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Corrected Euler-Maclaurin Type Inequalities for Differentiable s-Convex Functions

Saliha Djenaoui[1*] and Badreddine Meftah[2]

[1] *Laboratory of Analysis and Control of Differential Equations "ACED", Facuty MISM, Department of Mathematics, University of 8 May 1945 Guelma, P.O. Box 401, 24000 Guelma, Algeria*

[2] *LMAM, Département des Mathématiques, Faculté des mathématiques, de l'informatique et des sciences de la matière, Université 8 mai 1945 Guelma, Algeria.*

[*] *Corresponding author*

Abstract

Integral inequalities are generally applicable in many branches of mathematics such as real, complex, and numerical analysis, as well as in other disciplines outside mathematics. In this work, we first prove a new identity. Based on this equality, we establish some new corrected Euler-Maclaurin type inequalities for functions whose first derivatives are s-convex. The case where the first derivative is bounded as well as Lipschitzians are also discussed. Some applications to quadrature formulas and inequalities involving means are provided.

Keywords: *Lipschitzian functions, bounded functions, Corrected Euler-Maclaurin inequality, s-convex functions.*

2010 Mathematics Subject Classification: *Mathematics subject classification (2010): 26D10, 26D15, 26A51.*

1. Introduction

Definition 1.1 ([36]). *A function $\mathscr{C}: I \to \mathbb{R}$ is said to be convex, if*

$$\mathscr{C}(z\lambda + (1-z)\tau) \leq z\mathscr{C}(\lambda) + (1-z)\mathscr{C}(\tau)$$

holds for all $\lambda, \tau \in I$ and all $z \in [0,1]$.

The Hermite-Hadamard inequality, which is expressed as follows: for every convex function $\mathscr{C}$ on the interval $[\mathrm{V}, \mathrm{L}]$ with $\mathrm{V} < \mathrm{L}$

$$\mathscr{C}\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right) \leq \frac{1}{\mathrm{L}-\mathrm{V}} \int_{\mathrm{V}}^{\mathrm{L}} \mathscr{C}(u)\,du \leq \frac{\mathscr{C}(\mathrm{V})+\mathscr{C}(\mathrm{L})}{2}, \tag{1.1}$$

is without a doubt the fundamental inequality convex functions. In the other way, (1.1) holds in the reversed direction if the function $\mathscr{C}$ is concave (see [36]).

The above inequality has received renewed attention in past decades, several generalizations and refinements have been studied see [3, 11-14, 27-30, 42-44].

Convexity is a fundamental idea that is crucial to numerous fields, including game theory, economics, finance, and optimization. This notion has been expanded and developed in a number of ways as a result of its wide range of uses. We highlight the s-convexity among those generalizations, which has the following definition.

Definition 1.2 ([2]). *For every given $s \in (0,1]$, a nonnegative function $\mathscr{C}: I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second meaning, if*

$$\mathscr{C}(z\lambda + (1-z)\tau) \leq z^s\mathscr{C}(\lambda) + (1-z)^s\mathscr{C}(\tau)$$

holds for all $\lambda, \tau \in I$ and $z \in [0,1]$.

Email addresses: djenaoui.saliha@univ-guelma.dz (Saliha Djenaoui), badrimeftah@yahoo.fr (B. Meftah)

Convexity is closely related to the evolution of inequality theory, it represents a crucial instrument to study the characteristics of solutions of differential equations also allows obtaining estimates of error bounds of quadrature formulas. Regarding some articles dealing with some quadratures, see [1,4-10,16,18-26,31-35,37-41] and references therein.
The most popular three-point Newton-Cotes quadrature, called Simpson's inequality is given as follows:

$$\left|\frac{1}{6}\left(\mathscr{C}(\mathrm{V})+4\mathscr{C}\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)+\mathscr{C}(\mathrm{L})\right)-\frac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|\leq\frac{(\mathrm{L}-\mathrm{V})^4}{2880}\left\|\mathscr{C}^{(4)}\right\|_\infty,$$

where $\mathscr{C}$ is four-times continuously differentiable function on $[\mathrm{V},\mathrm{L}]$, and $\left\|\mathscr{C}^{(4)}\right\|_\infty=\sup_{x\in[\mathrm{V},\mathrm{L}]}\left|\mathscr{C}^{(4)}(x)\right|$.
In recent years, several authors have studied the estimates of the error limits of quadrature rules. Motivated and inspired by some papers, we plan to examine the corrected Euler-Maclaurin formula (see [17]), which can be stated as follows:

$$\left|\frac{1}{80}\left(27\mathscr{C}\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\frac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|\leq\frac{2401(\mathrm{L}-\mathrm{V})}{28800}\left\|\mathscr{C}'\right\|_\infty.$$

To this end, we first prove a new integral identity. Based on this identity, we establish several corrected Euler-Maclaurin inequalities for functions whose first derivatives are s-convex in the second sense. We also treat the case where the first derivative is bounded as well as Lipschitzian. In conclusion, some applications are presented.

2. Main results

In order to prove our results, we need the following lemma.

Lemma 2.1. *Let $\mathscr{C}: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $\mathrm{V},\mathrm{L}\in I^\circ$ with $\mathrm{V}<\mathrm{L}$, and $\mathscr{C}'\in L^1[\mathrm{V},\mathrm{L}]$. Then the following equality holds*

$$\frac{1}{80}\left(27\mathscr{C}\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\frac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du$$

$$=\frac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\mathscr{C}'\left((1-z)\mathrm{V}+z\frac{5\mathrm{V}+\mathrm{L}}{6}\right)dz+\int_0^1\left(4z-\frac{41}{20}\right)\mathscr{C}'\left((1-z)\frac{5\mathrm{V}+\mathrm{L}}{6}+z\frac{\mathrm{V}+\mathrm{L}}{2}\right)dz\right.$$

$$\left.+\int_0^1\left(4z-\frac{39}{20}\right)\mathscr{C}'\left((1-z)\frac{\mathrm{V}+\mathrm{L}}{2}+z\frac{\mathrm{V}+5\mathrm{L}}{6}\right)dz+\int_0^1(z-1)\mathscr{C}'\left((1-z)\frac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)dz\right).$$

