(U_2),X_1\big)-g_1(h\tilde{\nabla}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,X_1)$$
$$+g_1\big(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),X_1\big)-g_1\big(\tilde{\mathcal{T}}_{[U_1,U_2]}U_3,X_1\big)+g_1(\eta(U_3)\varphi([U_1,U_2]),X_1),\qquad(2.28)$$

$$g_1\big(\tilde{R}(X_1,X_2)X_3,X_4\big)=g_1(R'(X_1,X_2)X_3,X_4)+g_1\big(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_1}X_3\big)-g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_2}X_3)$$
$$+g_1(\eta\big(\mathcal{A}_{X_2}X_3\big)h\varphi(X_1),X_4)-\ g_1(\eta\big(\mathcal{A}_{X_1}X_3\big)h\varphi(X_2),X_4)+g_1\big(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),X_4\big)$$
$$-g_1\big(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),X_4\big),\qquad(2.29)$$

$$g_1\big(\tilde{R}(X_1,X_2)X_3,U_1\big)=g_1\big(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}X_3,U_1\big)+g_1(\mathcal{A}_{X_1}h\tilde{\nabla}_{X_2}X_3,U_1)+g_1\big(\eta\big(h\tilde{\nabla}_{X_2}X_3\big)v\varphi(X_1),U_1\big)$$
$$+g_1\big(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),U_1\big)-g_1\big(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}X_3,U_1\big)-g_1\big(\mathcal{A}_{X_2}h\tilde{\nabla}_{X_1}X_3,U_1\big)$$
$$-g_1\big(\eta\big(h\tilde{\nabla}_{X_1}X_3\big)v\varphi(X_2),U_1\big)-g_1\big(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),U_1\big)-g_1(\eta(X_3)v\varphi([X_1,X_2]),U_1),\qquad(2.30)$$

$$g_1\big(\tilde{R}(X_1,U_1)X_2,U_2\big)=g_1\big(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_1}X_2,U_2\big)+g_1\big(\mathcal{A}_{X_1}h\tilde{\nabla}_{U_1}X_2,U_2\big)+g_1(\eta(h\tilde{\nabla}_{U_1}X_2)v\varphi(X_1),U_2)$$
$$+g_1\big(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_1),U_2\big)-g_1\big(\hat{\nabla}_{U_1}\mathcal{A}_{X_1}X_2,U_2\big)+g_1(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_1),U_2)-g_1(\mathcal{T}_{U_1}h\tilde{\nabla}_{X_1}X_2,U_2)$$
$$-\eta\big(h\tilde{\nabla}_{X_1}X_2\big)g_1(v\varphi(U_1),U_2)-g_1(\tilde{\nabla}_{U_1}\eta(X_2)v\varphi(X_1),U_2)$$
$$-g_1\big(\mathcal{T}_{[X_1,U_1]}X_2,U_2\big)-g_1(\eta(X_2)v\varphi([X_1,U_1]),U_2),\qquad(2.31)$$

$$g_1(\tilde{R}(X_1,X_2)U_1,X_3)=g_1(h\big(\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}U_1,X_3\big)+g_1\big(\mathcal{A}_{X_1}v\tilde{\nabla}_{X_2}U_1,X_3\big)+g_1(\eta(v\tilde{\nabla}_{X_2}U_1)h\varphi(X_1),X_3)$$
$$+g_1(\tilde{\nabla}_{X_1}\eta(U_1)h\varphi(X_2),X_3)-g_1(h\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}U_1,X_3)-g_1\big(\mathcal{A}_{X_2}v\tilde{\nabla}_{X_1}U_1,X_3\big)$$
$$-g_1(\eta\big(v\tilde{\nabla}_{X_1}U_1\big)h\varphi(X_2),X_3)-g_1(\tilde{\nabla}_{X_2}\eta(U_1)h\varphi(X_1),X_3)-g_1\big(\mathcal{A}_{[X_1,X_2]}U_1,X_3\big)$$
$$-g_1(\eta(U_1)h\varphi([X_1,X_2]),X_3),\qquad(2.32)$$

$$g_1\big(\tilde{R}(X_1,U_1)U_2,U_3\big)=g_1\big(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_1}U_2,U_3\big)-g_1\big(\mathcal{A}_{X_1}U_3,\tilde{\mathcal{T}}_{U_1}U_2\big)+g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_2)v\varphi(X_1),U_3)$$
$$-g_1\big(\tilde{\nabla}_{X_1}\eta(U_2)\varphi(U_1),U_3\big)+g_1\big(\mathcal{T}_{U_1}U_3,\mathcal{A}_{X_1}U_2\big)-g_1\big(\eta\big(\mathcal{A}_{X_1}U_2\big)v\varphi(U_1),U_3\big)-g_1(\hat{\nabla}_{U_1}v\tilde{\nabla}_{X_1}U_2,U_3)$$
$$+g_1\big(\eta\big(v\tilde{\nabla}_{X_1}U_2\big)\varphi(U_1),U_3\big)-g_1\big(\tilde{\nabla}_{U_1}\eta(U_2)h\varphi(X_1),U_3\big)-g_1\big(\hat{\nabla}_{[X_1,U_1]}U_2,U_3\big)$$
$$+g_1(\eta(U_2)\varphi([X_1,U_1]),U_3),$$

(2.33)

where $U_1,U_2,U_3,U_4\in\Gamma(\mathcal{V}),\ X_1,X_2,X_3,X_4\in\Gamma(\mathcal{H})$.

Proof: If the Riemannian curvature tensor from Definition(2.2) is used for $U_1,U_2,U_3\in\Gamma(\mathcal{V})$, then the relation $\tilde{R}(U_1,U_2)U_3=\tilde{\nabla}_{U_1}\tilde{\nabla}_{U_2}U_3-\tilde{\nabla}_{U_2}\tilde{\nabla}_{U_1}U_3-\tilde{\nabla}_{[U_1,U_2]}U_3$ is obtained. By multiplying this equation by $U_4\in\Gamma(\mathcal{V})$ and using equation (2.23) and (2.24), the equation (2.27) is easily established. The other equations can be derived in a similar manner.

Proposition 2.7 Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, with $f:(M_1,g_1)\to(M_2,g_2)$ being a Riemannian submersion. Let $\tilde{S},\ S'$ and $\hat{S}$ be the Ricci tensors of $M_1,\ M_2$, and the fiber

$(f^{-1}(x), \hat{g}_{1x})$ with respect to a Q-SNMC $\tilde{\nabla}$, respectively. In this case, the following equations are obtained:

$$\tilde{S}(U_1,U_2) = \hat{S}(U_1,U_2) - g_1(\tilde{\mathcal{N}}, \tilde{\mathcal{T}}_{U_1}U_2) + g_1(\eta(\tilde{\mathcal{N}})v\varphi(U_1),U_2) + \sum_i \{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_1),U_2)$$
$$-g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_2) + g_1(\eta(\hat{\nabla}_{U_1}U_i)(\varphi(U_i),U_2) + g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_2) - g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_2)$$
$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_2) + g_1(\eta(U_i)\varphi([U_1,U_i]),U_2)\} - \sum_j \{g_1\left(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_2\right)$$
$$-g_1\left(\mathcal{A}_{X_j}U_2, h\tilde{\nabla}_{U_1}X_j\right) + g_1(\eta(h\tilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_2) + g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_2\right)$$
$$-g_1\left(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_2\right) + g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)g_1(\varphi(U_1),U_2)\right) + g_1\left(\mathcal{T}_{U_1}U_2, h\tilde{\nabla}_{X_j}X_j\right)$$
$$-g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)g_1(v\varphi(U_1),U_2)\right) - g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_2) - g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_2\right)$$
$$-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_2)\}, \tag{2.34}$$

$$\tilde{S}(X_1,X_2) = S'(X_1',X_2')\circ f + g_1(\mathcal{N}, h\tilde{\nabla}_{X_1}X_2) - \sum_i \{g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_2,U_i) - g_1(\mathcal{A}_{X_1}U_i, h\tilde{\nabla}_{U_i}X_2)$$
$$+g_1(\eta(h\tilde{\nabla}_{U_i}X_2)v\varphi(X_1),U_i) + g_1(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_i),U_i) - g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_2,U_i)$$
$$+g_1(\eta(\mathcal{A}_{X_1}X_2)\varphi(U_i),U_i) - g_1(\eta(h\tilde{\nabla}_{X_1}X_2)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_2)v\varphi(X_1),U_i)$$
$$-g_1(\mathcal{T}_{[X_1,U_i]}X_2,U_i) - g_1(\eta(X_2)v\varphi([X_1,U_i]),U_i)\} + \sum_j \{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_2}X_j) - g_1(\mathcal{A}_{X_1}X_2,\mathcal{A}_{X_j}X_j)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_2\right) - g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_2) + g_1(\tilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_2)$$
$$-g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_2\right)\},$$

(2.35)

$$\tilde{S}(X_1,U_1) = \sum_i \{g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1) - g_1(\mathcal{A}_{X_1}U_1, h\tilde{\nabla}_{X_i}X_i) + g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_1)$$
$$+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1) - g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1) + g_1(\mathcal{A}_{X_i}U_1, h\tilde{\nabla}_{X_1}X_i) - g_1(\eta(h\tilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_1)$$
$$-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1) - g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)\} + \sum_j \{g_1\left(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1\right)$$
$$-g_1(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1) + g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j\right) - g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_1) - g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1)$$
$$+g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi(U_j),U_1) - g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1) - g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)$$
$$+g_1(\eta(U_j)\varphi([X_1,U_j],U_1) - g_1(\tilde{\mathcal{N}},\mathcal{A}_{X_1}U_1) + g_1(\eta(\tilde{\mathcal{N}})v\varphi(X_1),U_1)\}, \tag{2.36}$$

where, $U_1,U_2 \in \Gamma(\mathcal{V})$, $X_1,X_2 \in \Gamma(\mathcal{H})$, $\tilde{\mathcal{N}} = \sum_{j=1}^{r} \tilde{\mathcal{T}}_{U_j}U_j$, and $\{X_i,U_j\}$ is an f −adaptable frame on (M_1,g_1).

Proof: Let $\tilde{S}(U_1,U_2)=\sum_i g_1(\tilde{R}(U_1,U_i)U_i,U_2)+\sum_j g_1(\tilde{R}(U_1,X_j)X_j,U_2)$.
By utilizing equations (2.27) and (2.31), equation (2.34) can be derived with ease. The proofs of the other equations can similarly be demonstrated.

Theorem 2.8 Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, with $f:(M_1,g_1)\to(M_2,g_2)$ being a Riemannian submersion. Let $\tilde{\tau}$, τ', $\hat{\tau}$ denote the scalar curvatures of M_1, M_2, and $(f^{-1}(x),\hat{g}_{1x})$ fibre respectively, and let $\{X_i,U_j\}$ be an f −adaptable frame on (M_1,g_1). Then, the scalar curvature of the Riemannian manifold M_1 with Q-SNMC $\tilde{\nabla}$ is as follows:

$$\begin{aligned}\tilde{\tau}=\tau'\circ f+\hat{\tau}-g_1(\widetilde{\mathcal{N}},\mathcal{N})+\sum_i\{&g_1(\mathcal{N},h\widetilde{\nabla}_{X_i}X_i)-g_1(v\widetilde{\nabla}_{X_i}\mathcal{T}_{U_i}X_i,U_i)-g_1(\mathcal{A}_{X_i}U_i,h\widetilde{\nabla}_{U_i}X_i)+\\ &g_1(\eta(h\widetilde{\nabla}_{U_i}X_i)v\varphi(X_i),U_i)+g_1(\widetilde{\nabla}_{X_i}\eta(X_i)v\varphi(U_i),U_i)-g_1(\widehat{\nabla}_{U_i}\mathcal{A}_{X_i}X_i,U_i)+g_1(\eta(\mathcal{A}_{X_i}X_i)\varphi(U_i),U_i)\\ &-g_1(\eta(h\widetilde{\nabla}_{X_i}X_i)v\varphi(U_i),U_i)-g_1(\widetilde{\nabla}_{U_i}\eta(X_i)v\varphi(X_i),U_i)-g_1(\mathcal{T}_{[X_i,U_i]}X_i,U_i)\\ &-g_1(\eta(X_i)v\varphi([X_i,U_i]),U_i)\}+\sum_{i,j}\{-g_1(\mathcal{A}_{X_i}X_j,\mathcal{A}_{X_i}X_j)-g_1(\mathcal{A}_{X_i}X_i,\mathcal{A}_{X_j}X_j)\\ &+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_i),X_i\right)-g_1(\eta(\mathcal{A}_{X_i}X_j)h\varphi(X_j),X_i)+g_1(\widetilde{\nabla}_{X_i}\eta(X_j)v\varphi(X_j),X_i)\\ &-g_1\left(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_i),X_i\right)-g_1(\eta(\widehat{\nabla}_{U_i}U_i)\varphi(U_j),U_j)-g_1(\widehat{\nabla}_{U_j}\eta(U_i)\varphi(U_i),U_j)+\\ &g_1\left(\eta\left(\widehat{\nabla}_{U_j}U_i\right)\varphi(U_i),U_j\right)+g_1\left(\tilde{\mathcal{T}}_{U_j}U_i,\mathcal{T}_{U_i}U_j\right)-g_1\left(\eta\left(\tilde{\mathcal{T}}_{U_j}U_i\right)v\varphi(U_i),U_j\right)+g_1(\widehat{\nabla}_{U_i}\eta(U_i)\varphi(U_j),U_j)\\ &+g_1(\eta(U_i)\varphi([U_j,U_i]),U_j)\}-\sum_j\{g_1(v\widetilde{\nabla}_{X_j}\mathcal{T}_{U_j}X_j,U_j)-g_1(\mathcal{A}_{X_j}U_j,h\widetilde{\nabla}_{U_j}X_j)\\ &+g_1\left(\eta\left(h\widetilde{\nabla}_{U_j}X_j\right)v\varphi(X_j),U_j\right)+g_1\left(\widetilde{\nabla}_{X_j}\ \eta(X_j)v\varphi(U_j),U_j\right)-g_1\left(\widehat{\nabla}_{U_j}\mathcal{A}_{X_j}X_j,U_j\right)\\ &+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_j),U_j\right)+g_1\left(\mathcal{N},h\widetilde{\nabla}_{X_j}X_j\right)-g_1\left(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_j),U_j\right)\\ &-g_1\left(\widetilde{\nabla}_{U_j}\eta(X_j)v\varphi(X_j),U_j\right)-g_1\left(\mathcal{T}_{[X_j,U_j]}X_j,U_j\right)-g_1(\eta(X_j)v\varphi([X_j,U_j]),U_j)\\ &+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_j),U_j)\}.\end{aligned}\tag{2.37}$$

