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RESEARCH PAPER

Analysis of Cauchy problems for variable-order derivatives with Mittag-Leffler kernel

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Abstract

In this paper, the Cauchy problem for variable-order fractional differential equations incorporating the Mittag-Leffler kernel is explored. The variable-order derivative is modeled as a bounded function that adapts to the underlying dynamics of the system. The existence of a solution by utilizing a fixed-point theorem along with an iterative series that converges to the precise solution is established. The uniqueness of the solution is guaranteed by enforcing conditions like generalized Lipschitz continuity and linear growth conditions. This study contributes to the broader understanding of fractional calculus and its applications in complex systems where classical models are insufficient.

Keywords: Cauchy problems; variable-order derivative; existence and uniqueness; Mittag-Leffler kernel

AMS 2020 Classification: 26A33; 34A34; 35B44

1 Introduction

Differential equations formulated using variable-order derivatives have recently attracted significant interest from researchers due to their ability to more accurately understand and model complex phenomena in fields such as physics, chemistry, engineering, biology, and economics. Contrary to constant-order derivatives, mathematical models created using variable-order derivatives have more clearly demonstrated memory effects or characteristics [1–4]. Variable-order derivatives allow for a more practical modeling approach that can better capture complex phenomena that exhibit memory effects or time-dependent behavior. Unlike constant-order derivatives, variable-order derivatives based on the changing nature of the system, offering a more realistic description of systems where the memory effect varies over time. For example, Patnaik et al.

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(2020) discuss how variable-order fractional operators realistic the modeling of complex physical phenomena that cannot be adequately described by constant-order models [5].

Studies on fractional-order derivatives began in the 17th century with the work of Leibniz and were later developed by Liouville and Riemann. This marked the rise of fractional calculus. Research in this area has become a powerful mathematical tool for modeling processes with memory effects. We know that factional calculus lies in the use of local and nonlocal operators. Local operators, such as integer-order derivatives, do not take into account for past states of a system, whereas nonlocal operators, including fractional derivatives, include memory effects, making them more realistic for describing processes [6–10]. This can be presented as the motivation for studying fractional-order derivatives.

One of the most commonly utilized functions in the analysis of fractional-order differential equations is the Mittag-Leffler function, a generalization of the exponential function. The Mittag-Leffler function is essential for solving fractional differential equations and plays a role similar to that of the exponential function used in solving integer-order equations. In past years, the Mittag-Leffler kernel with variable-order derivatives have been incorporated into the work of many researchers, and the results obtained have made a significant contribution to the literature [9–14].

Studying differential equations with variable-order derivatives and Mittag-Leffler kernels through the Cauchy problem have proven beneficial for understanding many issues. It is possible to extend the classical Cauchy problem, which specifies initial conditions for an unknown function and its derivatives, to fractional and variable-order derivatives. Thanks to this generalization, researchers interested in fractional order differential equations can study the dynamic properties of systems in more detail. This approach allows for more accurate modeling of real-world problems. At the same time, this approach based on the Cauchy problem not only investigates the fundamental properties of equation solutions but also provides a systematic approach to the numerical solutions of these systems. So this study presents a different way for examining existence and uniqueness conditions within this context, supported with numerical examples.

We now provide a brief summary of fractional integrals and derivatives with Mittag-Leffler kernels [11] in this subsection.

Several definitions of Mittag-Leffler operator

Definition 1 *Let $h \in H^1(c,d)$, $\alpha \in (0,1)$ then the Atangana-Baleanu derivative in the Caputo sense is defined as follows:*

$$ {}_{c}^{ABC}D_{s}^{\alpha} h\left(s\right) = \frac{AB(\alpha)}{1-\alpha} \int\limits_{c}^{s} h^{'}(\tau) E_{\alpha}\left[-\alpha \frac{(s-\tau)^{\alpha}}{1-\alpha}\right] d\tau, \tag{1} $$

where ${}_{c}^{ABC}D_{s}^{\alpha}$ is fractional operator with Mittag-Leffler kernel in the Caputo sense with order α with respect to s and

$$ AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}, \tag{2} $$

is a normalization function where

$$ AB(0) = AB(1) = 1. \tag{3} $$

Definition 2 *Let $h \in H^1(c,d)$, $\alpha \in (0,1)$ and not differentiable then, the Atangana-Baleanu derivative in the Riemann-Liouville sense is defined as follows:*

$$ {}_{c}^{ABR}D_s^{\alpha} h\left(s\right) = \frac{B(\alpha)}{1-\alpha}\frac{d}{ds}\int_c^s h(\tau) E_\alpha \left[-\alpha\frac{(s-\tau)^\alpha}{1-\alpha}\right] d\tau, \tag{4} $$

where Mittag-Leffler operator defined as below:

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}. \tag{5} $$

The related fractional integral is given by:

$$ {}_{c}^{AB}I_s^{\alpha} h(s) = \frac{1-\alpha}{AB(\alpha)} h(s) + \frac{\alpha}{\Gamma(\alpha)AB(\alpha)}\int_c^s h(\tau)(s-\tau)^{\alpha-1} d\tau. \tag{6} $$

2 Existence and uniqueness in Cauchy Problems for variable-order derivatives with Mittag-Leffler kernels

In this section by replacing the fractional order in derivatives and integrals with a bounded function, we will consider our Cauchy problem via variable-order fractional derivatives and integrals. Similar results were also developed in [15] by Atangana and Koca. Consequently, we examine the Cauchy problems for a variable-order derivative with a Mittag-Leffler kernel as follows:

$$ \begin{cases} {}_{0}^{ABC}D^{\alpha(t)} u(t) = f(t, u(t)), \\ u(0) = u_0. \end{cases} \tag{7} $$

Now applying the Atangana-Baleanu integral, this implies,

$$ u(t) = \frac{1-\alpha(t)}{AB(\alpha(t))} f(t,u(t)) + \frac{\alpha(t)}{\Gamma(\alpha(t))AB(\alpha(t))}\int_{t_0}^t f(\tau,u(\tau))(t-\tau)^{\alpha(\tau)-1} d\tau. \tag{8} $$

To facilitate application in physical models, the order function is usually restricted such that $0 < \alpha\left(t\right) \leq 1$. In this paper, we examine $\alpha\left(t\right)$ over two subintervals, where

$$ m = \min\left\{\alpha\left(t\right), t \in I\right\}, $$

and

$$ M = \max\left\{\alpha\left(t\right), t \in I\right\}, $$

where the minimum and maximum value of order function $\alpha\left(t\right)$ over interval I, respectively. In this work also we defined the following norm

$$\|u\|_{\infty} = \sup_{t \in D_y} |u(t)|. \tag{9}$$

Let us give now sufficient conditions for which the Cauchy problems for variable-order fractional derivative with power kernel has a unique equation if the nonlinear function $f(t, u(t))$ satisfies the following conditions [16, 17].

1) $\forall t \in [0, T]$, the function $\alpha(t)$ is a differentiable and bounded nonzero and non-constant functions and

$$0 < m < \alpha(t) < M \leq 1.$$

2) $\forall t \in [0, T]$, $\alpha'(t) \neq 0$.

3) $f(t, u(t))$ is a nonlinear function and twice differentiable and bounded.

4) $|f(t, u(t))|^2 < G\left(1 + |u|^2\right)$ (Linear Growth Condition).

5) $|f(t, u_1(t)) - f(t, u_2(t))|^2 < \overline{G}|u_1 - u_2|^2$ (Lipschitz Condition).

Then the Cauchy problem has a unique solution in $L^2([t_0, T], R)$. Now we give the proof of existence and uniqueness of the solution of the Cauchy problem for global derivative with Mittag-Leffler kernel.

We would like to note that we will consider the following equation to help readers better understand the proofs later on. Let us define

$$I_t^{\alpha(t)} = \frac{(T - t_0)^{2\alpha(t)-1}}{\Gamma^2(\alpha(t))(2\alpha(t) - 1)}, \tag{10}$$

$$I_t^M = \frac{(T - t_0)^{2M-1}}{\Gamma^2(M)(2M - 1)}. \tag{11}$$

$\forall_l \geq 1$, we define the following sequence:

$$u_l(t) = \frac{1 - \alpha(t)}{AB(\alpha(t))} f(t, u_l(t)) + \frac{\alpha(t)}{\Gamma(\alpha(t)) AB(\alpha(t))} \int_{t_0}^{t} f(\tau, u_l(\tau))(t - \tau)^{\alpha(\tau)-1} d\tau, \tag{12}$$

and

$$\begin{aligned} |u_l(t)|^2 \leq\ & 2\left|\frac{1 - \alpha(t)}{AB(\alpha(t))} f(t, u_l(t))\right|^2 \\ +\ & 2\left|\frac{\alpha(t)}{\Gamma(\alpha(t)) AB(\alpha(t))} \int_{t_0}^{t} f(\tau, u_l(\tau))(t - \tau)^{\alpha(\tau)-1} d\tau\right|^2. \end{aligned} \tag{13}$$