Proof. Let

$$I_1=\int_0^1 z\mathscr{C}'\left((1-z)\mathrm{V}+z\frac{5\mathrm{V}+\mathrm{L}}{6}\right)dz,$$

$$I_2=\int_0^1\left(4z-\frac{41}{20}\right)\mathscr{C}'\left((1-z)\frac{5\mathrm{V}+\mathrm{L}}{6}+z\frac{\mathrm{V}+\mathrm{L}}{2}\right)dz,$$

$$I_3=\int_0^1\left(4z-\frac{39}{20}\right)\mathscr{C}'\left((1-z)\frac{\mathrm{V}+\mathrm{L}}{2}+z\frac{\mathrm{V}+5\mathrm{L}}{6}\right)dz$$

and

$$I_4=\int_0^1(z-1)\mathscr{C}'\left((1-z)\frac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)dz.$$

Integrating by parts I_1, we get

$$\begin{aligned}I_1 &= \frac{6}{\mathrm{L}-\mathrm{V}}z\mathscr{C}\left((1-z)\mathrm{V}+z\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\Bigg|_{z=0}^{z=1}-\frac{6}{\mathrm{L}-\mathrm{V}}\int_0^1\mathscr{C}\left((1-z)\mathrm{V}+z\frac{5\mathrm{V}+\mathrm{L}}{6}\right)dz\\ &= \frac{6}{\mathrm{L}-\mathrm{V}}\mathscr{C}\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)-\frac{6}{\mathrm{L}-\mathrm{V}}\int_0^1\mathscr{C}\left((1-z)\mathrm{V}+z\frac{5\mathrm{V}+\mathrm{L}}{6}\right)dz\\ &= \frac{6}{\mathrm{L}-\mathrm{V}}\mathscr{C}\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)-\frac{36}{(\mathrm{L}-\mathrm{V})^2}\int_{\mathrm{V}}^{\frac{5\mathrm{V}+\mathrm{L}}{6}}\mathscr{C}(u)\,du. \end{aligned}\tag{2.1}$$

Similarly, we have

$$\begin{aligned} I_2 &= \tfrac{3}{\mathrm{L}-\mathrm{V}}\left(4z-\tfrac{41}{20}\right)\mathscr{C}\left((1-z)\,\tfrac{5\mathrm{V}+\mathrm{L}}{6}+z\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\Big|_{z=0}^{z=1}-\tfrac{12}{\mathrm{L}-\mathrm{V}}\int_0^1\mathscr{C}\left((1-z)\,\tfrac{5\mathrm{V}+\mathrm{L}}{6}+z\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)dz \\ &= \tfrac{117}{20(\mathrm{L}-\mathrm{V})}\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+\tfrac{123}{20(\mathrm{L}-\mathrm{V})}\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)-\tfrac{36}{(\mathrm{L}-\mathrm{V})^2}\int_{\frac{5\mathrm{V}+\mathrm{L}}{6}}^{\frac{\mathrm{V}+\mathrm{L}}{2}}\mathscr{C}(u)\,du, \end{aligned} \tag{2.2}$$

$$\begin{aligned} I_3 &= \tfrac{3}{\mathrm{L}-\mathrm{V}}\left(4z-\tfrac{39}{20}\right)\mathscr{C}\left((1-z)\,\tfrac{\mathrm{V}+\mathrm{L}}{2}+z\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\Big|_{z=0}^{z=1}-\tfrac{12}{\mathrm{L}-\mathrm{V}}\int_0^1\mathscr{C}\left((1-z)\,\tfrac{\mathrm{V}+\mathrm{L}}{2}+z\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)dz \\ &= \tfrac{123}{20(\mathrm{L}-\mathrm{V})}\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)+\tfrac{117}{20(\mathrm{L}-\mathrm{V})}\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)-\tfrac{36}{(\mathrm{L}-\mathrm{V})^2}\int_{\frac{\mathrm{V}+\mathrm{L}}{2}}^{\frac{\mathrm{V}+5\mathrm{L}}{6}}\mathscr{C}(u)\,du \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} I_4 &= \tfrac{6}{\mathrm{L}-\mathrm{V}}(z-1)\,\mathscr{C}\left((1-z)\,\tfrac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)\Big|_{z=0}^{z=1}-\tfrac{6}{\mathrm{L}-\mathrm{V}}\int_0^1\mathscr{C}\left((1-z)\,\tfrac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)dz \\ &= \tfrac{6}{\mathrm{L}-\mathrm{V}}\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)-\tfrac{36}{(\mathrm{L}-\mathrm{V})^2}\int_{\frac{\mathrm{V}+5\mathrm{L}}{6}}^{\mathrm{L}}\mathscr{C}(u)\,du. \end{aligned} \tag{2.4}$$

Summing (2.1)-(2.4), and then multiplying the resulting equality by $\frac{\mathrm{L}-\mathrm{V}}{36}$, we get the desired result. □

Theorem 2.2. *Assume $\mathscr{C}:[\mathrm{V},\mathrm{L}]\to\mathbb{R}$ be a differentiable function with $0\leq\mathrm{V}<\mathrm{L}$ and $\mathscr{C}'\in L^1[\mathrm{V},\mathrm{L}]$. If $|\mathscr{C}'|$ is s-convex in the second sense for some fixed $s\in(0,1]$. Then we have*

$$\begin{aligned} &\left|\tfrac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\tfrac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right| \\ &\leq \tfrac{\mathrm{L}-\mathrm{V}}{36(s+1)(s+2)}\left(\left|\mathscr{C}'(\mathrm{V})\right|+\left|\mathscr{C}'(\mathrm{L})\right|+\left(\tfrac{39s-2}{10}+16\left(\tfrac{41}{80}\right)^{s+2}\right)\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|\right. \\ &\quad\left.+\left(\tfrac{61s+22}{20}+8\left(\tfrac{39}{80}\right)^{s+2}\right)\left(\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|+\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|\right)\right). \end{aligned}$$

