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Research Article

Sub-fractional G-Brownian motion: Properties and simulations

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Abstract

In this article, we introduce a new stochastic process called the sub-fractional G -Brownian motion, which serves as an intermediate between the G -Brownian motion and the fractional G -Brownian motion. Although the sub-fractional G-Brownian motion shares some properties with the fractional G-Brownian motion, it features nonstationary increments. We then examine key characteristics of the process, such as self-similarity, Hölder continuity, and long-range dependence. Additionally, we propose a method for simulating sample paths of sub-fractional G-Brownian motion and conclude by simulating linear stochastic differential equations driven by sub-fractional G-Brownian motion.

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1. Introduction

Modelling the evolution of stochastic systems has become extremely important in many fields, such as telecommunication networks, finance, turbulence, etc. In recent decades, it has been frequently assumed that stochastic processes used in modeling are Markovian. However, this assumption is not always valid, as many studies have shown that real data exhibit long-range dependence, that is, the state of the process at a given time t depends not only on the situation at time t, but also on the entire history up to time t. This property cannot be neglected, as it significantly affects the expected behavior of the system. The most widely used process that exhibits long-range dependence is the fractional Brownian motion (fBm), see [3, 14].

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Peng [11] developed the G-framework for risk measures and pricing under uncertainty. Therefore, the fractional G-Brownian motion (fGBm) was defined by Chen [17] as a centered G-Gaussian process with stationary increments, self-similarity, and long-range dependence properties. Since then many stochastic systems driven by fGBm have been introduced [7]. However, the environments we deal with are not always stable and change at all times, making stationary increments property in real data seldom. Obviously, the use of inappropriate models leads to inaccuracies in solutions and, subsequently, in optimal control tasks. Therefore, a stochastic process with non-stationary increments is needed that preserves the other key properties to enhance the fineness of the stochastic model.

Motivated by the above discussion, we aim to define a new stochastic process that retains the main properties of the fGBm but features non-stationary increments. This process is a centered G-Gaussian process with self-similarity and long-range dependence properties, which we refer to as a "sub-fractional G-Brownian motion" (sfGBm). One example that motivates our work is climate modeling, where mathematical models are used to simulate and predict the behavior of the Earth's climate system. These models help to understand past and present climate conditions and predict future climate change under different scenarios. This field is characterized by inherent uncertainties due to the complexity of the climate system, limited observational data, and imperfect representations of physical processes. One of the challenges in climate modeling is accurately representing long-term dependencies and self-similar behaviors observed in climate variables such as temperature, precipitation, and atmospheric pressure [4, 10]. These characteristics are essential for capturing the persistence of anomalies (e.g. heat waves or cold spells) and the memory effect of past climatic events on future conditions. Traditional stochastic models such as fractional Gaussian processes have been used to address these features but are limited by their assumption of stationary increments, which does not always align with real-world climate data. In addition, climate systems are influenced by uncertainties that arise from multiple sources, including variability in external forcings (e.g., greenhouse gas emissions), incomplete knowledge of internal dynamics, and measurement errors.

Classical stochastic frameworks struggle to incorporate such uncertainties comprehensively, often leading to oversimplified models. To address these challenges, we propose the sub-fractional G-Brownian motion (sfGBm) as a novel tool in modeling. For example, climate data sets are often non-stationary due to natural variability such as El Niño and La Niña events. Non-stationary stochastic processes can be used to model these variations [6, 8]. In addition, they can simulate and predict the impacts of future climate change, taking into account factors such as greenhouse gas emissions and changes in land use. This can assist in planning adaptation and mitigation strategies. Moreover, climate changes typically exhibit long-term memory, as current climate states can have long-term influences on states in the far future. The non-stationarity, the long-memory, and the high uncertainty of these dynamics can be modeled using stochastic differential equation driven by a sub-fractional G-Brownian motion.

The remainder of this paper is organized as follows. In Section 2, we provide the essential concepts of the G-framework. In Section 3, we define sfGBm by means of fGBm. In addition, we present the moving average representation of the sfGBm. We also investigate the main theoretical properties such as self-similarity, Hölder's continuity, and long-range dependence. Section 4 is dedicated to the simulation of sample paths of the sfGBm.