We remember that $\forall t \in [0, T]$, the function $\alpha(t)$ is a differentiable and bounded nonzero and non-constant function and

$$0 < m < \alpha(t) < M \leq 1.$$

So we can rewrite the inequality as below:

$$|u_l(t)|^2 \leq 2\left|\frac{1-M}{AB(M)}f(t,u_l(t))\right|^2 + 2\left|\frac{M}{\Gamma(M)AB(M)}\int_{t_0}^{t} f(\tau,u_l(\tau))(t-\tau)^{M-1}d\tau\right|^2.$$

Leveraging the linear growth assumption in conjunction with Hölder's inequality, which provides a powerful tool for estimating integrals and controlling the growth of functions, we derive the following result. The advantages of Hölder's inequality, particularly its ability to handle non-linear terms effectively, play a crucial role in our analysis. So by applying the linear growth assumption and Hölder's inequality, we obtain

$$\begin{aligned} |u_l(t)|^2 &\leq 2G\left(1+|u_l(t)|^2\right)\left(\frac{1-M}{AB(M)}\right)^2 \\ &+ 2G\frac{M}{AB(M)\Gamma(M)}I_t^M\int_{t_0}^{t}\left(1+|u_l(\tau)|^2\right)d\tau, \end{aligned} \tag{14}$$

$$\begin{aligned} \max_{t_0\leq k\leq t}|u_l(k)|^2 &\leq 2G\left(1+\max_{t_0\leq k\leq t}|u_l(k)|^2\right)\left(\frac{1-M}{AB(M)}\right)^2 \\ &+ 2G\frac{M}{AB(M)\Gamma(M)}I_t^M\int_{t_0}^{t}\left(1+\max_{t_0\leq r\leq \tau}|u_l(r)|^2\right)d\tau. \end{aligned} \tag{15}$$

By taking the expectation, we obtain

$$\begin{aligned} E\left(\max_{t_0\leq k\leq t}|u_l(k)|^2\right) &\leq 2G\left(1+E\left(\max_{t_0\leq k\leq t}|u_l(k)|^2\right)\right)\left(\frac{1-M}{AB(M)}\right)^2 \\ &+ 2G\frac{M}{AB(M)\Gamma(M)}I_t^M\int_{t_0}^{t}\left(1+E\left(\max_{t_0\leq r\leq \tau}|u_l(r)|^2\right)\right)d\tau. \end{aligned} \tag{16}$$

If we move equations into a more acceptable position, then we will get

$$\begin{aligned} E\left(\max_{t_0\leq k\leq t}|u_l(k)|^2\right) &\leq 2G\left(1+E\left(\max_{t_0\leq k\leq t}|u_l(k)|^2\right)\right)\left(\frac{1-M}{AB(M)}\right)^2 \\ &+ 2G\frac{M}{AB(M)\Gamma(M)}I_t^M\int_{t_0}^{t}\left(1+E\left(\max_{t_0\leq r\leq \tau}|u_l(r)|^2\right)\right)d\tau. \end{aligned} \tag{17}$$

Adding 1 to both sides gives us

$$\begin{aligned} 1+E\left(\max_{t_0\leq k\leq t}|u_l(k)|^2\right) &\leq 2G\left(1+E\left(\max_{t_0\leq k\leq t}|u_l(k)|^2\right)\right)\left(\frac{1-M}{AB(M)}\right)^2 \\ &+ 2G\frac{M}{AB(M)\Gamma(M)}I_t^M\int_{t_0}^{t}\left(1+E\left(\max_{t_0\leq r\leq \tau}|u_l(r)|^2\right)\right)d\tau. \end{aligned} \tag{18}$$

So finally we get

$$E\left(\max_{t_0\le k\le t}|u_l(k)|^2\right) \le 1+2G\left(1+E\left(\max_{t_0\le k\le t}|u_l(k)|^2\right)\right)\left(\frac{1-M}{AB(M)}\right)^2 + \exp\left[2G\frac{M}{AB(M)\Gamma(M)}I_t^M(T-t_0)\right]. \tag{19}$$