Proof. From Lemma 1, properties of modulus and s-convexity in the second sense of $|\mathscr{C}'|$, we have

$$\begin{aligned} &\left|\tfrac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\tfrac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right| \\ &\leq \tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\left|\mathscr{C}'\left((1-z)\,\mathrm{V}+z\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|dz\right. \\ &\quad+\int_0^1\left|4z-\tfrac{41}{20}\right|\left|\mathscr{C}'\left((1-z)\,\tfrac{5\mathrm{V}+\mathrm{L}}{6}+z\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|dz \\ &\quad+\int_0^1\left|4z-\tfrac{39}{20}\right|\left|\mathscr{C}'\left((1-z)\,\tfrac{\mathrm{V}+\mathrm{L}}{2}+z\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|dz \\ &\quad\left.+\int_0^1(1-z)\left|\mathscr{C}'\left((1-z)\,\tfrac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)\right|dz\right) \\ &\leq \tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\left((1-z)^s\left|\mathscr{C}'(\mathrm{V})\right|+z^s\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|\right)dz\right. \\ &\quad+\int_0^{\frac{41}{80}}\left(\tfrac{41}{20}-4z\right)\left((1-z)^s\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|+z^s\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|\right)dz \end{aligned}$$

$$+\int_{\frac{41}{80}}^{1}\left(4z-\tfrac{41}{20}\right)\left((1-z)^s\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|+z^s\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|\right)dz$$

$$+\int_{0}^{\frac{39}{80}}\left(\tfrac{39}{20}-4z\right)\left((1-z)^s\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|+z^s\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|\right)dz$$

$$+\int_{\frac{39}{80}}^{1}\left(4z-\tfrac{39}{20}\right)\left((1-z)^s\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|+z^s\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|\right)dz$$

$$\left.+\int_{0}^{1}(1-z)\left((1-z)^s\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|+z^s\left|\mathscr{C}'(\mathrm{L})\right|\right)dz\right)$$

$$=\tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\left|\mathscr{C}'(\mathrm{V})\right|\int_0^1 z(1-z)^s\,dz+\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|\int_0^1 z^{s+1}dz\right.$$

$$+\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|\int_0^{\frac{41}{80}}\left(\tfrac{41}{20}-4z\right)(1-z)^s\,dz+\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|\int_0^{\frac{41}{80}}\left(\tfrac{41}{20}-4z\right)z^s dz$$

$$+\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|\int_{\frac{41}{80}}^{1}\left(4z-\tfrac{41}{20}\right)(1-z)^s\,dz+\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|\int_{\frac{41}{80}}^{1}\left(4z-\tfrac{41}{20}\right)z^s dz$$

$$+\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|\int_0^{\frac{39}{80}}\left(\tfrac{39}{20}-4z\right)(1-z)^s\,dz+\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|\int_0^{\frac{39}{80}}\left(\tfrac{39}{20}-4z\right)z^s dz$$

$$+\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|\int_{\frac{39}{80}}^{1}\left(4z-\tfrac{39}{20}\right)(1-z)^s\,dz+\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|\int_{\frac{39}{80}}^{1}\left(4z-\tfrac{39}{20}\right)z^s dz$$

$$\left.+\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|\int_0^1(1-z)^{s+1}\,dz+\left|\mathscr{C}'(\mathrm{L})\right|\int_0^1(1-z)z^s dz\right)$$

$$=\tfrac{\mathrm{L}-\mathrm{V}}{36(s+1)(s+2)}\left(\left|\mathscr{C}'(\mathrm{V})\right|+\left|\mathscr{C}'(\mathrm{L})\right|+\left(\tfrac{39s-2}{10}+16\left(\tfrac{41}{80}\right)^{s+2}\right)\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|\right.$$

$$\left.+\left(\tfrac{61s+22}{20}+8\left(\tfrac{39}{80}\right)^{s+2}\right)\left(\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|+\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|\right)\right),$$

where we have used the fact

$$\int_0^1 z(1-z)^s\,dz=\int_0^1(1-z)z^s dt=\tfrac{1}{(s+1)(s+2)},\tag{2.5}$$

$$\int_0^1 z^{s+1}dz=\int_0^1(1-z)^{s+1}\,dz=\tfrac{1}{s+2},\tag{2.6}$$

$$\int_0^{\frac{41}{80}}\left(\tfrac{41}{20}-4z\right)(1-z)^s\,dz=\int_{\frac{39}{80}}^{1}\left(4z-\tfrac{39}{20}\right)z^s dz=\tfrac{41s+2}{20(s+1)(s+2)}+\tfrac{4}{(s+1)(s+2)}\left(\tfrac{39}{80}\right)^{s+2},\tag{2.7}$$

$$\int_0^{\frac{41}{80}}\left(\tfrac{41}{20}-4z\right)z^s dz=\int_{\frac{39}{80}}^{1}\left(4z-\tfrac{39}{20}\right)(1-z)^s\,dz=\tfrac{4}{(s+1)(s+2)}\left(\tfrac{41}{80}\right)^{s+2},\tag{2.8}$$

$$\int_{\frac{41}{80}}^{1}\left(4z-\tfrac{41}{20}\right)(1-z)^s\,dz=\int_0^{\frac{39}{80}}\left(\tfrac{39}{20}-4z\right)z^s dz=\tfrac{4}{(s+1)(s+2)}\left(\tfrac{39}{80}\right)^{s+2}\tag{2.9}$$

and

$$\int_{\frac{41}{80}}^{1}\left(4z-\tfrac{41}{20}\right)z^{s}dz=\int_{0}^{\frac{39}{80}}\left(\tfrac{39}{20}-4z\right)(1-z)^{s}dz=\tfrac{39s-2}{20(s+1)(s+2)}+\tfrac{4}{(s+1)(s+2)}\left(\tfrac{41}{80}\right)^{s+2}. \tag{2.10}$$

The proof is finished. □

Corollary 2.3. *If we take $s=1$ in Theorem 1, we obtain*

$$\left|\tfrac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\tfrac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|$$

$$\leq\tfrac{2401(\mathrm{L}-\mathrm{V})}{28800}\left(\tfrac{64000|\mathscr{C}'(\mathrm{V})|+324919\left|\mathscr{C}'\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|+374642\left|\mathscr{C}'\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)\right|+324919\left|\mathscr{C}'\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|+64000|\mathscr{C}'(\mathrm{L})|}{1152480}\right).$$

