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Minimum Covering Seidel Laplacian Energy of a Graph

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Abstract. This work proposes a matrix called the minimum covering Seidel Laplacian matrix and a new type of graph energy called the minimum covering Seidel Laplacian energy $ES_{Lc}(\mathcal{G})$, which depends on the appropriate minimum covering set of the graph $\mathcal{G}$. Upper and lower bounds on $ES_{Lc}(\mathcal{G})$ are presented.

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1. Introduction and Preliminaries

In this work, $\mathcal{G}$ is considered as a simple graph that has vertex set V and edge set E of cardinality n, e. The adjacent vertices v_i, v_j and degree of a vertex v_i are consecutively indicated as $v_i \sim v_j$ and d_i. The ordinary graph energy is considered in [5] as sum of the absolute values of the graph eigenvalues. Inspired by this concept, many researchers have worked on other types of graph energy related to minimum covering set [1, 8, 9]. The reader can follow more background on graph energy and applications from [6, 7, 10, 14] and the references therein.

Let $\varnothing \neq C \subset V$. The set C is called vertex cover, if each edge of $\mathcal{G}$ has at least one of member of C as an endpoint. The smallest vertex cover for a graph, called the minimum vertex cover or minimum covering set, which will be denoted by C. The minimum covering Seidel matrix $S_c = S_c(\mathcal{G}) = \left(s_{ij}\right)$ is defined [8] as

$$s_{ij} = \begin{cases} -1 & v_i \sim v_j \\ 1 & v_i \nsim v_j \\ 1 & i = j \ \& \ v_i \in C \\ 0 & i = j \ \& \ v_i \notin C \end{cases}$$

and the minimum covering Seidel energy $ES_c(\mathcal{G})$ is known as

$$ES_c(\mathcal{G}) = \sum_{i=1}^{n} |\theta_i|,$$

where $\theta_1, \theta_2, ..., \theta_n$ are the eigenvalues of S_c.

We introduce the minimum covering Seidel Laplacian matrix $S_{Lc} = S_{Lc}(\mathcal{G}) = \left(s_{ij}^c\right)$ of $\mathcal{G}$ as

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$$s_{ij}^{c} = \begin{cases} 1 & v_i \sim v_j \\ -1 & v_i \nsim v_j \\ n-2-2d_i & i = j \ \& \ v_i \in C \\ n-1-2d_i & i = j \ \& \ v_i \notin C \end{cases}$$

with eigenvalues $\rho_1, \rho_2, ..., \rho_n$. The reader can conclude that $S_{Lc} = D - S_c$, where $D = diag(n-1-2d_1, n-1-2d_2 ..., n-1-2d_n)$.

This paper focuses on the bounds for the minimum covering Seidel Laplacian energy of a graph. The presented bounds are related to many graph parameters. This new type of energy will be examined from a mathematical perspective. It is possible that this new energy may have applications in chemistry and other fields.

Theorem 1.1 ([3]). *Let $M, N \in \mathbb{C}^{n\times n}$ be symmetric matrices. Then,*

$$\sigma_i (M+N) \leq \sigma_i (M) + \sigma_i (N),$$

where $\sigma_i (.)$, $i = 1, 2, ..., n$ stands for singular value of a matrix.

2. Main Results

This section is concerned with the bounds on the minimum covering Seidel Laplacian energy, which will be defined here. Let us give the below theorem related to the trace of S_{Lc}, which is essential for the study we will carry out.

Theorem 2.1. *The assertions below hold.*

$$\begin{aligned} tr(S_{Lc}) =& n(n-1) - 4e - |C|, \\ tr\left((S_{Lc})^2\right) =& \sum_{i=1}^{n} (n-1-2d_i-\varepsilon_i)^2 + n(n-1), \end{aligned} \tag{2.1}$$

where $\varepsilon_i = \begin{cases} 1, & v_i \in C \\ 0, & v_i \notin C \end{cases}$.