Let $u_1(.)$ and $u_2(.)$ be the solution of the Cauchy problem. Then $u_1(.), u_2(.) \in L^2([t_0,T],R)$. So we have

$$|u_1(t)-u_2(t)|^2 \le 2\left|\frac{1-M}{AB(M)}\left(f(t,u_1(t))-f(t,u_2(t))\right)\right|^2 + 2\left|\int_{t_0}^{t}\frac{M}{AB(M)\Gamma(M)}\left(f(\tau,u_1(\tau))-f(\tau,u_2(\tau))\right)d\tau\right|^2, \tag{20}$$

using the Lipschitz condition for function $f(t,u(t))$ then we obtain

$$|u_1(t)-u_2(t)|^2 \le 2\overline{G}\left(\frac{1-M}{AB(M)}\right)^2|u_1(t)-u_2(t)|^2 + 2\overline{G}\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^M\int_{t_0}^{t}|u_1(\tau)-u_2(\tau)|^2 d\tau, \tag{21}$$

thus

$$\begin{aligned} E\left(\sup_{t_0\le k\le t}|u_1(k)-u_2(k)|^2\right) &\le 2\overline{G}\left(\frac{1-M}{AB(M)}\right)^2 E\left(\sup_{t_0\le k\le t}|u_1(k)-u_2(k)|^2\right) \\ &+ 2\overline{G}\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^M\int_{t_0}^{t}E\left(\sup_{t_0\le r\le\tau}|(u_1(r)-u_2(r))|^2\right)d\tau \\ &\le 2\overline{G}\left(\frac{1-M}{AB(M)}\right)^2 E\left(\sup_{t_0\le k\le t}|u_1(k)-u_2(k)|^2\right) \\ &+ 2\overline{G}\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^M\int_{t_0}^{t}E\left(\sup_{t_0\le r\le\tau}|(u_1(r)-u_2(r))|^2\right)d\tau, \end{aligned} \tag{22}$$

$$\left(1-2\overline{G}\left(\frac{1-M}{AB(M)}\right)^2\right)E\left(\sup_{t_0\le t\le T}|u_1(t)-u_2(t)|^2\right) \le 2\overline{G}\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^M\int_{t_0}^{t}E\left(\sup_{t_0\le r\le\tau}|(u_1(r)-u_2(r))|^2\right)d\tau, \tag{23}$$

$$E\left(\sup_{t_0\le t\le T}|u_1(t)-u_2(t)|^2\right)\le \frac{2\overline{G}\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^M\int_{t_0}^{t}E\left(\sup_{t_0\le r\le \tau}|(u_1(r)-u_2(r))|^2\right)d\tau}{\left(1-2\overline{G}\left(\frac{1-M}{AB(M)}\right)^2\right)}, \tag{24}$$

$$E\left(\sup_{t_0\le t\le T}|u_1(t)-u_2(t)|^2\right)\le \varpi_1\int_{t_0}^{t}E\left(\sup_{t_0\le r\le \tau}|(u_1(r)-u_2(r))|^2\right)d\tau, \tag{25}$$

where

$$\varpi_1=\frac{2\overline{G}\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^M}{\left(1-2\overline{G}\left(\frac{1-M}{AB(M)}\right)^2\right)}, \tag{26}$$

with following condition

$$2\overline{G}\left(\frac{1-M}{AB(M)}\right)^2\neq 1. \tag{27}$$

Applying Gronwall inequality

$$E\left(\sup_{t_0\le t\le T}|u_1(t)-u_2(t)|^2\right)=0, \forall t\in[t_0,T]. \tag{28}$$

So we have

$$u_1(.)=u_2(.)\ \forall t\in[t_0,T]. \tag{29}$$

Our proof is completed for the existence of the solution.

Existence of solution

Same as the earlier part, we put Picard's recursive approach

$$u_l(t)=\frac{1-\alpha(t)}{AB(\alpha(t))}f(t,u_{l-1}(t))+\frac{\alpha(t)}{AB(\alpha(t))\Gamma(\alpha(t))}\int_{t_0}^{t}f(\tau,u_{l-1}(\tau))(t-\tau)^{\alpha(\tau)-1}d\tau. \tag{30}$$

Now we have to show that $\forall_l\ge 0$, $u_l(t)\in L^2([t_0,T],R)$. Here we suppose that $u_0(.)=u_0$ is the initial condition. For the case when $l=1$,

$$\begin{aligned} u_1(t) &= \frac{1-\alpha(t)}{AB(\alpha(t))}f(t,u_0(t))+\frac{\alpha(t)}{AB(\alpha(t))\Gamma(\alpha(t))}\int_{t_0}^{t}f(\tau,u_0(\tau))(t-\tau)^{\alpha(\tau)-1}d\tau, \\ u_1(t) &\le \frac{1-M}{AB(M)}f(t,u_0(t))+\frac{M}{AB(M)\Gamma(M)}\int_{t_0}^{t}f(\tau,u_0(\tau))(t-\tau)^{M-1}d\tau. \end{aligned} \tag{31}$$