2. Basic settings

In this section, we review some facts and basic notions about G-expectation theory, more details may be found in [11–13].

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real-valued functions defined on Ω. We suppose that $\mathcal{H}$ contains all constants and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. The space $\mathcal{H}$ is considered the space of random variables.

Definition 2.1. A sub-linear expectation $\mathbb{E}$ is a functional $\mathbb{E} : \mathcal{H} \longrightarrow \mathbb{R}$ such that

(i) monotonicity:

$$\text{if} \quad X \geq Y, \quad \text{then} \quad \mathbb{E}[X] \geq \mathbb{E}[Y];$$

(ii) constant preserving:

$$\mathbb{E}[c] = c \ \text{ for } \ c \in \mathbb{R};$$

(iii) sub-additivity: For each $X,Y \in \mathcal{H}$,

$$\mathbb{E}[X+Y] \quad \leq \quad \mathbb{E}[X] + \mathbb{E}[Y];$$

(iv) positive homogeneity:

$$\mathbb{E}[\lambda X] \quad = \quad \lambda \mathbb{E}[X] \ \text{ for } \ \lambda \geq 0.$$

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. Assume that $\varphi(X) \in \mathcal{H}$ if $X \in \mathcal{H}$ for each $\varphi \in \mathcal{C}_{l,lip}(\mathbb{R}^n)$, where $\mathcal{C}_{l,lip}(\mathbb{R}^n)$ is the space of $\mathbb{R}$-valued functions φ satisfying

$$|\varphi(x) - \varphi(y)| \quad \leq \quad C(1 + |x|^m + |y|^m)\,|x - y| \ \text{ for all } x, y \in \mathbb{R}^n$$

for some $C > 0$ and $m \in \mathbb{N}$ depending on φ.

Definition 2.2. Let X and Y be two real-valued random variables defined on sublinear expectation spaces $(\Omega, \mathcal{H}, \mathbb{E})$ and $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})$, respectively. They are called identically distributed, denoted by $X \overset{d}{=} Y$ if

$$\mathbb{E}[\varphi(X)] \quad = \quad \widetilde{\mathbb{E}}[\varphi(Y)] \quad \text{for any } \varphi \in \mathcal{C}_{l,lip}(\mathbb{R}^n).$$

Definition 2.3. In a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random vector $Y \in \mathcal{H}^m$ is said to be independent of another random vector $X \in \mathcal{H}^n$, if for each function test $\varphi \in \mathcal{C}_{l,lip}(\mathbb{R}^{n+m})$, we have

$$\mathbb{E}[\varphi(X,Y)] \quad = \quad \mathbb{E}[\mathbb{E}[\varphi(x,Y)]_{x=X}].$$

Definition 2.4. A real valued random variable $X \in \mathcal{H}$ is G-normal distributed, if for each $\varphi \in \mathcal{C}_{l,lip}(\mathbb{R}^n)$, the function $u(t,x) = \mathbb{E}[\varphi x + \sqrt{t}X]$ is the unique viscosity solution to the following parabolic G-heat equation:

$$\begin{cases} \partial_t u - G(\partial^2_{xx} u) & = & 0, \ \text{ for } (t,x) \in]0,T] \times \mathbb{R} \\ u(0,x) & = & \varphi(x), \ x \in \mathbb{R} \end{cases}$$

where G is a generator defined by $G(\alpha) = \frac{1}{2}\left(\overline{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-\right)$, $\alpha^+ := \max(\alpha, 0)$, and $\alpha^- := \max(-\alpha, 0)$.