Proof. We have

$$\begin{aligned} tr(S_{Lc}) =& \sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} s_{ii}^{c} \\ =& \sum_{i=1}^{n} (n-1-2d_i-\varepsilon_i) \\ =& n(n-1) - 2\sum_{i=1}^{n} d_i - \sum_{i=1}^{n} \varepsilon_i \\ =& n(n-1) - 4e - |C|. \end{aligned}$$

Also, we get

$$\begin{aligned}
tr\left((S_{Lc})^2\right) &= \sum_{i=1}^{n} \rho_i^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(s_{ij}^c s_{ji}^c\right) \\
&= \sum_{i\neq j}\left(s_{ij}^c\right)^2 + \sum_{i=1}^{n}(s_{ii}^c)^2 \\
&= 2\sum_{i<j}^{n}\left(s_{ij}^c\right)^2 + \sum_{i=1}^{n}(n-1-2d_i-\varepsilon_i)^2 \\
&= 2\left(e(1)^2 + \left(\frac{n(n-1)}{2} - e\right)(-1)^2\right) + \sum_{i=1}^{n}(n-1-2d_i-\varepsilon_i)^2 \\
&= n(n-1) + \sum_{i=1}^{n}(n-1-2d_i-\varepsilon_i)^2,
\end{aligned}$$

which completes the proof. □

Proposition 2.2. *If $\mathcal{G}$ is a k-regular graph and $\theta_1, \theta_2, ..., \theta_n$ are the eigenvalues of S_c, then $n-1-2k-\theta_j$, $j = 1, 2, ..., n$ are the eigenvalues of S_{Lc}.*

Proof. As $S_{Lc} = (n-1-2k) I_n - S_c$, the proof is clear. □

The study in [12] implies that the energies of graphs can be considered as special cases of matrix norms defined on $\mathbb{C}^{n\times n}$. Trace norm of $B \in \mathbb{C}^{n\times n}$ is defined as $\|B\|_* = \sum_{i=1}^{n}\sigma_i(B)$, where σ_i denote singular values. Clearly,

$$\left\|B - \frac{tr(B)}{n} I_n\right\|_* = \sum_{i=1}^{n}\left|\vartheta_i - \frac{tr(B)}{n}\right| \tag{2.2}$$

holds, where $\vartheta_1, \vartheta_2, ..., \vartheta_n$ are the eigenvalues of B. By setting A (adjacency matrix) instead of B in (2.2) and $tr(A) = 0$, then graph energy coincides with $\|A\|_*$. The energy of $B \in \mathbb{C}^{n\times n}$ is introduced as

$$E(B) = \sum_{i=1}^{n}\left|\vartheta_i - \frac{tr(B)}{n}\right| \tag{2.3}$$

in [2], which is a generalization of the graph energy.

By considering (2.1) and (2.3), we may define the minimum covering Seidel Laplacian energy of $\mathcal{G}$ as

$$ES_{Lc}(\mathcal{G}) = \sum_{i=1}^{n}\left|\rho_i - \frac{n(n-1) - 4e - |C|}{n}\right|. \tag{2.4}$$

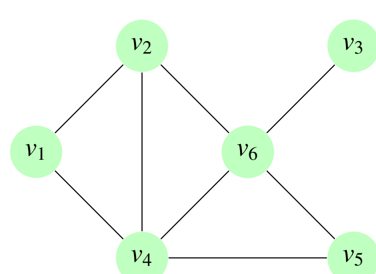

Figure 1. $\mathcal{G}$

Example 2.3. The graph in Figure 1 with possible minimum covering sets $C_1 = \{v_2, v_4, v_6\}$ and $C_2 = \{v_1, v_4, v_6\}$ has the following corresponding matrices.

$$S^1_{Lc} = \begin{bmatrix} 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -2 & -1 & 1 & -1 & 1 \\ -1 & -1 & 3 & -1 & -1 & 1 \\ 1 & 1 & -1 & -4 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & -4 \end{bmatrix}, S^2_{Lc} = \begin{bmatrix} 0 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 3 & -1 & -1 & 1 \\ 1 & 1 & -1 & -4 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & -4 \end{bmatrix}.$$