As a result of the preceding steps, we consequently arrive at the following conclusion. This outcome follows directly from the logical progression of the argument and the application of the relevant mathematical tools as before

$$E\left(\sup_{t_0\le t\le T}|u_1(t)|^2\right) \le 2G\left(\frac{1-M}{AB(M)}\right)^2\left(1+E|u_0|^2\right) + 2G\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^M\int_{t_0}^{t}\left(1+E|u_0|^2\right)d\tau, \tag{32}$$

$$E\left(\sup_{t_0\le t\le T}|u_1(t)|^2\right) \le 2G\left(\frac{1-M}{AB(M)}\right)^2\left(1+E|u_0|^2\right) + 2G\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^M\left(1+E|u_0|^2\right)(T-t_0). \tag{33}$$

We know that $u_0\in L^2([t_0,T],R)$, thus $E|u_0|^2<\infty$.
In that case

$$E\left(\sup_{t_0\le t\le T}|u_1(t)|^2\right)<\infty. \tag{34}$$

We make the assumption that $\forall_l\ge 1, y_l(t)\in L^2([t_0,T],R)$. It is necessary to prove that $y_{l+1}(t)\in L^2([t_0,T],R)$.

$$u_{l+1}(t) = \frac{1-\alpha(t)}{AB(\alpha(t))}f(t,u_l(t)) + \frac{\alpha(t)}{AB(\alpha(t))\Gamma(\alpha(t))}\int_{t_0}^{t}f(\tau,u_l(\tau))(t-\tau)^{\alpha(\tau)-1}d\tau, \tag{35}$$

$$u_{l+1}(t) \le \frac{1-M}{AB(M)}f(t,u_l(t)) + \frac{M}{AB(M)\Gamma(M)}\int_{t_0}^{t}f(\tau,u_l(\tau))(t-\tau)^{M-1}d\tau.$$

So we get

$$|u_{l+1}(t)|^2 \le 2G\left(\frac{1-M}{AB(M)}\right)^2\left(1+|u_l(t)|^2\right) + 2G\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^{\alpha(t)}\left(1+|u_l(\tau)|^2\right)d\tau, \tag{36}$$

$$E\left(\sup_{t_0\le t\le T}|u_{l+1}(t)|^2\right) \le 2G\left(\frac{1-M}{AB(M)}\right)^2\left(1+E\left(\sup_{t_0\le t\le T}|u_l(t)|^2\right)\right) + 2G\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^{\alpha(t)}\int_{t_0}^{t}\left(1+E\left(\sup_{t_0\le r\le\tau}|u_l(r)|^2\right)\right)d\tau. \tag{37}$$

By inductive hypothesis, $u_l(t) \in L^2([t_0,T],R)$ hence we have $E\left(\sup_{t_0\le t\le T}|u_l(t)|^2\right) \le \Omega$. So we obtain

$$E\left(\sup_{t_0\le t\le T}|u_{l+1}(t)|^2\right) \le 2G\,(1+\Omega)\left\{\left(\frac{1-M}{AB(M)}\right)^2+\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^{\alpha(t)}\,(T-t_0)\right\} < \infty. \tag{38}$$

Therefore we get

$$u_{l+1}(t) \in L^2([t_0,T],R). \tag{39}$$

According to the principles of induction, we can conclude that $\forall_l \ge 0$, $u_l(t) \in L^2([t_0,T],R)$. We now analyze,

$$\begin{aligned} E\left(|u_1(t)-u_0|^2\right) &\le 2G\left(\frac{1-M}{AB(M)}\right)^2\left(1+E\left(|u_0|^2\right)\right) \\ &+ 2\,(T-t_0)\,G\left(1+E\left(|u_0|^2\right)\right)\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^M, \end{aligned} \tag{40}$$

and

$$E\left(|u_1(t)-u_0|^2\right) \le \gamma_2, \tag{41}$$

where

$$\begin{aligned} \gamma_1 &= 2G\left(\frac{1-M}{AB(M)}\right)^2\left(1+E\left(|u_0|^2\right)\right) \\ &+ 2\,(T-t_0)\,G\left(1+E\left(|u_0|^2\right)\right)\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^{\alpha(t)}. \end{aligned} \tag{42}$$