Proof: By applying equations (2.34) and (2.35) to the equation

$$\tilde{\tau}=\sum_i\widetilde{S}(X_i,X_i)+\sum_j\widetilde{S}(U_j,U_j)$$

the equation (2.37) is easily obtained.

3. Calculations of Curvature Tensors in Riemannian Submersions with Quarter-Symmetric Non-Metric Connections

3.1 Weyl projective curvature tensor

In this section using Q-SNMC in the context of Riemannian submersions the relationships related to the Weyl projective curvature tensor between the base space, the total space, and the fiber bundles are examined.

Definition 3.1 Let M_1 be an $n-$ dimensional C^∞ manifold. In this scenario, within the $n-$dimensional space V^n, for every $X_1, X_2, X_3 \in \chi(M_1)$, the Weyl projective curvature tensor field of M_1 is defined as follows:

$$\tilde{P}(X_1,X_2)X_3 = \tilde{R}(X_1,X_2)X_3 - \frac{1}{n-1}\{\tilde{S}(X_2,X_3)X_1 - \tilde{S}(X_1,X_3)X_2\}, \tag{3.1}$$

where $\tilde{R}$ and $\tilde{S}$ denote the Riemannian curvature tensor and Ricci curvature tensor of the total space, respectively [23].

Theorem 3.2 Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, with, $f:(M_1,g_1) \to (M_2,g_2)$ representing a Riemannian submersion, and $\tilde{R}, R'$, and $\hat{R}$ denote the Riemannian curvature tensors, $\tilde{S}, S'$, and $\hat{S}$ represent the Ricci tensors of M_1, M_2, and the fiber $(f^{-1}(x), \hat{g}_{1x})$ respectively. Then, for every $U_1,U_2,U_3,U_4 \in \varkappa^v(M_1)$ and $X_1,X_2,X_3,X_4 \in \varkappa^h(M_1)$, the following Weyl projective curvature tensor relations hold:

$$\begin{aligned}
g_1\big(\tilde{P}(X_1,X_2)X_3,X_4\big) &= g_1(R'(X_1,X_2)X_3,X_4) + g_1\big(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_1}X_3\big) - g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_2}X_3) \\
&+\eta\big(\mathcal{A}_{X_2}X_3\big)g_1(h\varphi(X_1),X_4) - g_1\big(\eta\big(\mathcal{A}_{X_1}X_3\big)h\varphi(X_2),X_4\big) + g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),X_4) \\
&-g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),X_4) - \frac{1}{(n-1)}\{g_1(X_1,X_4)[S'(X_2',X_3')\circ f + g_1\big(\mathcal{N},h\tilde{\nabla}_{X_2}X_3\big) \\
&- \textstyle\sum_i \{g_1(v\tilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_3,U_i) - g_1(\mathcal{A}_{X_2}U_i,h\tilde{\nabla}_{U_i}X_3)+g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_3\big)v\varphi(X_2),U_i\big) \\
&+g_1\big(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(U_i),U_i\big) - g_1\big(\tilde{\nabla}_{U_i}\mathcal{A}_{X_2}X_3,U_i\big) + g_1(\eta(\mathcal{A}_{X_2}X_3)\varphi(U_i),U_i) \\
&-g_1\big(\eta\big(h\tilde{\nabla}_{X_2}X_3\big)v\varphi(U_i),U_i\big) - g_1\big(\tilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_2),U_i\big) - g_1(\mathcal{T}_{[X_2,U_i]}X_3,U_i) \\
&-g_1(\eta(X_3)v\varphi([X_2,U_i]),U_i)\} + \textstyle\sum_j \{-g_1\big(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_3}X_j\big) - g_1(\mathcal{A}_{X_2}X_3,\ \mathcal{A}_{X_j}X_j) \\
&+g_1\left(\eta(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_3) - g_1(\eta\big(\mathcal{A}_{X_2}X_j\big)h\varphi(X_j),X_3)+g_1(\tilde{\nabla}_{X_2}\eta(X_j)v\varphi(X_j),X_3) \\
&-g_1(\tilde{\nabla}_{X_j}\ \eta(X_j)v\varphi(X_2),X_3)\}] - g_1(X_2,X_4)[S'(X_1',X_3')\circ f + g_1(\mathcal{N},h\tilde{\nabla}_{X_1}X_3) \\
&-\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_3,U_i) - g_1(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_3) + g_1(\eta(h\tilde{\nabla}_{U_i}X_3)v\varphi(X_1),U_i) \\
&+g_1\big(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(U_i),U_i\big) - g_1\big(\tilde{\nabla}_{U_i}\mathcal{A}_{X_1}X_3,U_i\big) + g_1\big(\eta\big(\mathcal{A}_{X_1}X_3\big)\varphi(U_i),U_i\big) \\
&-g_1\big(\eta\big(h\tilde{\nabla}_{X_1}X_3\big)v\varphi(U_i),U_i\big) - g_1\big(\tilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_1),U_i\big) - g_1(\mathcal{T}_{[X_1,U_i]}X_3,U_i) \\
&-g_1(\eta(X_3)v\varphi([X_1,U_i]),U_i) + \sum_j\{-g_1\big(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_3}X_j\big) - g_1(\mathcal{A}_{X_1}X_3,\mathcal{A}_{X_j}X_j) \\
&+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_3\right) - g_1\big(\eta\big(\mathcal{A}_{X_1}X_j\big)h\varphi(X_j),X_3\big) + g_1\big(\tilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_3\big) \\
&-g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_3)\}],
\end{aligned} \tag{3.2}$$

$$g_1(\tilde{P}(X_1,X_2)X_3,U_1) = g_1(v\widetilde{\nabla}_{X_1}\mathcal{A}_{X_2}X_3,U_1) + g_1(\mathcal{A}_{X_1}h\widetilde{\nabla}_{X_2}X_3,U_1) + g_1(\eta(h\widetilde{\nabla}_{X_2}X_3)v\varphi(X_1),U_1)$$
$$+g_1(\widetilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),U_1) - g_1(v\widetilde{\nabla}_{X_2}\mathcal{A}_{X_1}X_3,U_1) - g_1(\mathcal{A}_{X_2}h\widetilde{\nabla}_{X_1}X_3,U_1)$$
$$-g_1(\eta(h\widetilde{\nabla}_{X_1}X_3)v\varphi(X_2),U_1) - g_1(\widetilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),U_1) - g_1(\eta(X_3)v\varphi([X_1,X_2]),U_1), \quad (3.3)$$

$$g_1(\tilde{P}(X_1,U_1)X_2,U_2) = g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_1}X_2,U_2) + g_1(\mathcal{A}_{X_1}h\tilde{\nabla}_{U_1}X_2,U_2) + g_1(\eta(h\tilde{\nabla}_{U_1}X_2)v\varphi(X_1),U_2)$$
$$+g_1(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_1),U_2) - g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_1}X_2,U_2) + g_1(\eta(\mathcal{A}_{X_1}X_2)\varphi(U_1),U_2) - g_1(\mathcal{T}_{U_1}h\widetilde{\nabla}_{X_1}X_2,U_2)$$
$$-g_1(\eta(h\widetilde{\nabla}_{X_1}X_2)v\varphi(U_1),U_2) - g_1(\tilde{\nabla}_{U_1}\eta(X_2)v\varphi(X_1),U_2) - g_1(\mathcal{T}_{[X_1,U_1]}X_2,U_2)$$
$$-g_1(\eta(X_2)v\varphi([X_1,U_1]),U_2) + \frac{1}{(n-1)}\{g_1(U_1,U_2)[S'(X_1',X_2')\circ f + g_1(\mathcal{N},h\widetilde{\nabla}_{X_1}X_2)$$
$$-\sum_i \{g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_2,U_i) - g_1(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_2) + g_1(\eta(h\tilde{\nabla}_{U_i}X_2)v\varphi(X_1),U_i)$$
$$+g_1(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_i),U_i) - g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_2,U_i) + g_1(\eta(\mathcal{A}_{X_1}X_2)\varphi(U_i),U_i)$$
$$-g_1(\eta(h\widetilde{\nabla}_{X_1}X_2)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_2)v\varphi(X_1),U_i) - g_1(\mathcal{T}_{[X_1,U_i]}X_2,U_i)$$
$$-g_1(\eta(X_2)v\varphi([X_1,U_i]),U_i)\} + \sum_j \{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_2}X_j) - g_1(\mathcal{A}_{X_1}X_2,\mathcal{A}_{X_j}X_j)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_2\right) - g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_2) + g_1(\widetilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_2)$$
$$-g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_2\right)\}], \quad (3.4)$$

$$g_1(\tilde{P}(U_1,U_2)U_3,X_1) = g_1(\tilde{\mathcal{T}}_{U_1}\hat{\nabla}_{U_2}U_3,X_1) - g_1(\eta(\hat{\nabla}_{U_2}U_3)\varphi(U_1),X_1) + g_1(h\widetilde{\nabla}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,X_1)$$
$$-g_1(\widetilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),X_1) - g_1(\tilde{\mathcal{T}}_{U_2}\hat{\nabla}_{U_1}U_3,X_1) + g_1(\eta(\hat{\nabla}_{U_1}U_3)\varphi(U_2),X_1) - g_1(h\widetilde{\nabla}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,X_1)$$
$$+g_1(\widetilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),X_1) - g_1(\tilde{\mathcal{T}}_{[U_1,U_2]}U_3,X_1) + g_1(\eta(U_3)\varphi([U_1,U_2],X_1), \quad (3.5)$$

$$g_1(\tilde{P}(U_1,U_2)U_3,U_4) = g_1(\hat{R}(U_1,U_2)U_3,U_4) - g_1(\eta(\hat{\nabla}_{U_2}U_3)\varphi(U_1),U_4) + g_1(\mathcal{T}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,U_4)$$
$$+g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_3)v\varphi(U_1),U_4) - g_1(\widetilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),U_4) + g_1(\eta(\hat{\nabla}_{U_1}U_3)\varphi(U_2),U_4)$$
$$-g_1(\mathcal{T}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,U_4) - g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_3)v\varphi(U_2),U_4) + g_1(\widetilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),U_4)$$
$$+g_1(\eta(U_3)\varphi([U_1,U_2]),U_4) - \frac{1}{n-1}\{g_1(U_1,U_4).[\hat{S}(U_2,U_3) - g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_2}U_3)$$
$$+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_2),U_3) + \sum_i \{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_2),U_3) - g_1(\tilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_3)$$
$$+g_1(\eta(\hat{\nabla}_{U_2}U_i)\varphi(U_i),U_3) + g_1(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_3) - g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_i)v\varphi(U_i),U_3) + g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_3)$$
$$+g_1(\eta(U_i)\varphi([U_2,U_i]),U_3)\} - \sum_j \{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_3) - g_1(\mathcal{A}_{X_j}U_3,h\widetilde{\nabla}_{U_2}X_j)$$
$$+g_1(\eta(h\widetilde{\nabla}_{U_2}X_j)v\varphi(X_j),U_3) + g_1\left(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_2),U_3\right) - g_1\left(\hat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_3\right)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_3\right) + g_1\left(\mathcal{T}_{U_2}U_3,h\widetilde{\nabla}_{X_j}X_j\right) - g_1\left(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_3\right)$$
$$-g_1(\tilde{\nabla}_{U_2}\eta(X_j)v\varphi(X_j),U_3) - g_1\left(\mathcal{T}_{[X_j,U_2]}X_j,U_3\right) - g_1(\eta(X_j)v\varphi([X_j,U_2]),U_3)\}$$
$$-g_1(U_2,U_4)[\hat{S}(U_1,U_3) - g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_1}U_3) + g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_1),U_3) + \sum_i \{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_1),U_3)$$

$$-g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_3)+g_1\left(\eta\left(\hat{\nabla}_{U_i}\,U_i\right)\varphi(U_i),U_3\right)+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_3)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_3)$$