S^1_{Lc} has eigenvalues $\rho_1 \approx -5.5311$, $\rho_2 \approx -4.6032$, $\rho_3 \approx -1.2864$, $\rho_4 \approx -0.2451$, $\rho_5 \approx 2.5311$, $\rho_6 \approx 4.1347$ and S^2_{Lc} has eigenvalues $\rho_1 \approx -5.5604$, $\rho_2 = -4.3521$, $\rho_3 \approx -1.0280$, $\rho_4 \approx -0.4433$, $\rho_5 \approx 2.3222$, $\rho_6 \approx 4.0615$. We have $\frac{n(n-1)-4e-|C|}{n} \approx -0,8333$. Consequently, $ES_{Lc_1}(\mathcal{G}) \approx 17,8414$ and $ES_{Lc_2}(\mathcal{G}) \approx 16,8809$. Thus, the minimum covering Seidel Laplacian energy depends on C.

Theorem 1.1 helps us to find the bound below involving $ES_c(\mathcal{G})$ and the average degree.

Theorem 2.4.

$$ES_{Lc}(\mathcal{G}) \le ES_c(\mathcal{G}) + \sum_{i=1}^{n} \left| 2\left(\overline{d} - d_i\right) + \frac{1}{n}|C| \right|, \tag{2.5}$$

where $\overline{d}$ is the average degree.

Proof. We have $S_{Lc} - \left(\frac{n(n-1)-4e-|C|}{n}\right) I_n = \left[D - \left(\frac{n(n-1)-4e-|C|}{n}\right) I_n\right] + (-S_c)$, as $S_{Lc} = D + (-S_c)$. The diagonal matrix $\left[D - \left(\frac{n(n-1)-4e-|C|}{n}\right) I_n\right]$ has eigenvalues $2\left(\overline{d} - d_i\right) + \frac{1}{n}|C|$, where $\overline{d} = \frac{2e}{n}$. By Theorem 1.1, we have

$$\begin{aligned} ES_{Lc}(\mathcal{G}) &= \sum_{i=1}^{n} \sigma_i \left(S_{Lc} - \left(\frac{n(n-1) - 4e - |C|}{n} \right) I_n \right) \\ &= \sum_{i=1}^{n} \sigma_i \left(D - \left(\frac{n(n-1) - 4e - |C|}{n} \right) I_n + (-S_c) \right) \\ &\le \sum_{i=1}^{n} \sigma_i \left(D - \left(\frac{n(n-1) - 4e - |C|}{n} \right) I_n \right) + \sum_{i=1}^{n} \sigma_i (-S_c) \\ &= \sum_{i=1}^{n} \left| 2\left(\overline{d} - d_i\right) + \frac{1}{n}|C| \right| + ES_c(\mathcal{G}). \end{aligned}$$

Thus, (2.5) is obtained. □

Let $\beta_i := \rho_i - \frac{n(n-1)-4e-|C|}{n}$ in (2.4). Then, we express $ES_{Lc}(\mathcal{G})$ as

$$ES_{Lc}(\mathcal{G}) = \sum_{j=1}^{n} \left|\beta_j\right|. \tag{2.6}$$

The following lemma serves to find new bounds on $ES_{Lc}(\mathcal{G})$.

Lemma 2.5. *The assertions below hold.*

$$\sum_{j=1}^{n} \beta_j = 0,$$

$$\sum_{j=1}^{n} \beta_j^2 = \sum_{j=1}^{n} \left(n - 1 - 2d_j - \varepsilon_j\right)^2 + n(n-1) - \frac{1}{n}\left(n(n-1) - 4e - |C|\right)^2 := M. \tag{2.7}$$

Proof. By Theorem 2.1

$$\begin{aligned} \sum_{j=1}^{n} \beta_j &= \sum_{j=1}^{n} \rho_j - \frac{1}{n} \sum_{j=1}^{n} \left(n(n-1) - 4e - |C|\right) \\ &= n(n-1) - 4e - |C| - n(n-1) + 4e + |C| = 0. \end{aligned}$$