Now $\forall_l \ge 1$,

$$E\left(\sup_{t_0\le t\le T}|u_1(t)|^2\right) \le \left(2G\left(1+E\,|u_0|^2\right)\right)\left\{\left(\frac{1-M}{AB(M)}\right)^2+\left(\frac{M}{AB(M)\Gamma(M)}\right)^2(T-t_0)\,I_t^{\alpha(t)}\right\}. \tag{43}$$

$$\begin{aligned} |u_{l+1}(t)-u_l(t)|^2 &\le 2\overline{G}\left(\frac{1-M}{AB(M)}\right)^2|u_l(t)-u_{l-1}(t)|^2 \\ &+ 2\overline{G}\left(\frac{M}{AB(M)\Gamma(M)}\right)^2 I_t^M\int_{t_0}^{t}|u_l(\tau)-u_{l-1}(\tau)|^2\,d\tau. \end{aligned} \tag{44}$$

Considering induction for $\forall_l \geq 0$, we get

$$E\left(\sup_{t_0\leq t\leq T}|u_{l+1}(t)-u_l(t)|^2\right)\leq \gamma_2\frac{(\beta_2(t-t_0))^l}{l!},\ t_0\leq t\leq T. \tag{45}$$

We assume that the inequality holds for all $l \geq 1$. We need to demonstrate its proof at t_{l+1}. At t_{l+1}, we have

$$\begin{aligned} E\left(\sup_{t_0\leq t\leq T}|u_{l+2}(t)-u_{l+1}(t)|^2\right) &\leq \gamma_2\frac{(\beta_2(t-t_0))^l}{(l+1)!}(t-t_0)\leq \gamma_2\frac{(\beta_2(t-t_0))^{l+1}}{(l+1)!}, \\ t_0 &\leq t\leq T. \end{aligned}$$

So at $l+1$, the inequality is proven through the principle of mathematical induction.

We can conclude that the Borel-Contelli lemma helps to find a positive integer number

$$l_0 = l_0(\varepsilon), \tag{46}$$

$\forall \varepsilon \in \Pi$ that

$$\sup_{t_0\leq t\leq T}|u_{l+1}(t)-u_l(t)|^2\leq \frac{1}{2^l},\ l\geq l_0. \tag{47}$$

It continues that the sum

$$u_0(t)+\sum_{k=0}^{l-1}[u_{k+1}(t)-u_k(t)] = u_l(t), \tag{48}$$

converges uniformly in $[0,T]$. Now, if we take

$$\lim_{n\to\infty} u_l(t) = u(t). \tag{49}$$

Therefore, we obtain

$$E|u_{l+1}(t)-u_l(t)|^2\leq \beta_2|u_l(t)-u(t)|^2. \tag{50}$$

Taking as $l\to\infty$, the right side of equality goes to zero, so we obtain

$$\begin{aligned} u(t) &= \frac{1-M}{AB(M)}f(t,u(t))+\frac{M}{AB(M)\Gamma(M)}\int_{t_0}^{t}f(\tau,u(\tau))(t-\tau)^{M-1}d\tau, \\ u(t) &= \frac{1-\alpha(t)}{AB(\alpha(t))}f(t,u(t))+\frac{\alpha(t)}{AB(\alpha(t))\Gamma(\alpha(t))}\int_{t_0}^{t}f(\tau,u(\tau))(t-\tau)^{\alpha(\tau)-1}d\tau. \end{aligned} \tag{51}$$

This completes the proof.

An illustrative application

In this subsection, we demonstrate the effectiveness of our approach by addressing a straightforward example of a Cauchy problem involving a variable-order fractional differential equation with the Atangana-Baleanu derivative. Let us consider the following Cauchy Problem

$$\begin{aligned} {}^{ABC}D^{\alpha(t)}u(t) &= \sin t + u(t)^2, \ t \in [0,T], \\ u(0) &= 0, \end{aligned} \tag{52}$$

where

$$\alpha(t) = 0.3 + 0.4 \sin t. \tag{53}$$

Let us check the conditions for which the Cauchy problems for variable-order fractional derivative with Atangana-Baleanu derivative has a unique equation if the nonlinear function $f(t, u(t))$ satisfies the following conditions.

1) We see that $\alpha(t) = 0.3 + 0.4 \sin t$, this function is differentiable and bounded for $t \in [0, T]$. It satisfies the condition $0.3 < \alpha\,(t) < 0.7$, thus

$$0 < m < \alpha\,(t) < M \le 1. \tag{54}$$

2) $\forall t \in [0,T]$, $\alpha'\,(t) = 0.4 \sin t \neq 0$.