Theorem 2.4. *Let f be as in Theorem 1. If $|\mathscr{C}'|^{q}$ is s-convex in the second sense for some fixed $s\in(0,1]$ where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then we have*

$$\left|\tfrac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\tfrac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|$$

$$\leq\tfrac{\mathrm{L}-\mathrm{V}}{36(p+1)^{\frac{1}{p}}}\left(\left(\tfrac{|\mathscr{C}'(\mathrm{V})|^{q}+\left|\mathscr{C}'\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{\left|\mathscr{C}'\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^{q}+|\mathscr{C}'(\mathrm{L})|^{q}}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\tfrac{1}{20}\left(\tfrac{39^{p+1}+41^{p+1}}{80}\right)^{\frac{1}{p}}\left(\left(\tfrac{\left|\mathscr{C}'\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^{q}+\left|\mathscr{C}'\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\tfrac{\left|\mathscr{C}'\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^{q}+\left|\mathscr{C}'\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\right)\right).$$

Proof. Lemma 1, properties of modulus, Hölder's inequality and s-convexity in the second sense of $|\mathscr{C}'|^{q}$, give us

$$\left|\tfrac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\tfrac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|$$

$$\leq\tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\left(\int_{0}^{1}z^{p}dz\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\mathscr{C}'\left((1-z)\mathrm{V}+z\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^{q}dz\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{0}^{1}\left|4z-\tfrac{41}{20}\right|^{p}dz\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\mathscr{C}'\left((1-z)\tfrac{5\mathrm{V}+\mathrm{L}}{6}+z\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^{q}dz\right)^{\frac{1}{q}}$$

$$+\left(\int_{0}^{1}\left|4z-\tfrac{39}{20}\right|^{p}dz\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+\mathrm{L}}{2}+z\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^{q}dz\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{0}^{1}(1-z)^{p}dz\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)\right|^{q}dz\right)^{\frac{1}{q}}\right)$$

$$\leq\tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\left(\int_{0}^{1}z^{p}dz\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left((1-z)^{s}|\mathscr{C}'(\mathrm{V})|^{q}+z^{s}\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^{q}\right)dz\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{0}^{\frac{41}{80}}\left(\tfrac{41}{20}-4z\right)^{p}dz+\int_{\frac{41}{80}}^{1}\left(4z-\tfrac{41}{20}\right)^{p}dz\right)^{\frac{1}{p}}$$

$$\times\left(\int_{0}^{1}\left((1-z)^{s}\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^{q}+z^{s}\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^{q}\right)dz\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{\frac{39}{80}}\left(\tfrac{39}{20}-4z\right)^p dz+\int_{\frac{39}{80}}^1\left(4z-\tfrac{39}{20}\right)^p dz\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left((1-z)^s\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q+z^s\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q\right)dz\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1(1-z)^p dz\right)^{\frac{1}{p}}\left(\int_0^1\left((1-z)^s\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q+z^s\left|\mathscr{C}'(\mathrm{L})\right|^q\right)dz\right)^{\frac{1}{q}}\Bigg)$$

$$=\frac{\mathrm{L}-\mathrm{V}}{36(p+1)^{\frac{1}{p}}}\left(\left(\frac{|\mathscr{C}'(\mathrm{V})|^q+\left|\mathscr{C}'\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathscr{C}'\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q+|\mathscr{C}'(\mathrm{L})|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\frac{1}{20}\left(\frac{39^{p+1}+41^{p+1}}{80}\right)^{\frac{1}{p}}\left(\left(\frac{\left|\mathscr{C}'\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q+\left|\mathscr{C}'\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\frac{\left|\mathscr{C}'\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q+\left|\mathscr{C}'\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right).$$

The proof is over. □

Corollary 2.5. *If we take $s=1$ in Theorem 2, we obtain*

$$\left|\frac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\frac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|$$

$$\leq\frac{\mathrm{L}-\mathrm{V}}{36(p+1)^{\frac{1}{p}}}\left(\left(\frac{|\mathscr{C}'(\mathrm{V})|^q+\left|\mathscr{C}'\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathscr{C}'\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q+|\mathscr{C}'(\mathrm{L})|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$+\frac{1}{20}\left(\frac{39^{p+1}+41^{p+1}}{80}\right)^{\frac{1}{p}}\left(\left(\frac{\left|\mathscr{C}'\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q+\left|\mathscr{C}'\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\frac{\left|\mathscr{C}'\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q+\left|\mathscr{C}'\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right)\right).$$

Theorem 2.6. *Let f be as in Theorem 1. If $|\mathscr{C}'|^q$ is s-convex in the second sense for some fixed $s\in(0,1]$ where $q\geq 1$. Then we have*

$$\left|\frac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\frac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|$$

$$\leq\frac{\mathrm{L}-\mathrm{V}}{36}\left(\left(\tfrac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+1)(s+2)}|\mathscr{C}'(\mathrm{V})|^q+\frac{1}{s+2}\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\tfrac{1601}{1600}\right)^{1-\frac{1}{q}}\left(\left(\frac{41s+2}{20(s+1)(s+2)}+\frac{8}{(s+1)(s+2)}\left(\tfrac{39}{80}\right)^{s+2}\right)\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q\right.$$

$$\left.+\left(\frac{39s-2}{20(s+1)(s+2)}+\frac{8}{(s+1)(s+2)}\left(\tfrac{41}{80}\right)^{s+2}\right)\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{1601}{1600}\right)^{1-\frac{1}{q}}\left(\left(\frac{39s-2}{20(s+1)(s+2)}+\frac{8}{(s+1)(s+2)}\left(\tfrac{41}{80}\right)^{s+2}\right)\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q\right.$$

$$\left.+\left(\frac{41s+2}{20(s+1)(s+2)}+\frac{8}{(s+1)(s+2)}\left(\tfrac{39}{80}\right)^{s+2}\right)\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q\right)^{\frac{1}{q}}$$

$$\left.+\left(\tfrac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+2}\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q+\frac{1}{(s+1)(s+2)}|\mathscr{C}'(\mathrm{L})|^q\right)^{\frac{1}{q}}\right).$$