$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_3)+g_1(\eta(U_i)\varphi([U_1,U_i]),U_3)\}\text{-}\textstyle\sum_j\{g_1\left(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_3\right)-g_1(\mathcal{A}_{X_j}U_3,h\tilde{\nabla}_{U_1}X_j)$$

$$+g_1(\eta(h\tilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_3)+g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_3\right)-g_1(\hat{\nabla}_{U_1}\;\mathcal{A}_{X_j}X_j,U_3)$$

$$+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_3)+g_1(\mathcal{T}_{U_1}U_3,h\tilde{\nabla}_{X_j}X_j)-g_1(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_3)$$

$$-g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_3)-g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_3\right)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_3)\}],\quad(3.6)$$

$$\begin{aligned}
g_1(\tilde{P}(X_1,X_2)U_1,X_3)&=g_1(h\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}U_1,X_3)+g_1(\mathcal{A}_{X_1}v\tilde{\nabla}_{X_2}U_1,X_3)+g_1(\eta(v\tilde{\nabla}_{X_2}U_1)h\varphi(X_1),X_3)\\
&+g_1(\tilde{\nabla}_{X_1}\eta(U_1)h\varphi(X_2),X_3)-g_1(h\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}U_1,X_3)-g_1(\mathcal{A}_{X_2}v\tilde{\nabla}_{X_1}U_1,X_3)\\
&-g_1(\eta(v\tilde{\nabla}_{X_1}U_1)h\varphi(X_2),X_3)-g_1(\tilde{\nabla}_{X_2}\eta(U_1)h\varphi(X_1),X_3)-g_1(\mathcal{A}_{[X_1,X_2]}U_1,X_3)\\
&-g_1(\eta(U_1)h\varphi([X_1,X_2]),X_3)-\frac{1}{n-1}\{g_1(X_1,X_3)[\sum\nolimits_i\{g_1(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1)-g_1(\mathcal{A}_{X_2}U_1,h\tilde{\nabla}_{X_i}X_i)\\
&+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_2),U_1)+g_1(\tilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1)-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_2}X_i)\\
&-g_1(\eta(h\tilde{\nabla}_{X_2}X_i)v\varphi(X_i),U_1)-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1)-g_1(\eta(X_i)v\varphi([X_2,X_i],U_1)\}\\
&\sum\nolimits_j\{g_1(v\tilde{\nabla}_{X_2}\hat{\nabla}_{U_j}U_j,U_1)-g_1(\tilde{\nabla}_{X_2}\eta(U_j)\varphi(U_j),U_1)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j)\\
&-g_1(\eta(\mathcal{A}_{X_2}U_j)v\varphi(U_j),U_1)-g_1\left(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_2}U_j,U_1\right)+g_1(\eta(v\tilde{\nabla}_{X_2}U_j)\varphi(U_j),U_1)\\
&-g_1\left(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_2),U_1\right)-g_1\left(\hat{\nabla}_{[X_2,U_j]}U_j,U_1\right)+g_1(\eta(U_j)\varphi([X_2,U_j]),U_1)\\
&-g_1(\tilde{\mathcal{N}},\mathcal{A}_{X_2}U_1)+g_1(\eta(\tilde{\mathcal{N}})v\varphi(X_2),U_1)\}]-g_1(X_2,X_3)[\textstyle\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)\\
&-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i)+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_1)+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)\\
&+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i)-g_1(\eta(h\tilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_1)-g_1(\tilde{\nabla}_{X_i}\;\eta(X_i)v\varphi(X_1),U_1)\\
&-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)\}+\sum\nolimits_j\{g_1\left(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1\right)-g_1(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1)\\
&+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j\right)-g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_1)-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1)+g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi(U_j),U_1)\\
&-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1)-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)+g_1(\eta(U_j)\varphi([X_1,U_j],U_1)-g_1(\tilde{\mathcal{N}},\mathcal{A}_{X_1}U_1)\\
&+g_1(\eta(\tilde{\mathcal{N}})v\varphi(X_1),U_1)\}].\qquad(3.7)
\end{aligned}$$

Proof. We can easily prove the first equation by using equations (2.29) and (2.35) in equation (3.1) for the product of $\tilde{P}$ and X_4. The proofs of the other equations can be done similarly.

We note that if $\mathcal{T}_U V = g_1(U,V).H$ then a Riemannian submersion is referred to as a Riemannian submersion with totally umbilical fibers, where H is the fiber's mean curvature vector field and $U,V \in \Gamma(kerf_*)$ [33]. Moreover based on Doğru's study [34], if $\mathcal{N} = 0$, then $\widetilde{\mathcal{N}}=0$.

Corollary 3.3 Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, and let $f:(M_1,g_1) \to (M_2,g_2)$ be a Riemannian submersion. If the Riemannian submersion contains total umbilical fibres $(\mathcal{N} = 0)$, then the Weyl projective curvature tensor is given by

$$g_1\big(\tilde{P}(X_1,X_2)X_3,X_4\big) = g_1(R'(X_1,X_2)X_3,X_4) + g_1\big(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_1}X_3\big) - g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_2}X_3)$$
$$+g_1\big(\eta\big(\mathcal{A}_{X_2}X_3\big)h\varphi(X_1),X_4\big) - g_1\big(\eta\big(\mathcal{A}_{X_1}X_3\big)h\varphi(X_2),X_4\big) + g_1(\widetilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),X_4)$$
$$-g_1(\widetilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),X_4) - \frac{1}{(n-1)}\{g_1(X_1,X_4)[S'(X_2',X_3')\circ f - \sum_i \{g_1(v\widetilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_3,U_i)$$
$$-g_1(\mathcal{A}_{X_2}U_i,h\widetilde{\nabla}_{U_i}X_3)+g_1\big(\eta\big(h\widetilde{\nabla}_{U_i}X_3\big)v\varphi(X_2),U_i\big)+g_1\big(\widetilde{\nabla}_{X_2}\eta(X_3)v\varphi(U_i),U_i\big) - g_1\big(\widehat{\nabla}_{U_i}\mathcal{A}_{X_2}X_3,U_i\big)$$
$$+g_1(\eta(\mathcal{A}_{X_2}X_3)\varphi(U_i),U_i)-g_1\big(\eta\big(h\widetilde{\nabla}_{X_2}X_3\big)v\varphi(U_i),U_i\big) - g_1\big(\widetilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_2),U_i\big)$$
$$-g_1(\mathcal{T}_{[X_2,U_i]}X_3,U_i) - g_1(\eta(X_3)v\varphi([X_2,U_i]),U_i)\} + \sum_j \{-g_1\big(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_3}X_j\big)$$
$$-g_1(\mathcal{A}_{X_2}X_3,\ \mathcal{A}_{X_j}X_j) + g_1(\eta(\mathcal{A}_{X_j}X_j)h\varphi(X_2),X_3) - g_1(\eta\big(\mathcal{A}_{X_2}X_j\big)h\varphi\big(X_j\big),X_3)$$
$$+g_1(\widetilde{\nabla}_{X_2}\eta\big(X_j\big)v\varphi\big(X_j\big),X_3) - g_1(\widetilde{\nabla}_{X_j}\ \eta\big(X_j\big)v\varphi(X_2),X_3)\}] - g_1(X_2,X_4)[S'(X_1',X_3')\circ f$$
$$-\sum_i \{g_1\big(v\widetilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_3,U_i\big) - g_1\big(\mathcal{A}_{X_1}U_i,h\widetilde{\nabla}_{U_i}X_3\big) + g_1(\eta\big(h\widetilde{\nabla}_{U_i}X_3\big)v\varphi(X_1),U_i)$$
$$+g_1\big(\widetilde{\nabla}_{X_1}\eta(X_3)v\varphi(U_i),U_i\big) - g_1\big(\widehat{\nabla}_{U_i}\mathcal{A}_{X_1}X_3,U_i\big) + g_1\big(\eta\big(\mathcal{A}_{X_1}X_3\big)\varphi(U_i),U_i\big)$$
$$-g_1\big(\eta\big(h\widetilde{\nabla}_{X_1}X_3\big)v\varphi(U_i),U_i\big) - g_1\big(\widetilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_1),U_i\big) - g_1(\mathcal{T}_{[X_1,U_i]}X_3,U_i)$$
$$-g_1(\eta(X_3)v\varphi([X_1,U_i]),U_i)\} + \sum_j \{-g_1\big(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_3}X_j\big) - g_1(\mathcal{A}_{X_1}X_3,\mathcal{A}_{X_j}X_j)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_3\right) - g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_3) + g_1(\widetilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_3)$$
$$-g_1(\widetilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_1),X_3)\}],$$

$$g_1\big(\tilde{P}(X_1,U_1)X_2,U_2\big) = g_1\big(v\widetilde{\nabla}_{X_1}\mathcal{T}_{U_1}X_2,U_2\big) + g_1\big(\mathcal{A}_{X_1}h\widetilde{\nabla}_{U_1}X_2,U_2\big) + g_1(\eta\big(h\widetilde{\nabla}_{U_1}X_2\big)v\varphi(X_1),U_2)$$
$$+g_1(\widetilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_1),U_2) - g_1(\widehat{\nabla}_{U_1}\mathcal{A}_{X_1}X_2,U_2) + g_1(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_1),U_2) - g_1(\mathcal{T}_{U_1}h\widetilde{\nabla}_{X_1}X_2,U_2)$$
$$-g_1\big(\eta\big(h\widetilde{\nabla}_{X_1}X_2\big)v\varphi(U_1),U_2\big)-g_1\big(\widetilde{\nabla}_{U_1}\eta(X_2)v\varphi(X_1),U_2\big) - g_1\big(\mathcal{T}_{[X_1,U_1]}X_2,U_2\big)$$
$$-g_1(\eta(X_2)v\varphi([X_1,U_1]),U_2) + \frac{1}{(n-1)}\{g_1(U_1,U_2)[S'(X_1',X_2')\circ f - \sum_i \{g_1(v\widetilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_2,U_i)$$
$$-g_1\big(\mathcal{A}_{X_1}U_i,h\widetilde{\nabla}_{U_i}X_2\big) + g_1\big(\eta\big(h\widetilde{\nabla}_{U_i}X_2\big)v\varphi(X_1),U_i\big) + g_1\big(\widetilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_i),U_i\big) - g_1\big(\widehat{\nabla}_{U_i}\mathcal{A}_{X_1}X_2,U_i\big)$$
$$+g_1\big(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_i),U_i\big) - g_1(\eta\big(h\widetilde{\nabla}_{X_1}X_2\big)v\varphi(U_i),U_i) - g_1\big(\widetilde{\nabla}_{U_i}\eta(X_2)v\varphi(X_1),U_i\big)$$
$$-g_1\big(\mathcal{T}_{[X_1,U_i]}X_2,U_i\big)-g_1(\eta(X_2)v\varphi([X_1,U_i]),U_i) + \sum_j \{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_2}X_j) - g_1(\mathcal{A}_{X_1}X_2,\mathcal{A}_{X_j}X_j)$$