$$
\begin{aligned}
\sum_{j=1}^{n}\beta_j^2 =& \sum_{j=1}^{n}\left(\rho_j - \tfrac{n(n-1)-4e-|C|}{n}\right)^2 \\
=& \sum_{j=1}^{n}\rho_j^2 - \frac{2}{n}\left(n\left(n-1\right)-4e-|C|\right)\sum_{j=1}^{n}\rho_j + \frac{1}{n^2}\sum_{j=1}^{n}\left(n\left(n-1\right)-4e-|C|\right)^2 \\
=& \sum_{j=1}^{n}\left(n-1-2d_j-\varepsilon_j\right)^2 + n\left(n-1\right) - \frac{2}{n}\left(n\left(n-1\right)-4e-|C|\right)^2 \\
&+ \frac{1}{n^2}\sum_{j=1}^{n}\left[\left(n\left(n-1\right)-4e\right)^2 - 2\left(n\left(n-1\right)-4e\right)|C| + |C|^2\right] \\
=& \sum_{j=1}^{n}\left(n-1-2d_j-\varepsilon_j\right)^2 + n\left(n-1\right) - \frac{2}{n}\left(n\left(n-1\right)-4e-|C|\right)^2 \\
&+ \frac{1}{n}\left(n\left(n-1\right)-4e-|C|\right)^2 \\
=& \sum_{j=1}^{n}\left(n-1-2d_j-\varepsilon_j\right)^2 + n\left(n-1\right) - \frac{1}{n}\left(n\left(n-1\right)-4e-|C|\right)^2.
\end{aligned}
$$

□

In the rest of the work, some bounds will be established on $ES_{Lc}(\mathcal{G})$ with the help of Lemma 2.5 and mathematical inequalities.

Theorem 2.6.

$$
\sqrt{M + n(n-1)\left|\prod_{i=1}^{n}\beta_i\right|^{\frac{2}{n}}} \leq ES_{Lc}(\mathcal{G}) \leq \sqrt{nM}.
$$

Proof. Using Cauchy-Schwarz inequality leads to

$$
\left(\sum_{j=1}^{n}\left|\beta_j\right|\right)^2 \leq \sum_{j=1}^{n}1\sum_{j=1}^{n}\beta_j^2.
$$

By (2.6) and (2.7), we obtain

$$
ES_{Lc}(\mathcal{G}) \leq \sqrt{nM}.
$$

From AM-GM inequality, we get

$$
\begin{aligned}
\frac{1}{n(n-1)}\sum_{i\neq j}|\beta_i|\left|\beta_j\right| \geq& \left(\prod_{i\neq j}|\beta_i|\left|\beta_j\right|\right)^{\frac{1}{n(n-1)}} \\
=& \left(\prod_{i=1}^{n}|\beta_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\
=& \left|\prod_{i=1}^{n}\beta_i\right|^{\frac{2}{n}}.
\end{aligned}
$$

Thus, $\sum_{i\neq j}|\beta_i|\left|\beta_j\right| \geq n(n-1)\left|\prod_{i=1}^{n}\beta_i\right|^{\frac{2}{n}}$. Using this fact with (2.7) implies

$$
\begin{aligned}
\left(ES_{Lc}(\mathcal{G})\right)^2 =& \sum_{i=1}^{n}|\beta_i|^2 + \sum_{i\neq j}|\beta_i|\left|\beta_j\right| \\
\geq& M + n(n-1)\left|\prod_{i=1}^{n}\beta_i\right|^{\frac{2}{n}}.
\end{aligned}
$$

□

Theorem 2.7. *Let $\mathcal{G}$ be a graph with $|\beta_1| \geq |\beta_2| \geq ... \geq |\beta_n| > 0$. Then,*

$$ES_{Lc}(\mathcal{G}) \geq \frac{|\beta_1| + |\beta_n|}{2\sqrt{nM\,|\beta_1|\,|\beta_n|}}.$$