Definition 2.5. A process $\{B(t)\}_{t \geqslant 0}$ on a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called G-Brownian motion if the following properties are fulfilled
(i) $B(0) = 0$ a.s.;
(ii) For each $t, \ s \geq 0$, the increment $B(t+s) - B(t)$ is $\mathcal{N}\left(\{0\}; [\underline{\sigma}^2 s, \overline{\sigma}^2 s]\right)$-distributed and independent of $(B(t_1), B(t_2), \ldots, B(t_n))$ for each $n \in \mathbb{N}$ and $t_1, t_2, \ldots, t_n \in [0,t]$, where

$$\underline{\sigma}^2 \quad = \quad -\mathbb{E}[-B^2(1)] \text{ and } \overline{\sigma}^2 \quad = \quad \mathbb{E}[B^2(1)].$$

Definition 2.6. A process $\{B_{1,2}(t) : t \in \mathbb{R}\}$ on a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a two-sided G-Brownian motion if for $\{B^{(2)}(t)\}_{t\geq 0}$ independent of $\{B^{(1)}(t)\}_{t\geq 0}$ we have

$$B_{1,2}(t) \quad = \quad \begin{cases} B^{(1)}(t) & t \geq 0 \\ B^{(2)}(-t) & t \leq 0 \end{cases}$$

where $\{B^{(1)}(t)\}_{t\geq 0}$ and $\{B^{(2)}(t)\}_{t\geq 0}$ are G-Brownian motions.

Definition 2.7. A process $\{X(t)\}_{t\in\mathbb{R}}$ in the space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a centred G-Gaussian process if for each fixed $t \in \mathbb{R}$, $X(t)$ is G-normal distributed by $\mathcal{N}(\{0\}; [\underline{\sigma}_t^2, \overline{\sigma}_t^2])$, where $0 \leq \underline{\sigma}_t \leq \overline{\sigma}_t$.

Definition 2.8. A centered G-Gaussian process $\{B_H(t), t \in \mathbb{R}\}$ defined on a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, is called a fractional G-Brownian motion with Hurst index $H \in (0, 1)$, if for each $s,\ t \in \mathbb{R}$, we have
(i) $B_H(0) = 0$, a.s.
(ii)

$$\begin{cases} \mathbb{E}[B_H(t)B_H(s)] &= \frac{1}{2}\overline{\sigma}^2\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \\ \\ -\mathbb{E}[-B_H(t)B_H(s)] &= \frac{1}{2}\underline{\sigma}^2\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \end{cases}$$

We denote the fractional G-Brownian motion by fGBm. The moving average representation of fGBm is defined as follows:

Definition 2.9. The fractional G-Brownian motion with Hurst index H is represented as

$$B_H(t) \quad = \quad c_H \int_{\mathbb{R}} N_H(t, u)dB(u),$$

where $N_H(t, u) = (t-u)_+^{H-1/2} + (u)_+^{H-1/2}$, $c_H = \frac{(2H\sin(\pi H)\Gamma(2H))^{1/2}}{\Gamma(H+1/2)}$, and $\{B(t)\}_{t\in\mathbb{R}}$ is a two-sided G-Brownian motion.

Definition 2.10. If there exists a weakly compact collection $\mathcal{P}$ of probability measures P, defined on $(\Omega, \mathfrak{F}(\Omega))$ then, the capacity $\widehat{C}(\cdot)$ in relation to $\mathcal{P}$ is given by

$$\widehat{C}(A) = \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathfrak{F}(\Omega),$$

where $\mathfrak{F}(\Omega)$ is the Borel σ-algebra of Ω, see [9].

Definition 2.11. A set A is said to be polar if $\widehat{C}(A) = 0$. A property holds quasi-surely $(q.s.)$ if it holds outside a polar.

3. Existence of sub-fractional G-Brownian motion

Similarly to [2], we define the following continuous process under uncertainty as the sub-fractional G -Brownian motion by means of the fractional G -Brownian motion.