$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_2\right)-g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_2)+g_1(\widetilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_2)$$

$$-g_1(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_2),$$

$$g_1(\tilde{P}(U_1,U_2)U_3,U_4)=g_1(\hat{R}(U_1,U_2)U_3,U_4)-g_1(\eta(\widehat{\nabla}_{U_2}U_3)\varphi(U_1),U_4)\;+g_1(\mathcal{T}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,U_4)$$
$$+g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_3)v\varphi(U_1),U_4)-g_1(\widetilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),U_4)+g_1(\eta(\widehat{\nabla}_{U_1}U_3)\varphi(U_2),U_4)$$
$$-g_1(\mathcal{T}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,U_4)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_3)v\varphi(U_2),U_4)+g_1(\widetilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),U_4)$$
$$+g_1(\eta(U_3)\varphi([U_1,U_2]),U_4)-\frac{1}{n-1}\{g_1(U_1,U_4).[\hat{S}(U_2,U_3)+\sum\nolimits_i\{-g_1(\eta(\widehat{\nabla}_{U_i}U_i)\varphi(U_2),U_3)$$
$$-g_1(\tilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_3)+g_1(\eta(\widehat{\nabla}_{U_2}U_i)\varphi(U_i),U_3)+g_1(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_3)-g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_i)v\varphi(U_i),U_3)$$
$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_3)$$
$$+g_1(\eta(U_i)\varphi([U_2,U_i]),U_3)\}-\sum\nolimits_j\{g_1(v\widetilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_3)-g_1(\mathcal{A}_{X_j}U_3,h\widetilde{\nabla}_{U_2}X_j)$$
$$+g_1(\eta(h\widetilde{\nabla}_{U_2}X_j)v\varphi(X_j),U_3)+g_1\left(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_2),U_3\right)-g_1\left(\widehat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_3\right)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_3\right)+g_1\left(\mathcal{T}_{U_2}U_3,h\widetilde{\nabla}_{X_j}X_j\right)-g_1\left(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_3\right)$$
$$-g_1(\tilde{\nabla}_{U_2}\eta(X_j)v\varphi(X_j),U_3)-g_1\left(\mathcal{T}_{[X_j,U_2]}X_j,U_3\right)-g_1(\eta(X_j)v\varphi([X_j,U_2]),U_3)\}$$
$$-g_1(U_2,U_4)[\hat{S}(U_1,U_3)+\;\textstyle\sum_i\{-g_1(\eta(\widehat{\nabla}_{U_i}\;U_i)\varphi(U_1),U_3)$$
$$-g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_3)+g_1\left(\eta\left(\widehat{\nabla}_{U_1}\;U_i\right)\varphi(U_i),U_3\right)+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_3)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_3)$$
$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_3)+g_1(\eta(U_i)\varphi([U_1,U_i]),U_3)\}\text{-}\textstyle\sum_j\{g_1\left(v\widetilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_3\right)-g_1(\mathcal{A}_{X_j}U_3,h\widetilde{\nabla}_{U_1}X_j)$$
$$+g_1(\eta(h\tilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_3)+g_1\left(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_3\right)-g_1(\widehat{\nabla}_{U_1}\;\mathcal{A}_{X_j}X_j,U_3)$$
$$+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_3)+g_1(\mathcal{T}_{U_1}U_3,h\widetilde{\nabla}_{X_j}X_j)-g_1(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_3)$$
$$-g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_3)-g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_3\right)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_3)\}],$$

$$g_1(\tilde{P}(X_1,X_2)U_1,X_3)=g_1(h\widetilde{\nabla}_{X_1}\mathcal{A}_{X_2}U_1,X_3)+g_1(\mathcal{A}_{X_1}v\widetilde{\nabla}_{X_2}U_1,X_3)+g_1(\eta(v\widetilde{\nabla}_{X_2}U_1)h\varphi(X_1),X_3)$$
$$+g_1(\widetilde{\nabla}_{X_1}\eta(U_1)h\varphi(X_2),X_3)-g_1(h\widetilde{\nabla}_{X_2}\mathcal{A}_{X_1}U_1,X_3)-g_1(\mathcal{A}_{X_2}v\widetilde{\nabla}_{X_1}U_1,X_3)$$
$$-g_1(\eta(v\widetilde{\nabla}_{X_1}U_1)h\varphi(X_2),X_3)-g_1(\widetilde{\nabla}_{X_2}\eta(U_1)h\varphi(X_1),X_3)-g_1(\mathcal{A}_{[X_1,X_2]}U_1,X_3)$$
$$-g_1(\eta(U_1)h\varphi([X_1,X_2]),X_3)-\frac{1}{n-1}\{g_1(X_1,X_3)[\sum\nolimits_i\{g_1(v\widetilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1)-g_1(\mathcal{A}_{X_2}U_1,h\widetilde{\nabla}_{X_i}X_i)$$
$$+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_2),U_1)+g_1(\widetilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1)-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\widetilde{\nabla}_{X_2}X_i)$$

$$-g_1\big(\eta\big(h\tilde{\nabla}_{X_2}X_i\big)v\varphi(X_i),U_1\big)-g_1\big(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1\big)-g_1(\eta(X_i)v\varphi([X_2,X_i],U_1)\}$$
$$\sum_j\{g_1(v\tilde{\nabla}_{X_2}\hat{\nabla}_{U_j}U_j,U_1)-g_1(\tilde{\nabla}_{X_2}\eta(U_j)\varphi(U_j),U_1)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j)$$
$$-g_1\big(\eta\big(\mathcal{A}_{X_2}U_j\big)v\varphi\big(U_j\big),U_1\big)-g_1\left(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_2}U_j,U_1\right)+g_1(\eta\big(v\tilde{\nabla}_{X_2}U_j\big)\varphi\big(U_j\big),U_1)$$
$$-g_1\left(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_2),U_1\right)-g_1\left(\hat{\nabla}_{[X_2,U_j]}U_j,U_1\right)+g_1\big(\eta\big(U_j\big)\varphi\big([X_2,U_j]\big),U_1\big)$$
$$-g_1(X_2,X_3)[\textstyle\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)-g_1\big(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i\big)+g_1\big(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_1),U_1\big)$$
$$+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i)-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_i\big)v\varphi(X_i),U_1)$$
$$-g_1(\tilde{\nabla}_{X_i}\ \eta(X_i)v\varphi(X_1),U_1)-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)\}+\sum_j\{g_1\left(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1\right)$$
$$-g_1(\tilde{\nabla}_{X_1}\eta\big(U_j\big)\varphi\big(U_j\big),U_1)+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j\right)-g_1\big(\eta\big(\mathcal{A}_{X_1}U_j\big)v\varphi\big(U_j\big),U_1\big)-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1)$$
$$+g_1(\eta\big(v\tilde{\nabla}_{X_1}U_j\big)\varphi\big(U_j\big),U_1)-g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_1),U_1)-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)$$
$$+g_1(\eta\big(U_j\big)\varphi\big([X_1,U_j],U_1\big)\}],$$
where $U_1,U_2,U_3,U_4\in\varkappa^v(M_1)$ and, $X_1,X_2,X_3,X_4\in\varkappa^h(M_1)$.

3.2 Concircular curvature tensor

In Riemannian submersions endowed with Q-SNMC, we present the curvature relations of the concircular curvature tensor here.

Definition 3.4 Let M_1 be an $n-$dimensional manifold. In this case, within the $n-$dimensional space V^n, for every $X_1,X_2,X_3\in\chi(M_1)$, the concircular curvature tensor field of M_1 is defined as follows:

$$\tilde{C}(X_1,X_2)X_3=\tilde{R}(X_1,X_2)X_3-\frac{\tilde{\tau}}{n(n-1)}\{g_1(X_2,X_3)X_1-g_1(X_1,X_3)X_2\},\qquad(3.8)$$

where $\tilde{\tau}$ represents a scalar tensor [23].

Theorem 3.5 Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, with, $f:(M_1,g_1)\to(M_2,g_2)$ representing a Riemannian submersion endowed with Q-SNMC, and $\tilde{R},R'$, and $\hat{R}$ denote the Riemannian curvature tensors, $\tilde{\tau},\tau'$, and $\hat{\tau}$ represent the scalar curvature tensors of M_1, M_2, and the fiber $(f^{-1}(x),\hat{g}_{1x})$ respectively. Then, for every $U_1,U_2,U_3,U_4\in\varkappa^v(M_1)$ and $X_1,X_2,X_3,X_4\in\varkappa^h(M_1)$, the following concircular curvature tensor relations hold:

$$g_1\big(\tilde{C}(X_1,X_2)X_3,X_4\big)=g_1(R'(X_1,X_2)X_3,X_4)-g_1\big(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_2}X_3\big)+g_1(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_1}X_3)$$
$$+\eta(\mathcal{A}_{X_2}X_3)g_1(h\varphi(X_1),X_4)-\eta(\mathcal{A}_{X_1}X_3)g_1(h\varphi(X_2),X_4)+g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),X_4)$$
$$-g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),X_4)-\frac{\tilde{\tau}}{n(n-1)}[g_1(X_1,X_4)g_1(X_2,X_3)-g_1(X_2,X_4)g_1(X_1,X_3)],$$

$$g_1(\tilde{C}(X_1,X_2)X_3,U_1)=g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}X_3,U_1)+g_1(\mathcal{A}_{X_1}h\tilde{\nabla}_{X_2}X_3,U_1)+g_1(\eta\big(h\tilde{\nabla}_{X_2}X_3\big)v\varphi(X_1),U_1)$$

$$+g_1(\widetilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),U_1) - g_1(v\widetilde{\nabla}_{X_2}\mathcal{A}_{X_1}X_3,U_1) - g_1(\mathcal{A}_{X_2}h\widetilde{\nabla}_{X_1}X_3,U_1)$$
$$-g_1(\eta\big(h\widetilde{\nabla}_{X_1}X_3\big)v\varphi(X_2),U_1) - g_1(\widetilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),U_1) - g_1(\eta(X_3)v\varphi([X_1,X_2]),U_1),$$

$$g_1\big(\tilde{C}(X_1,U_1)X_2,U_2\big) = g_1\big(v\widetilde{\nabla}_{X_1}\mathcal{T}_{U_1}X_2,U_2\big) + g_1\big(\mathcal{A}_{X_1}h\widetilde{\nabla}_{U_1}X_2,U_2\big) + g_1(\eta\big(h\widetilde{\nabla}_{U_1}X_2\big)v\varphi(X_1),U_2)$$
$$+g_1\big(\widetilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_1),U_2\big) - g_1(\widehat{\nabla}_{U_1}\mathcal{A}_{X_1}X_2,U_2) + g_1\big(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_1),U_2\big)$$
$$-g_1(\mathcal{T}_{U_1}h\widetilde{\nabla}_{X_1}X_2,U_2) - g_1(\eta(h\widetilde{\nabla}_{X_1}X_2)v\varphi(U_1),U_2) - g_1(\widetilde{\nabla}_{U_1}\eta(X_2)v\varphi(X_1),U_2) - g_1(\mathcal{T}_{[X_1,U_1]}X_2,U_2)$$
$$-g_1(\eta(X_2)v\varphi([X_1,U_1]),U_2) + \frac{\tilde{\tau}}{n(n-1)}g_1(X_1,X_2)g_1(U_1,U_2),$$

$$g_1(\tilde{C}(U_1,U_2)U_3,X_1) = g_1(\tilde{\mathcal{T}}_{U_1}\widehat{\nabla}_{U_2}U_3,X_1) - g_1(\eta(\widehat{\nabla}_{U_2}U_3)\varphi(U_1),X_1) + g_1(h(\widetilde{\nabla}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,X_1)$$
$$-g_1\big(\widetilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),X_1\big) - g_1\big(\tilde{\mathcal{T}}_{U_2}\widehat{\nabla}_{U_1}U_3,X_1\big) + g_1(\eta(\widehat{\nabla}_{U_1}U_3)\varphi(U_2),X_1) - g_1(h(\widetilde{\nabla}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,X_1)$$
$$+g_1(\widetilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),X_1) - g_1(\tilde{\mathcal{T}}_{[U_1,U_2]}U_3,X_1) + g_1(\eta(U_3)\varphi([U_1,U_2]),X_1),$$

$$g_1\big(\tilde{C}(U_1,U_2)U_3,U_4\big) = g_1\big(\hat{R}(U_1,U_2)U_3,U_4\big) - g_1(\eta\big(\widehat{\nabla}_{U_2}U_3\big)\varphi(U_1),U_4) + g_1(\mathcal{T}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,U_4)$$
$$+g_1\big(\eta\big(\tilde{\mathcal{T}}_{U_2}U_3\big)v\varphi(U_1),U_4\big) - g_1\big(\widetilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),U_4\big) + g_1(\eta\big(\widehat{\nabla}_{U_1}U_3\big)\varphi(U_2),U_4)$$
$$-g_1\big(\mathcal{T}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,U_4\big) - g_1\big(\eta\big(\tilde{\mathcal{T}}_{U_1}U_3\big)v\varphi(U_2),U_4\big) + g_1(\widetilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),U_4)$$
$$+g_1(\eta(U_3)\varphi([U_1,U_2],U_4) - \frac{\tilde{\tau}}{n(n-1)}\{g_1(U_2,U_3)g_1(U_1,U_4) - g_1(U_1,U_3)g_1(U_2,U_4)\},$$

$$g_1(\tilde{C}(X_1,X_2)U_1,X_3) = g_1(h(\widetilde{\nabla}_{X_1}\mathcal{A}_{X_2}U_1,X_3) + g_1(\mathcal{A}_{X_1}v\widetilde{\nabla}_{X_2}U_1,X_3) + \eta(v\widetilde{\nabla}_{X_2}U_1)g_1(h\varphi(X_1),X_3)$$
$$+g_1(\widetilde{\nabla}_{X_1}\eta(U_1)h\varphi(X_2),X_3) - g_1(h\big(\widetilde{\nabla}_{X_2}\mathcal{A}_{X_1}U_1,X_3\big) - g_1\big(\mathcal{A}_{X_2}v\widetilde{\nabla}_{X_1}U_1,X_3\big)$$
$$-\eta(v\widetilde{\nabla}_{X_1}U_1)g_1(h\varphi(X_2),X_3) - g_1(\widetilde{\nabla}_{X_2}\eta(U_1)h\varphi(X_1),X_3) - g_1(\mathcal{A}_{[X_1,X_2]}U_1,X_3)$$
$$- g_1(\eta(U_1)h\varphi([X_1,U_1]),X_3),$$

where $U_1,U_2,U_3,U_4 \in \varkappa^v(M_1)$ and $X_1,X_2,X_3,X_4 \in \varkappa^h(M_1)$.