Proof. Let $a_j, b_j \in \mathbb{R}^+$ $(1 \leq j \leq n)$. Then,

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 \leq \frac{1}{4}\left(\sqrt{\frac{U_1U_2}{u_1u_2}} + \sqrt{\frac{u_1u_2}{U_1U_2}}\right)^2 \left(\sum_{j=1}^{n} a_jb_j\right)^2, \tag{2.8}$$

where $U_1 = \max_{1\leq j\leq n}\left\{a_j\right\}$, $U_2 = \max_{1\leq j\leq n}\left\{b_j\right\}$; $u_1 = \min_{1\leq j\leq n}\left\{a_j\right\}$, $u_2 = \min_{1\leq j\leq n}\left\{b_j\right\}$ (see [11], p.60). Setting $a_j = 1$ and $b_j = \left|\beta_j\right|$ in (2.8)

$$\sum_{j=1}^{n} 1 \sum_{j=1}^{n} \left|\beta_j\right|^2 \leq \frac{1}{4}\left(\sqrt{\frac{|\beta_1|}{|\beta_n|}} + \sqrt{\frac{|\beta_n|}{|\beta_1|}}\right)^2 \left(\sum_{j=1}^{n} \left|\beta_j\right|\right)^2$$

holds. Then, by (2.7)

$$nM \leq \frac{1}{4}\frac{(|\beta_1| + |\beta_n|)^2}{|\beta_1|\,|\beta_n|}\left(ES_{Lc}(\mathcal{G})\right)^2,$$

i.e.,

$$ES_{Lc}(\mathcal{G}) \geq \frac{1}{2\sqrt{nM}}\left(\frac{|\beta_1| + |\beta_n|}{\sqrt{|\beta_1|\,|\beta_n|}}\right).$$

□

Theorem 2.8. *Let $\mathcal{G}$ be a graph with $|\beta_1| \geq |\beta_2| \geq ... \geq |\beta_n| > 0$. Then,*

$$ES_{Lc}(\mathcal{G}) \geq \sqrt{nM - \frac{n^2}{4}\left(|\beta_1| - |\beta_n|\right)^2}.$$

Proof. If a_j, b_j $(1 \leq j \leq n)$ are nonnegative real numbers, then

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{j=1}^{n} a_jb_j\right)^2 \leq \frac{n^2}{4}\left(U_1U_2 - u_1u_2\right)^2, \tag{2.9}$$

where U_j, u_j defined as in (2.8) (see [13]). By setting $a_j = \left|\beta_j\right|$, $b_j = 1$ in (2.9) leads to

$$n\sum_{j=1}^{n} \left|\beta_j\right|^2 - \left(\sum_{j=1}^{n} \left|\beta_j\right|\right)^2 \leq \frac{n^2}{4}\left(|\beta_1| - |\beta_n|\right)^2,$$

i.e.,

$$nM - \left(ES_{Lc}(\mathcal{G})\right)^2 \leq \frac{n^2}{4}\left(|\beta_1| - |\beta_n|\right)^2.$$

Now, the proof is clear. □

Theorem 2.9.

$$ES_{Lc}(\mathcal{G}) \geq |\beta_1| + 2(n-1)\left(\frac{\prod_{j=1}^{n}\left|\beta_j\right|^{\frac{2n-1}{2n(n-1)}}}{|\beta_1|^{\frac{1}{2(n-1)}}} - \frac{1}{2}\prod_{j=1}^{n}\left|\beta_j\right|^{\frac{1}{n}}\right).$$

Proof. Let $b_1, b_2, ..., b_n \geq 0$ and $k_1, k_2, ..., k_n \geq 0$ with $\sum_{j=1}^{n} k_j = 1$. Then,

$$\sum_{j=1}^{n} b_jk_j - \prod_{j=1}^{n} b_j^{k_j} \geq nK\left(\frac{1}{n}\sum_{j=1}^{n} b_j - \prod_{j=1}^{n} b_j^{\frac{1}{n}}\right), \tag{2.10}$$

where $K = \min\{k_1, k_2, ..., k_n\}$ [4]. By setting $b_j = |\beta_j|$, $1 \le j \le n$, $k_1 = \frac{1}{2n}$, $k_j = \frac{2n-1}{2n(n-1)}$ for $2 \le j \le n$ and $K = \frac{1}{2n}$ in (2.10) implies

$$\begin{aligned}\frac{|\beta_1|}{2n} + \frac{2n-1}{2n(n-1)}\sum_{j=2}^{n}|\beta_j| - |\beta_1|^{\frac{1}{2n}}\prod_{j=2}^{n}|\beta_j|^{\frac{2n-1}{2n(n-1)}} &\ge \frac{1}{2}\left(\frac{1}{n}\sum_{j=1}^{n}|\beta_j| - \prod_{j=1}^{n}|\beta_j|^{\frac{1}{n}}\right)\\ &= \frac{1}{2n}ES_{Lc}(\mathcal{G}) - \frac{1}{2}\prod_{j=1}^{n}|\beta_j|^{\frac{1}{n}}.\end{aligned} \tag{2.11}$$