Definition 3.1. Let $H \in (0, 1)$ and the process

$$S_H(t) = \frac{1}{\sqrt{2}}(B_H(t) + B_H(-t)), t \geq 0 \tag{3.1}$$

where $\{B_H(t) : t \in \mathbb{R}\}$ is fGBm. Let $t, s \geq 0$, the G-Gaussian process $(S_H(t))_{t\geq 0}$ with zero mean and "lower, upper" covariances:

$$\begin{cases} \mathbb{E}[S_H(t)S_H(s)] &= \overline{\sigma}^2(t^{2H} + s^{2H}) - \dfrac{1}{2}\overline{\sigma}^2[(t+s)^{2H} + |t-s|^{2H}] \\ \\ -\mathbb{E}[-S_H(t)S_H(s)] &= \underline{\sigma}^2(t^{2H} + s^{2H}) - \dfrac{1}{2}\underline{\sigma}^2[(t+s)^{2H} + |t-s|^{2H}] \end{cases} \tag{3.2}$$

is called sfGBm.

Remark 3.2. We can compare the worst-case scenario (high-risk) of the covariance in sfGBm (supremum over σ within the uncertainty set) with the standard covariance of classical sfBm as follows:

$$\mathbb{E}\left[S_H(t)S_H(s)\right] = \sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]} E_\sigma\left[S_H(t)S_H(s)\right] = \overline{\sigma}^2\zeta\left(t,s\right),$$

where each E_σ is a classical expectation, because in this case S_H behaves like a classical sfBm since the volatility is determined by σ, then it induces only one specific probability measure, and $\zeta\left(t,s\right)$ is the standard covariance function of the sfBm. If $\overline{\sigma} > 1$, the covariance of sfGBm under the worst-case scenario exceeds that of the standard sfBm. Similarly, we compare the best-case scenario (low-risk) of the covariance in sfGBm with the standard covariance of sfBm,

$$-\mathbb{E}\left[-S_H(t)S_H(s)\right] = \inf_{\sigma\in[\underline{\sigma},\overline{\sigma}]} E_\sigma\left[S_H(t)S_H(s)\right] = \underline{\sigma}^2\zeta\left(t,s\right).$$

If $\underline{\sigma} < 1$, the covariance of sfGBm is smaller than that of standard sfBm, reflecting a lower level of volatility.

Remark 3.3. The case $H = \frac{1}{2}$ corresponds to G-Brownian motion. In addition, from the above definition it is easy to check that

$$\mathbb{E}[S_H^2(t)] = \overline{\sigma}^2\left(2-2^{2H-1}\right)t^{2H};\; -\mathbb{E}[-S_H^2(t)] = \underline{\sigma}^2\left(2-2^{2H-1}\right)t^{2H}.$$

The following remark is very important, in the next calculus.

Remark 3.4. We have for all functions $f,\ g \in L^2\left(\mathbb{R}\right)$,

$$\int_{\mathbb{R}} f(s)dB(s) \rightsquigarrow \mathcal{N}\left(0, [\underline{\sigma}^2\int_{\mathbb{R}} f^2(s)ds, \overline{\sigma}^2\int_{\mathbb{R}} f^2(s)ds]\right),$$

consequently,

$$\begin{aligned}\mathbb{E}[\left(\int_{\mathbb{R}} f(s)dB(s)\right)^2] &= \overline{\sigma}^2\int_{\mathbb{R}} f^2(s)ds,\\ -\mathbb{E}[-\left(\int_{\mathbb{R}} f(s)dB(s)\right)^2] &= \underline{\sigma}^2\int_{\mathbb{R}} f^2(s)ds,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[\left(\int_{\mathbb{R}} f(s)dB(s)\right)\left(\int_{\mathbb{R}} g(s)dB(s)\right)] &= \overline{\sigma}^2\int_{\mathbb{R}} f(s)g(s)ds,\\ -\mathbb{E}[-\left(\int_{\mathbb{R}} f(s)dB(s)\right)\left(\int_{\mathbb{R}} g(s)dB(s)\right)] &= \underline{\sigma}^2\int_{\mathbb{R}} f(s)g(s)ds.\end{aligned}$$

3.1. Moving average representation of sfGBm

Theorem 3.5. *Let $H \in (0,1)$, for $t > 0$, the integral representation of sfGBm $\{S_H(t)\}_{t\geq 0}$, with respect to G-Brownian motion is given by*

$$S_H(t) = c'_H\int_{\mathbb{R}} K_H(t,u)dB(u),$$

where $K_H(t,u) = (t-u)_+^{H-1/2} + (-t-u)_+^{H-1/2} - 2(-u)_+^{H-1/2}$, and $c'_H = \frac{(\sin(\pi H)\Gamma(2H+1))^{1/2}}{\Gamma(H+1/2)\sqrt{2}}$.