Proof. The proof of the initial equation in Theorem 3.5 can be readily demonstrated by taking an inner product between $\tilde{C}$ and X_4 and applying (2.29) from (3.8). Similarly, analogous proofs can be conducted for other equations.

Corollary 3.6 Let (M_1,g_1) and (M_2,g_2) be a Riemannian manifold, and $f\colon (M_1,g_1) \to (M_2,g_2)$ be Riemannian submersion. Then the concircular curvature tensor of the Riemannian submersion endowed with Q-SNMC has no totally umbilical fibres.

3.3 Conharmonic curvature tensor

In this section, we present curvature relations of the Conharmonic curvature tensor in Riemannian submersions endowed with Q-SNMC.

Definition 3.7 Let M_1 be an $n-$dimensional manifold. In this case, within the $n-$dimensional space V^n, for every $X_1,X_2,X_3 \in \chi(M_1)$, the conharmonic curvature tensor field of M_1 is defined as follows:

$$\tilde{L}(X_1,X_2)X_3 = \tilde{R}(X_1,X_2)X_3 - \frac{1}{(n-2)}\{g_1(X_2,X_3)QX_1 - g_1(X_1,X_3)QX_2 + \tilde{S}(X_2,X_3)X_1 - \tilde{S}(X_1,X_3)X_2\}, \tag{3.9}$$

where $\tilde{Q}$, $\tilde{R}$ and $\tilde{S}$ denote the Ricci operator, Riemannian curvature tensor and Ricci curvature tensor, respectively [23].

Theorem 3.8 Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, with, $f:(M_1,g_1) \to (M_2,g_2)$ representing a Riemannian submersion endowed with Q-SNMC, and $\tilde{R}$, R' and $\hat{R}$ be Riemannian curvature tensors, $\tilde{S}$, S' and $\hat{S}$ be Ricci tensors of M_1 , M_2 and the fiber $(f^{-1}(x),\hat{g}_{1x})$ respectively. Then, for every $U_1,U_2,U_3,U_4 \in \varkappa^v(M_1)$ and $X_1,X_2,X_3,X_4 \in \varkappa^h(M_1)$, the following Conharmonic curvature tensor relations hold:

$$\begin{aligned}
g_1\big(\tilde{L}(X_1,X_2)X_3,X_4\big) &= g_1(R'(X_1,X_2)X_3,X_4) - g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_2}X_3) + g_1(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_1}X_3)\\
&+g_1\big(\eta\big(\mathcal{A}_{X_2}X_3\big)h\varphi(X_1),X_4\big) - g_1\big(\eta\big(\mathcal{A}_{X_1}X_3\big)h\varphi(X_2),X_4\big) + g_1\big(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),X_4\big)\\
&-g_1\big(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),X_4\big) - \frac{1}{(n-2)}\{g_1(X_2,X_3)[S'(X_1',X_4')\circ f + g_1\big(\mathcal{N},h\tilde{\nabla}_{X_1}X_4\big)\\
&- \textstyle\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_4,U_i) - g_1(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_4) + g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_4\big)v\varphi(X_1),U_i\big)\\
&+g_1\big(\tilde{\nabla}_{X_1}\eta(X_4)v\varphi(U_i),U_i\big) - g_1\big(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_4,U_i\big) + g_1\big(\eta\big(\mathcal{A}_{X_1}X_4\big)\varphi(U_i),U_i\big)\\
&-g_1\big(\eta\big(h\tilde{\nabla}_{X_1}X_4\big)v\varphi(U_i),U_i\big) - g_1\big(\tilde{\nabla}_{U_i}\ \eta(X_4)v\varphi(X_1),U_i\big) - g_1(\mathcal{T}_{[X_1,U_i]}X_4,U_i)\\
&-g_1(\eta(X_4)v\varphi([X_1,U_i]),U_i)\} + \sum_j\{-g_1\big(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_4}X_j\big) - g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_j}X_j)\\
&+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_4\right) - g_1(\eta\big(\mathcal{A}_{X_1}X_j\big)h\varphi\big(X_j\big),X_4) + g_1(\tilde{\nabla}_{X_1}\eta\big(X_j\big)v\varphi\big(X_j\big),X_4)\\
&-g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_1),X_4)\} - g_1(X_1,X_3)[S'(X_2',X_4')\circ f + g_1(\mathcal{N},h\tilde{\nabla}_{X_2}X_4)\}\\
&- \sum_i\{g_1(v\tilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_4,U_i) - g_1(\mathcal{A}_{X_2}U_i,h\tilde{\nabla}_{U_i}X_4) + g_1(\eta\big(h\tilde{\nabla}_{U_i}X_4\big)v\varphi(X_2),U_i)\\
&+g_1\big(\tilde{\nabla}_{X_2}\eta(X_4)v\varphi(U_i),U_i\big) - g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_2}X_4,U_i) + g_1\big(\eta\big(\mathcal{A}_{X_2}X_4\big)\varphi(U_i),U_i\big)\\
&-g_1(\eta(h\tilde{\nabla}_{X_2}X_4)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_4)v\varphi(X_2),U_i) - g_1(\mathcal{T}_{[X_2,U_j]}X_4,U_i)\\
&-g_1(\eta(X_4)v\varphi([X_2,U_i]),U_i)\} + \sum_j\{-g_1(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_4}X_j) - g_1(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_j}X_j)\\
&+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_4) - g_1(\eta\big(\mathcal{A}_{X_2}X_j\big)h\varphi\big(X_j\big),X_4) + g_1(\tilde{\nabla}_{X_2}\eta\big(X_j\big)v\varphi\big(X_j\big),X_4)\\
&-g_1(\tilde{\nabla}_{X_j}\ \eta\big(X_j\big)v\varphi(X_2),X_4)\} + g_1(X_1,X_4)[S'(X_2',X_3')\circ f + g_1(\mathcal{N},h\tilde{\nabla}_{X_2}X_3)\\
&- \sum_i\{g_1(v\tilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_3,U_i) - g_1(\mathcal{A}_{X_2}U_i,h\tilde{\nabla}_{U_i}X_3) + g_1(\eta\big(h\tilde{\nabla}_{U_i}X_3\big)v\varphi(X_2),U_i)\\
&+g_1\big(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(U_i),U_i\big) - g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_2}X_3,U_i) + g_1\big(\eta\big(\mathcal{A}_{X_2}X_3\big)\varphi(U_i),U_i\big)\\
&-g_1(\eta(h\tilde{\nabla}_{X_2}X_3)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_2),U_i) - g_1(\mathcal{T}_{[X_2,U_j]}X_3,U_j)
\end{aligned}$$

$$-g_1(\eta(X_3)v\varphi([X_2,U_i]),U_i)\}+\sum_j\{-g_1(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_3}X_j)-g_1(\mathcal{A}_{X_2}X_3,\mathcal{A}_{X_j}X_j)$$

$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_3\right)-g_1(\eta(\mathcal{A}_{X_2}X_j)h\varphi(X_j),X_3)+g_1(\widetilde{\nabla}_{X_2}\eta(X_j)v\varphi(X_j),X_3)$$

$$-g_1\left(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_2),X_3\right)-g_1(X_2,X_4)[S'(X_1',X_3')\circ f+$$

$$g_1(\mathcal{N},h\widetilde{\nabla}_{X_1}X_3)\ -\ \sum_i\{g_1(v\widetilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_3,U_i)$$
$$-g_1(\mathcal{A}_{X_1}U_i,h\widetilde{\nabla}_{U_i}X_3)+g_1(\eta(h\widetilde{\nabla}_{U_i}X_3)v\varphi(X_1),U_i)+g_1(\widetilde{\nabla}_{X_1}\eta(X_3)v\varphi(U_i),U_i)$$
$$-g_1(\widetilde{\nabla}_{U_i}\mathcal{A}_{X_1}X_3,U_i)+g_1(\eta(\mathcal{A}_{X_1}X_3)\varphi(U_i),U_i)-g_1(\eta(h\widetilde{\nabla}_{X_1}X_3)v\varphi(U_i),U_i)$$
$$-g_1(\widetilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_1),U_i)-g_1(\mathcal{T}_{[X_1,U_i]}X_3,U_i)-g_1(\eta(X_3)v\varphi([X_1,U_i]),U_i)$$

$$+\ \sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_3}X_j)-g_1(\mathcal{A}_{X_1}X_3,\mathcal{A}_{X_j}X_j)+\ g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_3)$$

$$-g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_3)+g_1(\widetilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_3)-g_1(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_3)\}]\},$$

$$g_1(\tilde{L}(X_1,X_2)X_3,U_1)=g_1(v\widetilde{\nabla}_{X_1}\mathcal{A}_{X_2}X_3,U_1)+g_1(\mathcal{A}_{X_1}h\widetilde{\nabla}_{X_2}X_3,U_1)+g_1(\eta(h\widetilde{\nabla}_{X_2}X_3)v\varphi(X_1),U_1)$$
$$+g_1(\widetilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),U_1)-g_1(v\widetilde{\nabla}_{X_2}\mathcal{A}_{X_1}X_3,U_1)-g_1(\mathcal{A}_{X_2}h\widetilde{\nabla}_{X_1}X_3,U_1)$$
$$-g_1(\eta(h\widetilde{\nabla}_{X_1}X_3)v\varphi(X_2),U_1)-g_1(\widetilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),U_1)-g_1(\eta(X_3)v\varphi([X_1,X_2]),U_1)$$
$$-\frac{1}{(n-2)}\{g_1(X_2,X_3)[\sum_i\{g_1(v\widetilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)-g_1(\mathcal{A}_{X_1}U_1,h\widetilde{\nabla}_{X_i}X_i)+g_1(\eta(h\widetilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_1)$$
$$+g_1(\widetilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)-g_1(v\widetilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\widetilde{\nabla}_{X_1}X_i)-g_1(\eta(h\widetilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_1)$$
$$-g_1(\widetilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)\}+\sum_j\{g_1(v\widetilde{\nabla}_{X_1}\widehat{\nabla}_{U_j}U_j,U_1)$$

$$-g_1(\widetilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j)-g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_1)-g_1(\widehat{\nabla}_{U_j}v\widetilde{\nabla}_{X_1}U_j,U_1)$$

$$+g_1(\eta(v\widetilde{\nabla}_{X_1}U_j)\varphi(U_j),U_1)-g_1(\widetilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1)-g_1(\widehat{\nabla}_{[X_1,U_j]}U_j,U_1)$$
$$+g_1(\eta(U_j)\varphi([X_1,U_j],U_1)-g_1(\widetilde{\mathcal{N}},\mathcal{A}_{X_1}U_j)+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(X_1),U_1)\}$$
$$-g_1(X_1,X_3)[\sum_i\{g_1(v\widetilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1)-g_1(\mathcal{A}_{X_2}U_1,h\widetilde{\nabla}_{X_i}X_i)+g_1(\eta(h\widetilde{\nabla}_{X_i}X_i)v\varphi(X_2),U_1)$$
$$+g_1(\widetilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1)-g_1(v\widetilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\widetilde{\nabla}_{X_2}X_i)-g_1(\eta(h\widetilde{\nabla}_{X_2}X_i)v\varphi(X_i),U_1)$$
$$-g_1(\widetilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1)-g_1(\eta(X_i)v\varphi([X_2,X_i],U_1)\}+\sum_j\{g_1(v\widetilde{\nabla}_{X_2}\widehat{\nabla}_{U_j}U_j,U_1)$$

$$-g_1(\widetilde{\nabla}_{X_2}\eta(U_j)\varphi(U_j),U_1)+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j\right)-g_1(\eta(\mathcal{A}_{X_2}U_j)v\varphi(U_j),U_1)$$

$$-g_1\left(\widehat{\nabla}_{U_j}v\widetilde{\nabla}_{X_2}U_j,U_1\right)+g_1(\eta(v\widetilde{\nabla}_{X_2}U_j)\varphi(U_j),U_1)-g_1(\widetilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_2),U_1)$$

$$-g_1\left(\widehat{\nabla}_{[X_2,U_j]}U_j,U_1\right)+g_1(\eta(U_j)\ \ \varphi([X_2,U_j],U_1)-g_1(\widetilde{\mathcal{N}},\mathcal{A}_{X_2}U_1)+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(X_2),U_1),$$