3) We have that $f(t, u(t)) = \sin t + u(t)^2$. This function is a nonlinear function and twice differentiable and bounded.

4) Let's examine in this section if the function $f(t, u(t))$ meets the linear growth criterion specified by

$$|f(t,u(t))|^2 < G\left(1 + |u|^2\right). \tag{55}$$

Now we consider the norm $\|\varphi\|_\infty = \sup\limits_{t \in D_\varphi} |\varphi(t)|$ and put the existence and uniqueness of the solution for $[0, T]$. For $[0, T_1]$, there exist two positive constant U_1 and $U_2 < \infty$ such that

$$\begin{aligned} \|u_1\|_\infty &< U_1, \\ \|u_2\|_\infty &< U_2. \end{aligned} \tag{56}$$

Let's reconsider the right-hand side of the Cauchy problem equation and apply the necessary steps in order.

$$\begin{aligned} |f(t,u(t))|^2 &= \left|\sin t + u(t)^2\right|^2 \\ &\le \left|\sin^2 t\right| + 2\,|\sin t|\,|u(t)|^2 + |u(t)|^4 . \end{aligned} \tag{57}$$

Now we need to ensure that

$$\left|\sin^2 t\right| + 2\,|\sin t|\,|u(t)|^2 + |u(t)|^4 \le G\left(1 + |u(t)|^2\right). \tag{58}$$

Here we have to find a constant G to satisfy the condition.

$$\begin{aligned} |f(t,u(t))|^2 &\leq \left|\sin^2 t\right| + 2\left|\sin t\right| \left|u(t)^2\right| + \left|u(t)^4\right| \\ &\leq 1 + 2\left|u(t)^2\right| + \sup_{t\in[0,T]} \left|u(t)^4\right| \\ &\leq 1 + 2\left|u(t)^2\right| + \left\|u^4\right\|_\infty \\ &\leq \left(1+\left\|u^4\right\|_\infty\right)\left(1+\frac{2}{1+\|u^4\|_\infty}|u(t)|^2\right) \\ &\leq G\left(1+|u(t)|^2\right). \end{aligned} \tag{59}$$

Here

$$G = \max\left(1+\left\|u^4\right\|_\infty\right), \tag{60}$$

and under the conditions below

$$\max\left(\frac{2}{1+\|u^4\|_\infty}\right) \leq 1. \tag{61}$$

5) Let's check in this section whether the function $f(t,u(t))$ satisfies the Lipschitz condition is given by:

$$|f(t,u_1(t)) - f(t,u_2(t))|^2 \leq \overline{G}\,|u_1-u_2|^2. \tag{62}$$

Our goal is to obtain the above expression for our own equation.

$$\begin{aligned} |f(t,u_1(t)) - f(t,u_2(t))|^2 &= \left|u_1(t)^2 - u_2(t)^2\right|^2 \\ &\leq |u_1(t)+u_2(t)|^2\,|u_1(t)-u_2(t)|^2 \\ &\leq \left\{\begin{array}{c} \left|u_1(t)^2\right| + 2\,|u_1(t)|\,|u_2(t)| \\ +\left|u_2(t)^2\right| \end{array}\right\}|u_1(t)-u_2(t)|^2 \\ &\leq \left\{\begin{array}{c} \sup_{t\in[0,T]}\left|u_1(t)^2\right| \\ +2\sup_{t\in[0,T]}|u_1(t)|\sup_{t\in[0,T]}|u_2(t)| \\ +\sup_{t\in[0,T]}\left|u_2(t)^2\right| \end{array}\right\}|u_1(t)-u_2(t)|^2 \\ &\leq \left\{\begin{array}{c} \|u_1\|_\infty^4 + 2\,\|u_1\|_\infty\,\|u_2\|_\infty \\ +\|u_2\|_\infty^4 \end{array}\right\}|u_1(t)-u_2(t)|^2 \\ &\leq \overline{G}\,|u_1(t)-u_2(t)|^2, \end{aligned} \tag{63}$$

where

$$\overline{G} = \max\left\{\|u_1\|_\infty^4 + 2\,\|u_1\|_\infty\,\|u_2\|_\infty + \|u_2\|_\infty^4\right\}. \tag{64}$$

The solution to our Cauchy problem exists and is unique since all conditions are satisfied.