Proof. Using Lemma 1, properties of modulus, power mean inequality and s-convexity in the second sense of $|\mathscr{C}'|^q$, we get

$$\left|\frac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\frac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|$$

$$\leq\frac{\mathrm{L}-\mathrm{V}}{36}\left(\left(\int_0^1 z\,dz\right)^{1-\frac{1}{q}}\left(\int_0^1 z\left|\mathscr{C}'\left((1-z)\mathrm{V}+z\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q dz\right)^{\frac{1}{q}}\right.$$

$$
\begin{aligned}
&+\left(\int_0^1\left|4z-\tfrac{41}{20}\right|dz\right)^{1-\frac{1}{q}}\left(\int_0^1\left|4z-\tfrac{41}{20}\right|\left|\mathscr{C}'\left((1-z)\tfrac{5\mathrm{V}+\mathrm{L}}{6}+z\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q dz\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|4z-\tfrac{39}{20}\right|dz\right)^{1-\frac{1}{q}}\left(\int_0^1\left|4z-\tfrac{39}{20}\right|\left|\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+\mathrm{L}}{2}+z\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q dz\right)^{\frac{1}{q}}\\
&+\left(\int_0^1(1-z)\,dz\right)^{1-\frac{1}{q}}\left(\int_0^1(1-z)\left|\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+5\mathrm{L}}{6}+t\mathrm{L}\right)\right|^q dz\right)^{\frac{1}{q}}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\leq\;& \tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\left(\int_0^1 z\,dz\right)^{1-\frac{1}{q}}\left(\left|\mathscr{C}'(\mathrm{V})\right|^q\int_0^1 z(1-z)^s dz+\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q\int_0^1 z^{s+1}dz\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^{\frac{41}{80}}\left(\tfrac{41}{20}-4z\right)dz+\int_{\frac{41}{80}}^1\left(4z-\tfrac{41}{20}\right)dz\right)^{1-\frac{1}{q}}\\
&\times\left(\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q\left(\int_0^{\frac{41}{80}}\left(\tfrac{41}{20}-4z\right)(1-z)^s dz+\int_{\frac{41}{80}}^1\left(4z-\tfrac{41}{20}\right)(1-z)^s dz\right)\right.\\
&\left.+\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q\left(\int_0^{\frac{41}{80}}\left(\tfrac{41}{20}-4z\right)z^s dz+\int_{\frac{41}{80}}^1\left(4z-\tfrac{41}{20}\right)z^s dz\right)\right)^{\frac{1}{q}}\\
&+\left(\int_0^{\frac{39}{80}}\left(\tfrac{39}{20}-4z\right)dz+\int_{\frac{39}{80}}^1\left(4z-\tfrac{39}{20}\right)dz\right)^{1-\frac{1}{q}}\\
&\times\left(\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q\left(\int_0^{\frac{39}{80}}\left(\tfrac{39}{20}-4z\right)(1-z)^s dz+\int_{\frac{39}{80}}^1\left(4z-\tfrac{39}{20}\right)(1-z)^s dz\right)\right.\\
&\left.+\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q\left(\int_0^{\frac{39}{80}}\left(\tfrac{39}{20}-4z\right)z^s dz+\int_{\frac{39}{80}}^1\left(4z-\tfrac{39}{20}\right)z^s dz\right)\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1(1-z)\,dz\right)^{1-\frac{1}{q}}\left(\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q\int_0^1(1-z)^{s+1}dz+\left|\mathscr{C}'(\mathrm{L})\right|^q\int_0^1(1-z)z^s dz\right)^{\frac{1}{q}}\right)\\
=\;& \tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\left(\tfrac{1}{2}\right)^{1-\frac{1}{q}}\left(\tfrac{1}{(s+1)(s+2)}\left|\mathscr{C}'(\mathrm{V})\right|^q+\tfrac{1}{s+2}\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&+\left(\tfrac{1601}{1600}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{41s+2}{20(s+1)(s+2)}+\tfrac{8}{(s+1)(s+2)}\left(\tfrac{39}{80}\right)^{s+2}\right)\left|\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q\right.\\
&\left.+\left(\tfrac{39s-2}{20(s+1)(s+2)}+\tfrac{8}{(s+1)(s+2)}\left(\tfrac{41}{80}\right)^{s+2}\right)\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\tfrac{1601}{1600}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{39s-2}{20(s+1)(s+2)}+\tfrac{8}{(s+1)(s+2)}\left(\tfrac{41}{80}\right)^{s+2}\right)\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q\right.\\
&\left.+\left(\tfrac{41s+2}{20(s+1)(s+2)}+\tfrac{8}{(s+1)(s+2)}\left(\tfrac{39}{80}\right)^{s+2}\right)\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q\right)^{\frac{1}{q}}\\
&\left.+\left(\tfrac{1}{2}\right)^{1-\frac{1}{q}}\left(\tfrac{1}{s+2}\left|\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q+\tfrac{1}{(s+1)(s+2)}\left|\mathscr{C}'(\mathrm{L})\right|^q\right)^{\frac{1}{q}}\right),
\end{aligned}
$$

where we used (2.5)-(2.10) and the fact that

$$\int_0^1\left|4z-\tfrac{41}{20}\right|dz=\int_0^1\left|4z-\tfrac{39}{20}\right|dz=\tfrac{1601}{1600}.$$

The proof is acquired. □

Corollary 2.7. *In Theorem 3, if we take $s=1$. Then we get*

$$\left|\frac{1}{80}\left(27\mathscr{C}\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\frac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|$$

$$\leq \frac{\mathrm{L}-\mathrm{V}}{72}\left(\left(\frac{|\mathscr{C}'(\mathrm{V})|^q+2\left|\mathscr{C}'\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|\mathscr{C}'\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q+|\mathscr{C}'(\mathrm{L})|^q}{3}\right)^{\frac{1}{q}}\right.$$

$$+\frac{1601}{800}\left(\frac{196919\left|\mathscr{C}'\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|^q+187321\left|\mathscr{C}'\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q}{384240}\right)^{\frac{1}{q}}$$

$$\left.+\frac{1601}{800}\left(\frac{187321\left|\mathscr{C}'\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)\right|^q+196919\left|\mathscr{C}'\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|^q}{384240}\right)^{\frac{1}{q}}\right).$$

3. Further results

Theorem 3.1. *Assume that $\mathscr{C}:[\mathrm{V},\mathrm{L}]\to\mathbb{R}$ is a differentiable function on $[\mathrm{V},\mathrm{L}]$ with $\mathrm{V}<\mathrm{L}$ and $\mathscr{C}'\in L^1[\mathrm{V},\mathrm{L}]$. If there exist constants $m,M\in\mathbb{R}$ with $\infty<m<M<+\infty$ such that, for every $x\in[\mathrm{V},\mathrm{L}]$, $m\leq\mathscr{C}'(x)\leq M$. Then we have*

$$\left|\frac{1}{80}\left(27\mathscr{C}\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\frac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|\leq\frac{2401(\mathrm{L}-\mathrm{V})(M-m)}{57600}.$$