$$g_1\big(\tilde{L}(X_1,U_1)X_2,U_2\big)=g_1\big(v\widetilde{\nabla}_{X_1}\mathcal{T}_{U_1}X_2,U_2\big)+g_1(\mathcal{A}_{X_1}h\widetilde{\nabla}_{U_1}X_2,U_2)+g_1(\eta(h\widetilde{\nabla}_{U_1}X_2)v\varphi(X_1),U_2)$$
$$+g_1\big(\widetilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_1),U_2\big)-g_1\big(\widehat{\nabla}_{U_1}\mathcal{A}_{X_1}X_2,U_2\big)+g_1\big(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_1),U_2\big)-g_1\big(\mathcal{T}_{U_1}h\widetilde{\nabla}_{X_1}X_2,U_2\big)$$
$$-g_1\big(\eta\big(h\widetilde{\nabla}_{X_1}X_2\big)v\varphi(U_1),U_2\big)-g_1(\widetilde{\nabla}_{U_1}\eta(X_2)v\varphi(X_1),U_2)-g_1(\mathcal{T}_{[X_1,U_1]}X_2,U_2)$$
$$-g_1(\eta(X_2)v\varphi([X_1,U_1]),U_2)+\frac{1}{(n-2)}\{g_1(X_1,X_2)[\hat{S}(U_1,U_2)-g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_1}U_2)$$
$$+g_1\big(\eta(\widetilde{\mathcal{N}})v\varphi(U_1),U_2\big)+\textstyle\sum_i\{-g_1(\eta(\widehat{\nabla}_{U_i}U_i)\varphi(U_1),U_2)-g_1(\widetilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_2)$$
$$+g_1\big(\eta\big(\widehat{\nabla}_{U_1}U_i\big)\varphi(U_i),U_2\big)+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_2)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_2)+g_1(\widetilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_2)$$
$$+g_1(\eta(U_i)\varphi([U_1,U_i]),U_2)-\sum_j\{g_1(v\widetilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_2)-g_1(\mathcal{A}_{X_j}U_2,h\widetilde{\nabla}_{U_1}X_j)$$
$$+g_1\big(\eta\big(h\widetilde{\nabla}_{U_1}X_j\big)v\varphi(X_j),U_2\big)+g_1\left(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_2\right)-g_1\left(\widehat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_2\right)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_2\right)+g_1\left(\mathcal{T}_{U_1}U_2,h\widetilde{\nabla}_{X_j}X_j\right)-g_1\left(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_2\right)$$
$$-g_1(\widetilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_2)-g_1(\mathcal{T}_{[X_j,U_1]}X_j,U_2)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_2)\}]$$
$$+g_1(U_1,U_2)[S'(X_1',X_2')\circ f+g_1\big(\mathcal{N},h\widetilde{\nabla}_{X_1}X_2\big)-\sum_i\{g_1(v\,\widetilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_2,U_i)-g_1(\mathcal{A}_{X_1}U_i,h\widetilde{\nabla}_{U_i}X_2)$$
$$+g_1(\eta\big(h\widetilde{\nabla}_{U_i}X_2\big)v\varphi(X_1),U_i)+g_1(\widetilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_i),U_i)-g_1(\widehat{\nabla}_{U_i}\mathcal{A}_{X_1}X_2,U_i)$$
$$+g_1(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_i),U_i)-g_1\big(\eta\big(h\widetilde{\nabla}_{X_1}X_2\big)v\varphi(U_i),U_i\big)$$
$$-g_1\big(\widetilde{\nabla}_{U_i}\eta(X_2)v\varphi(X_1),U_i\big)-g_1\big(\mathcal{T}_{[X_1,U_1]}X_2,U_i\big)-g_1(\eta(X_2)v\varphi([X_1,U_i]),U_i)$$
$$+\sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_2}X_j)-g_1\left(\mathcal{A}_{X_1}X_2,\mathcal{A}_{X_j}X_j\right)+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_2)$$
$$-g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_2)+\ g_1\big(\widetilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_2\big)-g_1(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_2)\}]\},$$

$$g_1\big(\tilde{L}(U_1,U_2)U_3,U_4\big)=g_1\big(\hat{R}(U_1,U_2)U_3,U_4\big)-g_1\big(\eta\big(\widehat{\nabla}_{U_2}U_3\big)\varphi(U_1),U_4\big)+g_1(\mathcal{T}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,U_4)$$
$$+g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_3)v\varphi(U_1),U_4)\ \ -g_1(\widetilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),U_4)\ \ +g_1(\eta(\widehat{\nabla}_{U_1}U_3)\varphi(U_2),U_4)$$
$$-g_1(\mathcal{T}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,U_4)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_3)v\varphi(U_2),U_4)+g_1(\widetilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),U_4)$$
$$+g_1(\eta(U_3)\varphi([U_1,U_2]),U_4)\ \ -\frac{1}{(n-2)}\{g_1(U_2,U_3)[\hat{S}(U_1,U_4)-g_1(\widetilde{\mathcal{N}},\tilde{\mathcal{T}}_{U_1}U_4)$$
$$+g_1\big(\eta(\widetilde{\mathcal{N}})v\varphi(U_1),U_4\big)+\sum_i\{-g_1\big(\eta\big(\widehat{\nabla}_{U_i}U_i\big)\varphi(U_1),U_4\big)-g_1\big(\widetilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_4\big)$$
$$+g_1(\eta\big(\widehat{\nabla}_{U_1}U_i\big)\varphi(U_i),U_4)+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_4)-g_1(\eta\big(\tilde{\mathcal{T}}_{U_1}U_i\big)v\varphi(U_i),U_4)$$
$$+g_1\big(\widetilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_4\big)+g_1(\eta(U_i)\varphi([U_1,U_i]),U_4)\}-\textstyle\sum_j\{\,g_1(v\widetilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_4)$$
$$-g_1(\mathcal{A}_{X_j}U_4,h\widetilde{\nabla}_{U_1}X_j)+g_1(\eta\big(h\widetilde{\nabla}_{U_1}X_j\big)v\varphi(X_j),U_4)+g_1(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_4)-g_1(\widehat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_4)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_4\right)+g_1(\mathcal{T}_{U_1}U_4,h\widetilde{\nabla}_{X_j}X_j)-g_1(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_4)$$
$$-g_1\big(\widetilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_4\big)-g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_4\right)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_4)\}]$$

$$-g_1(U_1,U_3)[\hat{S}(U_2,U_4)-g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_2}U_4)+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_2),U_4)+\sum_i\{-g_1(\eta(\widehat{\nabla}_{U_i}U_i)\varphi(U_2),U_4)$$
$$-g_1(\widetilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_4)+g_1(\eta(\widehat{\nabla}_{U_2}U_i)\varphi(U_i),U_4)+g_1(\tilde{\mathcal{T}}_{U_2}U_i,\ \mathcal{T}_{U_i}U_4)$$
$$-g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_j)v\varphi(U_j),U_4)+g_1(\widetilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_4)+g_1(\eta(U_i)\varphi([U_2,U_i]),U_4)$$
$$-\sum_j\{g_1\left(v\widetilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_4\right)-g_1\left(\mathcal{A}_{X_j}U_4,h\widetilde{\nabla}_{U_2}X_j\right)+g_1(\eta(h\widetilde{\nabla}_{U_2}X_j)v\varphi(X_j),U_4)$$
$$+g_1(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_2),U_4)-g_1(\widehat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_4)+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_4)$$
$$+g_1\left(\mathcal{T}_{U_2}U_4,h\widetilde{\nabla}_{X_j}X_j\right)-g_1(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_4)-g_1(\widetilde{\nabla}_{U_2}\eta(X_j)v\varphi(X_j),U_4)$$
$$-g_1(\mathcal{T}_{[X_j,U_2]}X_j,U_4)-g_1(\eta(X_j)v\varphi([X_j,U_2]),U_4)\}]+g_1(U_1,U_4)[\hat{S}(U_2,U_3)-g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_2}U_3)$$
$$+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_2),U_3)+\sum_i\{-g_1(\eta(\widehat{\nabla}_{U_i}U_i)\varphi(U_2),U_3)-g_1(\widetilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_3)$$
$$+g_1(\eta(\widehat{\nabla}_{U_2}U_i)\varphi(U_i),U_3)+g_1(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_3)-g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_i)v\varphi(U_i),U_3)$$
$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_3)+g_1(\eta(U_i)\varphi([U_2,U_i]),U_3)-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_3)$$
$$-g_1(\mathcal{A}_{X_j}U_3,h\widetilde{\nabla}_{U_2}X_j)+g_1(\eta(h\widetilde{\nabla}_{U_2}X_j)v\varphi(X_j),U_3)+g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_2),U_3)$$
$$-g_1(\widehat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_3)+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_3\right)+g_1(\mathcal{T}_{U_2}U_3,h\widetilde{\nabla}_{X_j}X_j)$$
$$-g_1(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_3)-g_1(\widetilde{\nabla}_{U_2}\eta(X_j)v\varphi(X_j),U_3)-g_1(\mathcal{T}_{[X_j,U_2]}X_j,U_3)$$
$$-g_1(\eta(X_j)v\varphi([X_j,U_2]),U_3)\}-g_1(U_2,U_4)[\ \hat{S}(U_1,U_3)-g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_1}U_3)+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_1),U_3)$$
$$\sum_i\{-g_1(\eta(\widehat{\nabla}_{U_i}U_i)\varphi(U_1),U_3)-g_1(\widetilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_3)+g_1(\eta(\widehat{\nabla}_{U_1}U_i)\varphi(U_i),U_3)$$
$$+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_3)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_3)+g_1(\widetilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_3)$$
$$+g_1(\eta(U_i)\varphi([U_1,U_i]),U_3)-\sum_j\{g_1(v\widetilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_3)-g_1(\mathcal{A}_{X_j}U_3,h\widetilde{\nabla}_{U_1}X_j)$$
$$+g_1(\eta(h\widetilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_3)+g_1\left(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_3\right)-g_1\left(\widehat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_3\right)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_3\right)+g_1\left(\mathcal{T}_{U_1}U_3,h\widetilde{\nabla}_{X_j}X_j\right)-g_1\left(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_3\right)$$
$$-g_1(\widetilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_3)-g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_3\right)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_3)\}]\},$$

$$g_1(\tilde{L}(X_1,X_2)U_1,X_3)=g_1(h(\widetilde{\nabla}_{X_1}\mathcal{A}_{X_2}U_1,X_3)+g_1(\mathcal{A}_{X_1}v\widetilde{\nabla}_{X_2}U_1,X_3)+g_1(\eta(v\widetilde{\nabla}_{X_2}U_1)h\varphi(X_1),X_3)$$
$$+g_1(\widetilde{\nabla}_{X_1}\eta(U_1)h\varphi(X_2),X_3)-g_1(h\widetilde{\nabla}_{X_2}\mathcal{A}_{X_1}U_1,X_3)-g_1(\mathcal{A}_{X_2}v\widetilde{\nabla}_{X_1}U_1,X_3)$$
$$-g_1(\eta(v\widetilde{\nabla}_{X_1}U_1)h\varphi(X_2),X_3)-g_1(\widetilde{\nabla}_{X_2}\eta(U_1)h\varphi(X_1),X_3)-g_1(\mathcal{A}_{[X_1,X_2]}U_1,X_3)$$

$$-g_1(\eta(U_1)h\varphi([X_1,X_2]),X_3) - \frac{1}{(n-2)}\{g_1(X_1,X_3)[\sum_i \{g_1(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1)$$
$$-g_1(\mathcal{A}_{X_2}U_1,h\tilde{\nabla}_{X_i}X_i) + g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_2),U_i) + g_1(\tilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1)$$
$$-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1) + g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_2}X_i) - g_1(\eta(h\tilde{\nabla}_{X_2}X_i)v\varphi(X_i),U_1) - g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_i),U_1)$$
$$-g_1(\eta(X_i)v\varphi([X_2,X_i],U_1)\} + \sum_j \{g_1\left(v\tilde{\nabla}_{X_2}\hat{\nabla}_{U_j}U_j,U_1\right) - g_1(\tilde{\nabla}_{X_2}\eta(U_j)\varphi(U_j),U_1)$$
$$+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j\right) - g_1(\eta(\mathcal{A}_{X_2}U_j)v\varphi(U_j),U_1) - g_1\left(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_2}U_j,U_1\right) + g_1(\eta(v\tilde{\nabla}_{X_2}U_j)\varphi(U_j),U_1)$$
$$-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_2),U_1) - g_1(\hat{\nabla}_{[X_2,U_j]}U_j,U_1) + g_1(\eta(U_j)\varphi([X_2,U_j]),U_1)\}$$
$$-g_1(\tilde{\mathcal{N}},\mathcal{A}_{X_2}U_1) + g_1(\eta(\tilde{\mathcal{N}})v\varphi(X_2),U_1)] - g_1(X_2,X_3)[\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)$$
$$-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i) + g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_1) + g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)$$
$$-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)$$
$$+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i) - g_1(\eta(h\tilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_1) - g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)$$
$$-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)\} + \sum_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1) - g_1(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1)$$
$$+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j\right) - g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_1) - g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1) + g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi(U_j),U_1)$$
$$-g_1\left(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1\right) - g_1\left(\hat{\nabla}_{[X_1,U_j]}U_j,U_1\right) + g_1(\eta(U_j)\varphi([X_1,U_j]),U_1)\}$$
$$-g_1(\tilde{\mathcal{N}},\mathcal{A}_{X_1}U_1) + g_1(\eta(\tilde{\mathcal{N}})v\varphi(X_1),U_1)].$$

Proof. We can easily demonstrate the first equation by using (2.29) and (2.35) in equation (3.9) for the inner product of $\tilde{L}$ and X_4. The proofs of other equations can be done similarly.