3 Conclusion

In this study, we have investigated the Cauchy problem for differential equations with variable-order derivatives and Mittag-Leffler kernels, emphasizing the significance of these mathematical tools in capturing complex, real-world phenomena. The investigations into the existence and uniqueness of solutions have demonstrated that variable-order derivatives, coupled with the Mittag-Leffler function, offer a powerful framework for extending traditional models to accommodate time-dependent and spatially-dependent memory effects. Our findings contribute to the growing body of knowledge in the field of variable-order systems, with potential applications spanning physics, engineering, finance, and beyond. Future research may focus on refining the stability analysis, developing more efficient numerical techniques, and exploring additional applications of these concepts in modeling complex dynamic systems.

Declarations

Use of AI tools

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

There are no external data associated with the manuscript.

Ethical approval (optional)

The author states that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

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References

[1] Atangana, A. *Fractional Operators with Constant and Variable Order with Application to Geo-hydrology*. Academic Press: United Kingdom, (2017).

[2] Goufo, E.F.D. and Atangana, A. Dynamics of traveling waves of variable order hyperbolic

Liouville equation: Regulation and control. *Discrete & Continuous Dynamical Systems-Series S*, 13(3), (2020).

[3] Khan, H., Alzabut, J., Gulzar, H., Tunç, O. and Pinelas, S. On system of variable order nonlinear p-Laplacian fractional differential equations with biological application. *Mathematics*, 11(8), 1913, (2023). [CrossRef]

[4] Zhuang, P., Liu, F., Anh, V. and Turner, I. Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term. *SIAM Journal on Numerical Analysis*, 47(3), 1760-1781, (2009). [CrossRef]

[5] Patnaik, S., Hollkamp, J.P. and Semperlotti, F. Applications of variable-order fractional operators: a review. *Proceedings of the Royal Society A*, 476(2234), 20190498, (2020). [CrossRef]

[6] Uçar, E., Uçar, S., Evirgen, F. and Özdemir, N. A fractional SAIDR model in the frame of Atangana–Baleanu derivative. *Fractal and Fractional*, 5(2), 32, (2021). [CrossRef]

[7] Uçar, E., Uçar, S., Evirgen, F. and Özdemir, N. Investigation of E-cigarette smoking model with mittag-leffler kernel. *Foundations of Computing and Decision Sciences*, 46(1), 97-109, (2021). [CrossRef]

[8] Venkatesh, A., Manivel, M. and Baranidharan, B. Numerical study of a new time-fractional Mpox model using Caputo fractional derivatives. *Physica Scripta*, 99(2), 025226, (2024). [CrossRef]

[9] Alkahtani, B.S.T. and Koca I. Fractional stochastic sır model. *Results in Physics*, 24, 104124, (2021). [CrossRef]

[10] Koca, I. Efficient numerical approach for solving fractional partial differential equations with non-singular kernel derivatives. *Chaos, Solitons & Fractals*, 116, 278-286, (2018). [CrossRef]

[11] Atangana, A. and Baleanu, D. New fractional derivatives with nonlocal and nonsingular kernel: theory and application to heat transfer model. *Thermal Science*, 20(2), 763-769, (2016). [CrossRef]

[12] Iwa, L.L., Omame, A. and Inyama, S.C. A fractional-order model of COVID-19 and Malaria co-infection. *Bulletin of Biomathematics*, 2(2), 133-161, (2024). [CrossRef]

[13] Din, A. and Abidin, M.Z. Analysis of fractional-order vaccinated Hepatitis-B epidemic model with Mittag-Leffler kernels. *Mathematical Modelling and Numerical Simulation with Applications*, 2(2), 59-72, (2022). [CrossRef]

[14] Sulaiman, T.A., Yavuz, M., Bulut, H. and Baskonus, H.M. Investigation of the fractional coupled viscous Burgers' equation involving Mittag-Leffler kernel. *Physica A: Statistical Mechanics and its Applications*, 527, 121126, (2019). [CrossRef]

[15] Atangana, A. and Koca, I. *Fractional Differential and Integral Operators with Respect to a Function: Theory, Methods and Applications*. Springer: Singapore, (2025).

[16] Atangana, A. Extension of rate of change concept: from local to nonlocal operators with applications. *Results in Physics*, 19, 103515, (2020). [CrossRef]

[17] Umarov, S. and Steinberg, S. Variable order differential equations with piecewise constant order-function and diffusion with changing modes. *Zeitschrift für Analysis und ihre Anwendungen*, 28(4), 431-450, (2009). [CrossRef]

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