Proof. From Lemma 1, we have

$$\frac{1}{80}\left(27\mathscr{C}\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\frac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du$$

$$=\frac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\left(\mathscr{C}'\left((1-z)\mathrm{V}+z\frac{5\mathrm{V}+\mathrm{L}}{6}\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)dz\right.$$

$$+\int_0^1\left(4z-\frac{41}{20}\right)\left(\mathscr{C}'\left((1-z)\frac{5\mathrm{V}+\mathrm{L}}{6}+z\frac{\mathrm{V}+\mathrm{L}}{2}\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)dz$$

$$+\int_0^1\left(4z-\frac{39}{20}\right)\left(\mathscr{C}'\left((1-z)\frac{\mathrm{V}+\mathrm{L}}{2}+z\frac{\mathrm{V}+5\mathrm{L}}{6}\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)dz$$

$$\left.+\int_0^1(z-1)\left(\mathscr{C}'\left((1-z)\frac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)dz\right)$$

$$=\frac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\left(\mathscr{C}'\left((1-z)\mathrm{V}+z\frac{5\mathrm{V}+\mathrm{L}}{6}\right)-\frac{m+M}{2}\right)dz+\frac{m+M}{2}\int_0^1 z\,dz\right.$$

$$+\int_0^1\left(4z-\frac{41}{20}\right)\left(\mathscr{C}'\left((1-z)\frac{5\mathrm{V}+\mathrm{L}}{6}+z\frac{\mathrm{V}+\mathrm{L}}{2}\right)-\frac{m+M}{2}\right)dz+\frac{m+M}{2}\int_0^1\left(4z-\frac{41}{20}\right)dz$$

$$+\int_0^1\left(4z-\frac{39}{20}\right)\left(\mathscr{C}'\left((1-z)\frac{\mathrm{V}+\mathrm{L}}{2}+z\frac{\mathrm{V}+5\mathrm{L}}{6}\right)-\frac{m+M}{2}\right)dz+\frac{m+M}{2}\int_0^1\left(4z-\frac{39}{20}\right)dz$$

$$\left.+\int_0^1(z-1)\left(\mathscr{C}'\left((1-z)\frac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)-\frac{m+M}{2}\right)dz+\frac{m+M}{2}\int_0^1(z-1)\,dz\right)$$

$$=\frac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\left(\mathscr{C}'\left((1-z)\mathrm{V}+z\frac{5\mathrm{V}+\mathrm{L}}{6}\right)-\frac{m+M}{2}\right)dz\right.$$

$$+\int_0^1\left(4z-\frac{41}{20}\right)\left(\mathscr{C}'\left((1-z)\frac{5\mathrm{V}+\mathrm{L}}{6}+z\frac{\mathrm{V}+\mathrm{L}}{2}\right)-\frac{m+M}{2}\right)dz$$

$$+\int_0^1\left(4z-\frac{39}{20}\right)\left(\mathscr{C}'\left((1-z)\frac{\mathrm{V}+\mathrm{L}}{2}+z\frac{\mathrm{V}+5\mathrm{L}}{6}\right)-\frac{m+M}{2}\right)dz$$

$$+\int_0^1 (z-1)\left(\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)-\tfrac{m+M}{2}\right)dz$$
$$+\tfrac{m+M}{2}\left(\int_0^1 zdz+\int_0^1\left(4z-\tfrac{41}{20}\right)dz+\int_0^1\left(4z-\tfrac{39}{20}\right)dz+\int_0^1 (z-1)\,dz\right)\Bigg)$$
$$\begin{aligned}
= \; & \tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\left(\mathscr{C}'\left((1-z)\,\mathrm{V}+z\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)-\tfrac{m+M}{2}\right)dz\right. \\
& +\int_0^1\left(4z-\tfrac{41}{20}\right)\left(\mathscr{C}'\left((1-z)\tfrac{5\mathrm{V}+\mathrm{L}}{6}+z\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)-\tfrac{m+M}{2}\right)dz \\
& +\int_0^1\left(4z-\tfrac{39}{20}\right)\left(\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+\mathrm{L}}{2}+z\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)-\tfrac{m+M}{2}\right)dz \\
& \left.+\int_0^1 (z-1)\left(\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)-\tfrac{m+M}{2}\right)dz\right), \qquad (3.1)
\end{aligned}$$

where we have used the fact

$$\int_0^1 zdz+\int_0^1\left(4z-\tfrac{41}{20}\right)dz+\int_0^1\left(4z-\tfrac{39}{20}\right)dz+\int_0^1 (z-1)\,dz=0.$$

Applying the absolute value in both sides of (3.1), we get

$$\begin{aligned}
& \left|\tfrac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\tfrac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right| \\
\leq \; & \tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\left|\mathscr{C}'\left((1-z)\,\mathrm{V}+z\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)-\tfrac{m+M}{2}\right|dz\right. \\
& +\int_0^1\left|4z-\tfrac{41}{20}\right|\left|\mathscr{C}'\left((1-z)\tfrac{5\mathrm{V}+\mathrm{L}}{6}+z\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)-\tfrac{m+M}{2}\right|dz \\
& +\int_0^1\left(\left|4z-\tfrac{39}{20}\right|\right)\left|\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+\mathrm{L}}{2}+z\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)-\tfrac{m+M}{2}\right|dz \\
& \left.+\int_0^1 (1-z)\left|\mathscr{C}'\left((1-t)\tfrac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)-\tfrac{m+M}{2}\right|dz\right), \qquad (3.2)
\end{aligned}$$

Since $m\leq\mathscr{C}'(x)\leq M$ for all $x\in[\mathrm{V},\mathrm{L}]$, we have

$$\left|\mathscr{C}'\left((1-z)\,\mathrm{V}+z\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)-\tfrac{m+M}{2}\right|\leq\tfrac{M-m}{2}, \qquad (3.3)$$

$$\left|\mathscr{C}'\left((1-z)\tfrac{5\mathrm{V}+\mathrm{L}}{6}+z\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)-\tfrac{m+M}{2}\right|\leq\tfrac{M-m}{2}, \qquad (3.4)$$

$$\left|\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+\mathrm{L}}{2}+z\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)-\tfrac{m+M}{2}\right|\leq\tfrac{M-m}{2} \qquad (3.5)$$

and

$$\left|\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)-\tfrac{m+M}{2}\right|\leq\tfrac{M-m}{2}. \qquad (3.6)$$