Corollary 3.9 Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, with, $f:(M_1,g_1)\to(M_2,g_2)$ representing a Riemannian submersion endowed with Q-SNMC, and $\tilde{R}$,R' and $\hat{R}$ be Riemannian curvature tensors, $\tilde{S}$,S' and $\hat{S}$ be Ricci tensors of M_1, M_2 and the fiber $(f^{-1}(x),\hat{g}_{1x})$ respectively. If the Riemannian submersion has total umblical fibres $(\mathcal{N}=0)$, then the conharmonic curvature tensor is given by

$$g_1(\tilde{L}(X_1,X_2)X_3,X_4) = g_1(R'(X_1,X_2)X_3,X_4) + g_1(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_1}X_3) - g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_2}X_3)$$
$$+g_1(\eta(\mathcal{A}_{X_2}X_3)h\varphi(X_1),X_4) - g_1(\eta(\mathcal{A}_{X_1}X_3)h\varphi(X_2),X_4) + g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),X_4)$$
$$-g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),X_4) - \frac{1}{(n-2)}\{g_1(X_2,X_3)[S'(X_1',X_4')\circ f - \sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_4,U_i)$$
$$-g_1(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_4) + g_1(\eta(h\tilde{\nabla}_{U_i}X_4)v\varphi(X_1),U_i) + g_1(\tilde{\nabla}_{X_1}\eta(X_4)v\varphi(U_i),U_i) -$$
$$g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_4,U_i)$$
$$+g_1(\eta(\mathcal{A}_{X_1}X_4)\varphi(U_i),U_i) - g_1(\eta(h\tilde{\nabla}_{X_1}X_4)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_4)v\varphi(X_1),U_i)$$
$$-g_1(\mathcal{T}_{[X_1,U_i]}X_4,U_i) - g_1(\eta(X_4)v\varphi([X_1,U_i]),U_i)\} + \sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_4}X_j)$$
$$-g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_j}X_j) + g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_4\right) - g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_4)$$

$$+g_1(\widetilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_4) \; - g_1(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_4)\} - g_1(X_1,X_3)[S'(X_2',X_4')\circ f$$

$$-\sum_i\{g_1(v\widetilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_4,U_i) - g_1(\mathcal{A}_{X_2}U_i,h\widetilde{\nabla}_{U_i}X_4) + g_1(\eta(h\widetilde{\nabla}_{U_i}X_4)v\varphi(X_2),U_i)$$
$$+g_1(\widetilde{\nabla}_{X_2}\eta(X_4)v\varphi(U_i),U_i) - g_1(\widehat{\nabla}_{U_i}\mathcal{A}_{X_2}X_4,U_i) + g_1(\eta(\mathcal{A}_{X_2}X_4)\varphi(U_i),U_i)$$
$$-g_1(\eta(h\widetilde{\nabla}_{X_2}X_4)v\varphi(U_i),U_i) - g_1(\widetilde{\nabla}_{U_i}\eta(X_4)v\varphi(X_2),U_i) - g_1(\mathcal{T}_{[X_2,U_i]}X_4,U_i)$$
$$-g_1(\eta(X_4)v\varphi([X_2,U_i]),U_i)\} + \sum_j\{-g_1(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_4}X_j) - g_1(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_j}X_j)$$

$$+g_1(\eta(\mathcal{A}_{X_j}X_j)h\varphi(X_2),X_4) \; -g_1(\eta(\mathcal{A}_{X_2}X_j)h\varphi(X_j),X_4) + g_1(\widetilde{\nabla}_{X_2}\eta(X_j)v\varphi(X_j),X_4)$$

$$-g_1(\widetilde{\nabla}_{X_j}\;\eta(X_j)v\varphi(X_2),X_4)\} + g_1(X_1,X_4)[S'(X_2',X_3')\circ f - \sum_i\{g_1(v\widetilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_3,U_i)$$
$$-g_1(\mathcal{A}_{X_2}U_i,h\widetilde{\nabla}_{U_i}X_3) + g_1(\eta(h\widetilde{\nabla}_{U_i}X_3)v\varphi(X_2),U_i)$$
$$+g_1(\widetilde{\nabla}_{X_2}\eta(X_3)v\varphi(U_i),U_i) - g_1(\widehat{\nabla}_{U_i}\mathcal{A}_{X_2}X_3,U_i) + g_1(\eta(\mathcal{A}_{X_2}X_3)\varphi(U_i),U_i)$$
$$-g_1(\eta(h\widetilde{\nabla}_{X_2}X_3)v\varphi(U_i),U_i) \; -g_1(\widetilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_2),U_i) - g_1(\mathcal{T}_{[X_2,U_i]}X_3,U_i)$$
$$-g_1(\eta(X_3)v\varphi([X_2,U_i]),U_i)\} + \sum_j\{-g_1(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_3}X_j) - g_1(\mathcal{A}_{X_2}X_3,\mathcal{A}_{X_j}X_j)$$

$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_3\right) - g_1(\eta(\mathcal{A}_{X_2}X_j)h\varphi(X_j),X_3) + g_1(\widetilde{\nabla}_{X_2}\eta(X_j)v\varphi(X_j),X_3)$$

$$-g_1\left(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_2),X_3\right) - g_1(X_2,X_4)[S'(X_1',X_3')\circ f - \sum_i\{g_1(v\widetilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_3,U_i)$$
$$-g_1(\mathcal{A}_{X_1}U_i,h\widetilde{\nabla}_{U_i}X_3) + g_1(\eta(h\widetilde{\nabla}_{U_i}X_3)v\varphi(X_1),U_i) + g_1(\widetilde{\nabla}_{X_1}\eta(X_3)v\varphi(U_i),U_i)$$
$$-g_1(\widehat{\nabla}_{U_i}\mathcal{A}_{X_1}X_3,U_i) + g_1(\eta(\mathcal{A}_{X_1}X_3)\varphi(U_i),U_i) - g_1(\eta(h\widetilde{\nabla}_{X_1}X_3)v\varphi(U_i),U_i)$$
$$-g_1(\widetilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_1),U_i) - g_1(\mathcal{T}_{[X_1,U_i]}X_3,U_i) - g_1(\eta(X_3)v\varphi([X_1,U_i]),U_i)$$

$$+\sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_3}X_j) - g_1(\mathcal{A}_{X_1}X_3,\mathcal{A}_{X_j}X_j) + g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_3)$$

$$-g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_3) + g_1(\widetilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_3) - g_1(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_3)\}]\},$$

$$g_1(\tilde{L}(X_1,X_2)X_3,U_1) = g_1(v\widetilde{\nabla}_{X_1}\mathcal{A}_{X_2}X_3,U_1) + g_1(\mathcal{A}_{X_1}h\widetilde{\nabla}_{X_2}X_3,U_1) + g_1(\eta(h\widetilde{\nabla}_{X_2}X_3)v\varphi(X_1),U_1)$$
$$+g_1(\widetilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),U_1) - g_1(v\widetilde{\nabla}_{X_2}\mathcal{A}_{X_1}X_3,U_1) - g_1(\mathcal{A}_{X_2}h\widetilde{\nabla}_{X_1}X_3,U_1)$$
$$-g_1(\eta(h\widetilde{\nabla}_{X_1}X_3)v\varphi(X_2),U_1) - g_1(\widetilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),U_1) - g_1(\eta(X_3)v\varphi([X_1,X_2]),U_1)$$
$$-\frac{1}{(n-2)}\{g_1(X_2,X_3)[\sum_i\{g_1(v\widetilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1) - g_1(\mathcal{A}_{X_1}U_1,h\widetilde{\nabla}_{X_i}X_i) + g_1(\eta(h\widetilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_1)$$
$$+g_1(\widetilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1) - g_1(v\widetilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1) + g_1(\mathcal{A}_{X_i}U_1,h\widetilde{\nabla}_{X_1}X_i) - g_1(\eta(h\widetilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_1)$$
$$-g_1(\widetilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1) - g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)\} + \sum_j\{g_1(v\widetilde{\nabla}_{X_1}\widehat{\nabla}_{U_j}U_j,U_1)$$

$$-g_1(\widetilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1) + g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j) - g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_1) - g_1(\widehat{\nabla}_{U_j}v\widetilde{\nabla}_{X_1}U_j,U_1)$$

$$+g_1(\eta(v\widetilde{\nabla}_{X_1}U_j)\varphi(U_j),U_1) - g_1(\widetilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1) - g_1(\widehat{\nabla}_{[X_1,U_j]}U_j,U_1)$$

$$+g_1(\eta(U_j)\varphi([X_1,U_j],U_1)\} - g_1(X_1,X_3)\Big[\sum_i\{g_1(v\widetilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1\Big) - g_1\big(\mathcal{A}_{X_2}U_1,h\widetilde{\nabla}_{X_i}X_i\big)$$
$$+g_1(\eta\big(h\widetilde{\nabla}_{X_i}X_i\big)v\varphi(X_2),U_1) + g_1(\widetilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1) - g_1(v\widetilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1) + g_1(\mathcal{A}_{X_i}U_1,h\widetilde{\nabla}_{X_2}X_i)$$
$$-g_1(\eta\big(h\widetilde{\nabla}_{X_2}X_i\big)v\varphi(X_i),U_1) - g_1\big(\widetilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1\big) - g_1(\eta(X_i)v\varphi([X_2,X_i],U_1)\}$$
$$+\sum_j\{g_1(v\widetilde{\nabla}_{X_2}\widehat{\nabla}_{U_j}U_j,U_1) - g_1(\widetilde{\nabla}_{X_2}\eta(U_j)\varphi(U_j),U_1) + g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j\right)$$
$$-g_1\big(\eta\big(\mathcal{A}_{X_2}U_j\big)v\varphi\big(U_j\big),U_1\big) - g_1\left(\widehat{\nabla}_{U_j}v\widetilde{\nabla}_{X_2}U_j,U_1\right) + g_1(\eta\big(v\widetilde{\nabla}_{X_2}U_j\big)\varphi\big(U_j\big),U_1)$$
$$-g_1(\widetilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_2),U_1) - g_1\left(\widehat{\nabla}_{[X_2,U_j]}U_j,U_1\right) + g_1(\eta\big(U_j\big)\ \varphi\big([X_2,U_j],U_1\big)\}]\},$$

$$g_1\big(\tilde{L}(X_1,U_1)X_2,U_2\big) = g_1\big(v\widetilde{\nabla}_{X_1}\mathcal{T}_{U_1}X_2,U_2\big) + g_1(\mathcal{A}_{X_1}h\widetilde{\nabla}_{U_1}X_2,U_2) + g_1(\eta(h\widetilde{\nabla}_{U_1}X_2)v\varphi(X_1,U_2)$$
$$+g_1\big(\widetilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_1),U_2\big) - g_1\big(\widehat{\nabla}_{U_1}\mathcal{A}_{X_1}X_2,U_2\big) + g_1\big(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_1),U_2\big) - g_1\big(\mathcal{T}_{U_1}h\widetilde{\nabla}_{X_1}X_2,U_2\big)$$
$$-g_1\big(\eta\big(h\widetilde{\nabla}_{X_1}X_2\big)v\varphi(U_1),U_2\big) - g_1(\widetilde{\nabla}_{U_1}\eta(X_2)v\varphi(X_1),U_2) - g_1(\mathcal{T}_{[X_1,U_1]}X_2,U_2)$$
$$-g_1(\eta(X_2)v\varphi([X_1,U_1]),U_2) + \frac{1}{(n-2)}\{g_1(X_1,X_2)[\hat{S}(U_1,U_2)$$
$$+\sum_i\{-g_1(\eta(\widehat{\nabla}_{U_i}U_i)\varphi(U_1),U_2) - g_1(\widetilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_2)$$
$$+g_1\big(\eta\big(\widehat{\nabla}_{U_1}U_i\big)\varphi(U_i),U_2\big) + g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_2) - g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_2) + g_1(\widetilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_2)$$
$$+g_1(\eta(U_i)\varphi([U_1,U_i]),U_2) - \sum_j\{g_1(v\widetilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_2) - g_1(\mathcal{A}_{X_j}U_2,h\widetilde{\nabla}_{U_1}X_j)$$
$$+g_1\big(\eta\big(h\widetilde{\nabla}_{U_1}X_j\big)v\varphi\big(X_j\big),U_2\big) + g_1\left(\widetilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(U_1),U_2\right) - g_1\left(\widehat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_2\right)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_2\right) + g_1\left(\mathcal{T}_{U_1}U_2,h\widetilde{\nabla}_{X_j}X_j\right) - g_1\left(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_2\right)$$
$$-g_1(\widetilde{\nabla}_{U_1}\eta\big(X_j\big)v\varphi\big(X_j\big),U_2) - g_1(\mathcal{T}_{[X_j,U_1]}X_j,U_2) - g_1(\eta\big(X_j\big)v\varphi\big([X_j,U_1]\big),U_2)$$
$$+g_1(U_1,U_2)[S'(X_1',X_2')\circ f - \sum_i\{g_1(v\,\widetilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_2,U_i) - g_1(\mathcal{A}_{X_1}U_i,h\widetilde{\nabla}_{U_i}X_2)$$
$$+g_1(\eta\big(h\widetilde{\nabla}_{U_i}X_2\big)v\varphi(X_1),U_i) + g_1(\widetilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_i),U_i) - g_1(\widehat{\nabla}_{U_i}\mathcal{A}_{X_1}X_2,U_i)$$
$$+g_1(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_i),U_i) - g_1\big(\eta\big(h\widetilde{\nabla}_{X_1}X_2\big)v\varphi(U_i),U_i\big)$$
$$-g_1\big(\widetilde{\nabla}_{U_i}\eta(X_2)v\varphi(X_1),U_i\big) - g_1\big(\mathcal{T}_{[X_1,U_1]}X_2,U_i\big) - g_1(\eta(X_2)v\varphi([X_1,U_i]),U_i)\}$$
$$+\sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_2}X_j) - g_1\left(\mathcal{A}_{X_1}X_2,\mathcal{A}_{X_j}X_j\right) + g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_2)$$
$$-g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi\big(X_j\big),X_2) + g_1(\widetilde{\nabla}_{X_1}\eta\big(X_j\big)v\varphi\big(X_j\big),X_2) - g_1(\widetilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_1),X_2)\}]\},$$