We also have $|\beta_1|^{\frac{1}{2n}}\prod_{j=2}^{n}|\beta_j|^{\frac{2n-1}{2n(n-1)}} = |\beta_1|^{-\frac{1}{2(n-1)}}\prod_{j=1}^{n}|\beta_j|^{\frac{2n-1}{2n(n-1)}} = \frac{\prod_{j=1}^{n}|\beta_j|^{\frac{2n-1}{2n(n-1)}}}{|\beta_1|^{\frac{1}{2(n-1)}}}$ and $\sum_{j=2}^{n}|\beta_j| = ES_{Lc}(\mathcal{G}) - |\beta_1|$. Using these facts in (2.11) yields

$$\left(\frac{1}{2n} - \frac{2n-1}{2n(n-1)}\right)|\beta_1| + \left(\frac{2n-1}{2n(n-1)} - \frac{1}{2n}\right)ES_{Lc}(\mathcal{G}) \ge \frac{\prod_{j=1}^{n}|\beta_j|^{\frac{2n-1}{2n(n-1)}}}{|\beta_1|^{\frac{1}{2(n-1)}}} - \frac{1}{2}\prod_{j=1}^{n}|\beta_j|^{\frac{1}{n}},$$

i.e.,

$$-\frac{1}{2(n-1)}|\beta_1| + \frac{1}{2(n-1)}ES_{Lc}(\mathcal{G}) \ge \frac{\prod_{j=1}^{n}|\beta_j|^{\frac{2n-1}{2n(n-1)}}}{|\beta_1|^{\frac{1}{2(n-1)}}} - \frac{1}{2}\prod_{j=1}^{n}|\beta_j|^{\frac{1}{n}}.$$

Then, we get

$$ES_{Lc}(\mathcal{G}) \ge |\beta_1| + 2(n-1)\left(\frac{\prod_{j=1}^{n}|\beta_j|^{\frac{2n-1}{2n(n-1)}}}{|\beta_1|^{\frac{1}{2(n-1)}}} - \frac{1}{2}\prod_{j=1}^{n}|\beta_j|^{\frac{1}{n}}\right).$$

Thus, the statement holds. □

Theorem 2.10.

$$ES_{Lc}(\mathcal{G}) \le \frac{M}{2} + \sqrt{Mn}.$$

Proof. By Minkowski inequality, we have

$$\left(\sum_{j=1}^{n}\left(1 + |\beta_j|\right)^2\right)^{\frac{1}{2}} \le \left(\sum_{j=1}^{n}|\beta_j|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n}1\right)^{\frac{1}{2}}. \tag{2.12}$$

Using Bernoulli inequality in (2.12) and applying (2.7) states that

$$(n + 2ES_{Lc}(\mathcal{G}))^{\frac{1}{2}} \le \sqrt{M} + \sqrt{n},$$

which implies

$$ES_{Lc}(\mathcal{G}) \le \frac{M}{2} + \sqrt{Mn}.$$

□

Conclusions

In this work, we introduce the minimum covering Seidel Laplacian energy of a graph $ES_{Lc}(\mathcal{G})$, which depends on both the underlying graph $\mathcal{G}$ and its particular minimum covering set C. We study the minimum covering Seidel Laplacian energy from the mathematical aspects and establish some upper and lower bounds on $ES_{Lc}(\mathcal{G})$ using mathematical inequalities. The obtained bounds involve $|C|$, vertex degree, average degree, and eigenvalues of the minimum covering Seidel Laplacian matrix.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have read and agreed to the published version of the manuscript.

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