Using (3.3)-(3.6) in (3.2) we get

$$\begin{aligned}
& \left|\tfrac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\tfrac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right| \\
\leq \; & \tfrac{(\mathrm{L}-\mathrm{V})(M-m)}{72}\left(\int_0^1 zdz+\int_0^1\left|4z-\tfrac{41}{20}\right|dz+\int_0^1\left|4z-\tfrac{39}{20}\right|dz+\int_0^1 (1-z)\,dz\right) \\
= \; & \tfrac{2401(\mathrm{L}-\mathrm{V})(M-m)}{57600},
\end{aligned}$$

which is the desired result. □

Theorem 3.2. *Assume that* $\mathscr{C}:[\mathrm{V},\mathrm{L}]\to\mathbb{R}$ *be a differentiable function on* $[\mathrm{V},\mathrm{L}]$ *with* $\mathrm{V}<\mathrm{L}$ *and* $\mathscr{C}'\in L^1[\mathrm{V},\mathrm{L}]$. *The fact that* $\mathscr{C}'$ *is L-Lipschitzian function on* $[\mathrm{V},\mathrm{L}]$, *gives us*

$$\left|\tfrac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\tfrac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|\leq\tfrac{4081(\mathrm{L}-\mathrm{V})^2}{172800}L.$$

Proof. From Lemma 1, we have

$$\begin{aligned}
&\tfrac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\tfrac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\\
=&\ \tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\left(\mathscr{C}'\left((1-z)\mathrm{V}+z\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)-\mathscr{C}'(\mathrm{V})+\mathscr{C}'(\mathrm{V})\right)dz\right.\\
&+\int_0^1\left(4z-\tfrac{41}{20}\right)\left(\mathscr{C}'\left((1-z)\tfrac{5\mathrm{V}+\mathrm{L}}{6}+z\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)-\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right)dz\\
&+\int_0^1\left(4z-\tfrac{39}{20}\right)\left(\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+\mathrm{L}}{2}+z\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)-\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right)dz\\
&\left.+\int_0^1(z-1)\left(\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)-\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)+\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)dz\right)\\
=&\ \tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\left(\mathscr{C}'\left((1-z)\mathrm{V}+z\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)-\mathscr{C}'(\mathrm{V})\right)dz+\mathscr{C}'(\mathrm{V})\int_0^1 z\,dz\right.\\
&+\int_0^1\left(4z-\tfrac{41}{20}\right)\left(\mathscr{C}'\left((1-z)\tfrac{5\mathrm{V}+\mathrm{L}}{6}+z\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)-\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right)dz\\
&+\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\int_0^1\left(4z-\tfrac{41}{20}\right)dz\\
&+\int_0^1\left(4z-\tfrac{39}{20}\right)\left(\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+\mathrm{L}}{2}+z\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)-\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right)dz\\
&+\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\int_0^1\left(4z-\tfrac{39}{20}\right)dz\\
&+\int_0^1(z-1)\left(\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)-\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)dz\\
&\left.+\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\int_0^1(z-1)\,dz\right)\\
&=\tfrac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\left(\mathscr{C}'\left((1-z)\mathrm{V}+z\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)-\mathscr{C}'(\mathrm{V})\right)dz\right.\\
&+\int_0^1\left(4z-\tfrac{41}{20}\right)\left(\mathscr{C}'\left((1-z)\tfrac{5\mathrm{V}+\mathrm{L}}{6}+z\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)-\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right)dz\\
&+\int_0^1\left(4z-\tfrac{39}{20}\right)\left(\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+\mathrm{L}}{2}+z\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)-\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)\right)dz\\
&+\int_0^1(z-1)\left(\mathscr{C}'\left((1-z)\tfrac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)-\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)dz\\
&\left.+\tfrac{1}{2}\left(\mathscr{C}'(\mathrm{V})-\mathscr{C}'\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)+\tfrac{1}{20}\left(\mathscr{C}'\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)-\mathscr{C}'\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)\right)\right).
\end{aligned}\tag{3.7}$$

By applying the absolute value in both sides of (3.7), and using the fact that $\mathscr{C}'$ is L-Lipschitzian function it yields

$$\left|\tfrac{1}{80}\left(27\mathscr{C}\left(\tfrac{5\mathrm{V}+\mathrm{L}}{6}\right)+26\mathscr{C}\left(\tfrac{\mathrm{V}+\mathrm{L}}{2}\right)+27\mathscr{C}\left(\tfrac{\mathrm{V}+5\mathrm{L}}{6}\right)\right)-\tfrac{1}{\mathrm{L}-\mathrm{V}}\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du\right|$$

$$\begin{aligned}
&\leq \frac{\mathrm{L}-\mathrm{V}}{36}\left(\int_0^1 z\left|\mathscr{C}'\left((1-z)\mathrm{V}+z\frac{5\mathrm{V}+\mathrm{L}}{6}\right)-\mathscr{C}'(\mathrm{V})\right|dz\right.\\
&+\int_0^1\left|4z-\frac{41}{20}\right|\left|\mathscr{C}'\left((1-z)\frac{5\mathrm{V}+\mathrm{L}}{6}+z\frac{\mathrm{V}+\mathrm{L}}{2}\right)-\mathscr{C}'\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|dz\\
&+\int_0^1\left|4z-\frac{39}{20}\right|\left|\mathscr{C}'\left((1-z)\frac{\mathrm{V}+\mathrm{L}}{2}+z\frac{\mathrm{V}+5\mathrm{L}}{6}\right)-\mathscr{C}'\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)\right|dz\\
&+\int_0^1(1-z)\left|\mathscr{C}'\left((1-z)\frac{\mathrm{V}+5\mathrm{L}}{6}+z\mathrm{L}\right)-\mathscr{C}'\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|dz\\
&\left.+\frac{1}{2}\left|\mathscr{C}'(\mathrm{V})-\mathscr{C}'\left(\frac{\mathrm{V}+5\mathrm{L}}{6}\right)\right|+\frac{1}{20}\left|\mathscr{C}'\left(\frac{\mathrm{V}+\mathrm{L}}{2}\right)-\mathscr{C}'\left(\frac{5\mathrm{V}+\mathrm{L}}{6}\right)\right|\right)\\
&\leq \frac{(\mathrm{L}-\mathrm{V})^2}{216}L\left(\int_0^1 z^2dz+2\int_0^1\left|4z-\frac{41}{20}\right|zdz+2\int_0^1\left|4z-\frac{39}{20}\right|zdz+\int_0^1(1-z)zdz+\frac{5}{2}+\frac{1}{10}\right)\\
&= \frac{4081(\mathrm{L}-\mathrm{V})^2}{172800}L,
\end{aligned}$$