$$g_1\big(\tilde{L}(U_1,U_2)U_3,U_4\big) = g_1\big(\hat{R}(U_1,U_2)U_3,U_4\big) - \eta\big(\widehat{\nabla}_{U_2}U_3\big)g_1(\varphi(U_1),U_4) + g_1(\mathcal{T}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,U_4)$$
$$+g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_3)v\varphi(U_1),U_4) - g_1(\widetilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),U_4) + g_1(\eta(\widehat{\nabla}_{U_1}U_3)\varphi(U_2),U_4)$$
$$-g_1(\mathcal{T}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,U_4) - g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_3)v\varphi(U_2),U_4) + g_1(\widetilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),U_4)$$

$$+g_1(\eta(U_3)\varphi([U_1,U_2]),U_4) \quad -\frac{1}{(n-2)}\{g_1(U_2,U_3)[\hat{S}(U_1,U_4)$$

$$+\sum_i \{-g_1\big(\eta\big(\widehat{\nabla}_{U_i}U_i\big)\varphi(U_1),U_4\big) - g_1\big(\widetilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_4\big)$$

$$+g_1(\eta\big(\widehat{\nabla}_{U_1}U_i\big)\varphi(U_i),U_4) + g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_4) - g_1(\eta\big(\tilde{\mathcal{T}}_{U_1}U_i\big)v\varphi(U_i),U_4)$$

$$+g_1\big(\widetilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_4\big) + g_1(\eta(U_i)\varphi([U_1,U_i]),U_4) - \sum_j\{\, g_1(v\widetilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_4)$$

$$-g_1(\mathcal{A}_{X_j}U_4,h\widetilde{\nabla}_{U_1}X_j) + g_1(\eta\big(h\widetilde{\nabla}_{U_1}X_j\big)v\varphi\big(X_j\big),U_4) + g_1(\widetilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(U_1),U_4) - g_1(\widehat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_4)$$

$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_4\right) + g_1(\mathcal{T}_{U_1}U_4,h\widetilde{\nabla}_{X_j}X_j) - g_1(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_4)$$

$$-g_1\big(\widetilde{\nabla}_{U_1}\eta\big(X_j\big)v\varphi\big(X_j\big),U_4\big) - g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_4\right) - g_1(\eta\big(X_j\big)v\varphi\big(\big[X_j,U_1\big]\big),U_4)\}]$$

$$-g_1(U_1,U_3)[\hat{S}(U_2,U_4) + \sum_i \{-g_1(\eta\big(\widehat{\nabla}_{U_i}U_i\big)\varphi(U_2),U_4) - g_1(\widetilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_4)$$

$$+g_1(\eta\big(\widehat{\nabla}_{U_2}U_i\big)\varphi(U_i),U_4) + g_1(\tilde{\mathcal{T}}_{U_2}U_i,\ \mathcal{T}_{U_i}U_4) - g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_j)v\varphi(U_j),U_4)$$

$$+g_1(\widetilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_4) + g_1(\eta(U_i)\varphi([U_2,U_i]),U_4)$$

$$-\sum_j\{\, g_1\left(v\widetilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_4\right) - g_1\left(\mathcal{A}_{X_j}U_4,h\widetilde{\nabla}_{U_2}X_j\right) + g_1(\eta\big(h\widetilde{\nabla}_{U_2}X_j\big)v\varphi\big(X_j\big),U_4)$$

$$+g_1(\widetilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(U_2),U_4) - g_1(\widehat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_4) + g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_4)$$

$$+g_1\left(\mathcal{T}_{U_2}U_4,h\widetilde{\nabla}_{X_j}X_j\right) - g_1(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_4) - g_1(\widetilde{\nabla}_{U_2}\eta\big(X_j\big)v\varphi\big(X_j\big),U_4)$$

$$-g_1(\mathcal{T}_{[X_j,U_2]}X_j,U_4) - g_1(\eta\big(X_j\big)v\varphi\big(\big[X_j,U_2\big]\big),U_4)\}] + g_1(U_1,U_4)[\hat{S}(U_2,U_3)$$

$$+\sum_i \{-g_1(\eta\big(\widehat{\nabla}_{U_i}U_i\big)\varphi(U_2),U_3) - g_1(\widetilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_3)$$

$$+g_1\big(\eta\big(\widehat{\nabla}_{U_2}U_i\big)\varphi(U_i),U_3\big) + g_1(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_3) - g_1(\eta\big(\tilde{\mathcal{T}}_{U_2}U_i\big)v\varphi(U_i),U_3)$$

$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_3) + g_1(\eta(U_i)\varphi([U_2,U_i]),U_3) - \sum_j \ \{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_3)$$

$$-g_1(\mathcal{A}_{X_j}U_3,h\widetilde{\nabla}_{U_2}X_j) + g_1(\eta\big(h\widetilde{\nabla}_{U_2}X_j\big)v\varphi\big(X_j\big),U_3) + g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(U_2),U_3)$$

$$-g_1(\widehat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_3) + \ g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_3\right) + g_1(\mathcal{T}_{U_2}U_3,h\widetilde{\nabla}_{X_j}X_j)$$

$$-g_1(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_3) - g_1(\widetilde{\nabla}_{U_2}\eta\big(X_j\big)v\varphi\big(X_j\big),U_3) - g_1(\mathcal{T}_{[X_j,U_2]}X_j,U_3)$$

$$-g_1(\eta\big(X_j\big)v\varphi\big(\big[X_j,U_2\big]\big),U_3)\} - g_1(U_2,U_4)[\ \hat{S}(U_1,U_3)$$

$$\sum_i \{-g_1(\eta\big(\widehat{\nabla}_{U_i}U_i\big)\varphi(U_1),U_3) - g_1(\widetilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_3) + g_1(\eta\big(\widehat{\nabla}_{U_1}U_i\big)\varphi(U_i),U_3)$$

$$+g_1\big(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_3\big) - g_1\big(\eta\big(\tilde{\mathcal{T}}_{U_1}U_i\big)v\varphi(U_i),U_3\big) + g_1\big(\widetilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_3\big)$$

$$+g_1(\eta(U_i)\varphi([U_1,U_i]),U_3)-\sum_j\{g_1(v\widetilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_3)-g_1(\mathcal{A}_{X_j}U_3,h\widetilde{\nabla}_{U_1}X_j)$$

$$+g_1(\eta(h\widetilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_3)+g_1\left(\widetilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_3\right)-g_1\left(\widehat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_3\right)$$

$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_3\right)+g_1\left(\mathcal{T}_{U_1}U_3,h\widetilde{\nabla}_{X_j}X_j\right)-g_1\left(\eta\left(h\widetilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_3\right)$$

$$-g_1(\widetilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_3)-g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_3\right)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_3)\}]\},$$

$$g_1(\tilde{L}(X_1,X_2)U_1,X_3)=g_1(h(\widetilde{\nabla}_{X_1}\mathcal{A}_{X_2}U_1,X_3)+g_1(\mathcal{A}_{X_1}v\widetilde{\nabla}_{X_2}U_1,X_3)+\eta(v\widetilde{\nabla}_{X_2}U_1)g_1(h\varphi(X_1),X_3)$$
$$+g_1(\widetilde{\nabla}_{X_1}\eta(U_1)h\varphi(X_2),X_3)-g_1(h\widetilde{\nabla}_{X_2}\mathcal{A}_{X_1}U_1,X_3)-g_1(\mathcal{A}_{X_2}v\widetilde{\nabla}_{X_1}U_1,X_3)$$
$$-g_1(\eta(v\widetilde{\nabla}_{X_1}U_1)h\varphi(X_2),X_3)-g_1(\widetilde{\nabla}_{X_2}\eta(U_1)h\varphi(X_1),X_3)-g_1(\mathcal{A}_{[X_1,X_2]}U_1,X_3)$$
$$-g_1(\eta(U_1)h\varphi([X_1,X_2]),X_3)-\frac{1}{(n-2)}\{g_1(X_1,X_3)[\sum_i\{g_1(v\widetilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1)$$
$$-g_1(\mathcal{A}_{X_2}U_1,h\widetilde{\nabla}_{X_i}X_i)+\eta(h\widetilde{\nabla}_{X_i}X_i)g_1(v\varphi(X_2),U_1)+g_1(\widetilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1)$$
$$-g_1\left(v\widetilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1\right)+g_1(\mathcal{A}_{X_i}U_1,h\widetilde{\nabla}_{X_2}X_i)-g_1(\eta(h\widetilde{\nabla}_{X_2}X_i)v\varphi(X_i),U_1)-g_1(\widetilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1)$$
$$-g_1(\eta(X_i)v\varphi([X_2,X_i],U_1)+\sum_j\{g_1\left(v\widetilde{\nabla}_{X_2}\widehat{\nabla}_{U_j}U_j,U_1\right)-g_1(\widetilde{\nabla}_{X_2}\eta(U_j)\varphi(U_j),U_1)$$

$$+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j\right)-g_1(\eta(\mathcal{A}_{X_2}U_j)v\varphi(U_j),U_1)-g_1\left(\widehat{\nabla}_{U_j}v\widetilde{\nabla}_{X_2}U_j,U_1\right)+g_1(\eta(v\widetilde{\nabla}_{X_2}U_j)\varphi(U_j),U_1)$$

$$-g_1(\widetilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_2),U_1)-g_1(\widehat{\nabla}_{[X_2,U_j]}U_j,U_1)\quad+g_1(\eta(U_j)\varphi([X_2,U_j]),U_1)\}]$$

$$-g_1(X_2,X_3)[\sum_i\{g_1(v\widetilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)-g_1(\mathcal{A}_{X_1}U_1,h\widetilde{\nabla}_{X_i}X_i)+g_1(\eta(h\widetilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_1)$$
$$+\ g_1(\widetilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)-g_1(v\widetilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\widetilde{\nabla}_{X_1}X_i)$$
$$-g_1(\eta(h\widetilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_1)-g_1(\widetilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)\}$$
$$+\sum_j\{g_1(v\widetilde{\nabla}_{X_1}\widehat{\nabla}_{U_j}U_j,U_1)-g_1(\widetilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1)+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j\right)$$

$$-g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_1)-g_1(\widehat{\nabla}_{U_j}v\widetilde{\nabla}_{X_1}U_j,U_1)+g_1(\eta(v\widetilde{\nabla}_{X_1}U_j)\varphi(U_j),U_1)$$

$$-g_1\left(\widetilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1\right)-g_1\left(\widehat{\nabla}_{[X_1,U_j]}U_j,U_1\right)+g_1(\eta(U_j)\varphi([X_1,U_j]),U_1)\}].$$

4. Conclusion and Discussion

This study presents significant geometric results by computing the Weyl projective curvature tensor, the concircular curvature tensor, and the conharmonic curvature tensor under quarter-symmetric non-metric connections in Riemannian submersions. In particular, the interplay among these curvature tensors sheds light on the underlying structure of submersions, especially when totally umbilic fibers are present. Therefore, the results indicate that quarter-symmetric non-metric connections play a pivotal role in shaping the curvature and geometry of Riemannian submersions.

The results obtained in this study are expected to inspire further investigations into the interrelations among various curvature tensors and potential generalizations to other geometric structures.

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Conflict of Interest

The authors declare that there are no conflicts of interest regarding this paper.

Research and Publication Ethics Statement

The authors declare that this study complies with research and publication ethics.

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