which is the desired result. □

4. Applications

In this section, we consider some applications using the correct Euler-Maclaurin formula as well as an application using means for arbitrary real numbers.

Let Υ be the partition of the interval $[\mathrm{V},\mathrm{L}]$ such that $\mathrm{V}=x_0<x_1<\ldots<x_n=\mathrm{L}$, and take the quadrature formula into consideration.

$$\int_{\mathrm{V}}^{\mathrm{L}}\mathscr{C}(u)\,du=\lambda(\mathscr{C},\Upsilon)+R(\mathscr{C},\Upsilon),$$

where

$$\lambda(\mathscr{C},\Upsilon)=\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{80}\left(27\mathscr{C}\left(\frac{5x_i+x_{i+1}}{6}\right)+26\mathscr{C}\left(\frac{x_i+x_{i+1}}{2}\right)+27\mathscr{C}\left(\frac{x_i+5x_{i+1}}{6}\right)\right)$$

and the accompanying approximation error is shown by $R(\mathscr{C},\Upsilon)$.

Proposition 4.1. *Assume that $\mathscr{C}:[\mathrm{V},\mathrm{L}]\to\mathbb{R}$ is a differentiable function on $[\mathrm{V},\mathrm{L}]$ with $\mathscr{C}'\in L^1[\mathrm{V},\mathrm{L}]$ and $n\in\mathbb{N}$. If $|\mathscr{C}'|$ is an s-convex function in the second sense for a given $s\in(0,1]$, we have*

$$\begin{aligned}
&|R(\mathscr{C},\Upsilon)|\\
&\leq \sum_{i=0}^{n-1}\frac{2401(x_{i+1}-x_i)^2}{28800}\left(\frac{64000}{1152480}\left(|\mathscr{C}'(x_i)|+|\mathscr{C}'(x_{i+1})|\right)+\frac{374642}{1152480}\left|\mathscr{C}'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right.\\
&\left.+\frac{324919}{1152480}\left(\left|\mathscr{C}'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|+\left|\mathscr{C}'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|\right)\right).
\end{aligned}$$

Proof. Applying Corollary 1 on the subintervals $[x_i,x_{i+1}]$ $(i=0,1,\ldots,n-1)$ of the partition Υ, we get

$$\begin{aligned}
&\left|\frac{1}{80}\left(27\mathscr{C}\left(\frac{5x_i+x_{i+1}}{6}\right)+26\mathscr{C}\left(\frac{x_i+x_{i+1}}{2}\right)+27\mathscr{C}\left(\frac{x_i+5x_{i+1}}{6}\right)\right)-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\mathscr{C}(u)\,du\right|\\
&\leq \frac{2401(x_{i+1}-x_i)}{28800}\left(\frac{64000}{1152480}\left(|\mathscr{C}'(x_i)|+|\mathscr{C}'(x_{i+1})|\right)+\frac{374642}{1152480}\left|\mathscr{C}'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right.\\
&\left.+\frac{324919}{1152480}\left(\left|\mathscr{C}'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|+\left|\mathscr{C}'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|\right)\right). \quad (4.1)
\end{aligned}$$

The required outcome is produced by multiplying both sides of (4.1) by $(x_{i+1}-x_i)$, adding the resulting inequalities for all $i=0,1,\ldots,n-1$, and applying the triangular inequality. □

Application to special means

For arbitrary real numbers $\mathrm{V},\mathrm{V}_1,\mathrm{V}_2,\ldots,\mathrm{V}_n,\mathrm{L}$ we have:

The Arithmetic mean: $A(\mathrm{V}_1,\mathrm{V}_2,\ldots,\mathrm{V}_n)=\frac{\mathrm{V}_1+\mathrm{V}_2+\ldots+\mathrm{V}_n}{n}$.

The p-Logarithmic mean: $L_p(\mathrm{V},\mathrm{L})=\left(\frac{\mathrm{L}^{p+1}-\mathrm{V}^{p+1}}{(p+1)(\mathrm{L}-\mathrm{V})}\right)^{\frac{1}{p}}$, $\mathrm{V},\mathrm{L}>0$, $\mathrm{V}\neq\mathrm{L}$ and $p\in\mathbb{R}\setminus\{-1,0\}$.

Proposition 4.2. *Let $\mathrm{V},\mathrm{L}\in\mathbb{R}$ with $0<\mathrm{V}<\mathrm{L}$, then we have*

$$\begin{aligned}
&\left|27A^2(\mathrm{V},\mathrm{V},\mathrm{V},\mathrm{V},\mathrm{V},\mathrm{L})+26A^2(\mathrm{V},\mathrm{L})+27A^2(\mathrm{V},\mathrm{L},\mathrm{L},\mathrm{L},\mathrm{L},\mathrm{L})-80L_2^2(\mathrm{V},\mathrm{L})\right|\\
&\leq \frac{2401(\mathrm{L}-\mathrm{V})^2}{360}.
\end{aligned}$$

Proof. When Theorem 4 is applied to the function $\mathscr{C}(x)=x^2$, the claim is obtained. □

5. Conclusion

In this study, we have constructed a new identity. Based on this equality, we have established some corrected Euler-Maclaurin inequalities for functions whose first derivatives are s-convex, bounded, and L-Lipschitzian. Some special cases have been discussed. Applications to numerical integration as well as to inequalities involving means have been presented. We hope that our results will be useful and stimulate researchers to explore this area and to study and develop other types of inequalities.

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