Proof. First, observe that $S_H(0) = \mathbb{E}[S_H(t)] = \mathbb{E}[-S_H(t)] = 0$. Let $s,\ t \geq 0$, according to Remark 3.4, we have

$$\begin{aligned}
\mathbb{E}[S_H(t)S_H(s)] &= (c'_H)^2\mathbb{E}\Big[\Big(\int_{\mathbb{R}} K_H(t,u)dB(u)\Big)\Big(\int_{\mathbb{R}} K_H(s,u)dB(u)\Big)\Big] \\
&= \overline{\sigma}^2(c'_H)^2\int_{\mathbb{R}} K_H(t,u)K_H(s,u)du \\
&= \overline{\sigma}^2\mathbb{E}[X_H(t)X_H(s)],
\end{aligned}$$

where $(X_H(t))_{t\geq 0}$, is the classical sub-fractional Brownian motion [15]. It is well-known that

$$\mathbb{E}[X_H(t)X_H(s)] = t^{2H} + s^{2H} - \frac{1}{2}[(t+s)^{2H} + |t-s|^{2H}],$$

thus, we obtain the first equation of (3.2). The second equation can be proved similarly by replacing the $\mathbb{E}[\cdot]$ by $-\mathbb{E}[-\cdot]$ in the above equation. $\square$

3.2. Properties of the sub-fractional G-Brownian motion

Definition 3.6. A process $\{X(t)\}_{t\in\mathbb{R}}$ in the sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called H-self-similar if

$$X(at) \overset{d}{=} a^H X(t) \quad \text{for } a > 0.$$

Theorem 3.7. *Let $H \in (0,1)$ and $S_H(\cdot)$, be a sfGBm. The following properties are fulfilled:*

(i) Self-similarity:

$$\{S_H(at) : t \geq 0\} \overset{d}{=} \Big\{a^H S_H(t) : t \geq 0\Big\} \text{ for } a > 0.$$

(ii) Hölder's continuity: *For $H \in (\frac{1}{2}, 1)$, there exists a modification of the process $\{S_H(t)\}_{t\geq 0}$, which is a quasi-surly continuous process, whose paths are γ-Hölder for every $\gamma \in [0, H - \frac{1}{2})$.*

(iii) Second moment of increments: *For $s \leq t$*

$$\begin{aligned}
\mathbb{E}[(S_H(t) - S_H(s))^2] &= -\overline{\sigma}^2 2^{2H-1}(t^{2H} + s^{2H}) \\
&\quad + \overline{\sigma}^2\big[(s+t)^{2H} + |t-s|^{2H}\big] \qquad (3.3)\\
-\mathbb{E}[-(S_H(t) - S_H(s))^2] &= -\underline{\sigma}^2 2^{2H-1}(t^{2H} + s^{2H}) \\
&\quad + \underline{\sigma}^2\big[(s+t)^{2H} + |t-s|^{2H}\big] \qquad (3.4)
\end{aligned}$$

(iv) Moving average representation: *Let $H \in (0,1)$, for $t > 0$, the sfGBm process $\{S_H(t)\}_{t\geq 0}$ admits the following representation with respect to G-Brownian motion*

$$S_H(t) = c'_H \int_{\mathbb{R}} K_H(t,u)dB(u),$$

where $K_H(t,u) = (t-u)_+^{H-1/2} + (-t-u)_+^{H-1/2} - 2(-u)_+^{H-1/2}$ and $c'_H = \frac{(\sin(\pi H)\Gamma(2H+1))^{1/2}}{\Gamma(H+1/2)\sqrt{2}}$.

Remark 3.8. If we set $\underline{\sigma}^2 = \overline{\sigma}^2 = 1$ in (3.3), then,

$$\mathbb{E}[(S_H(t) - S_H(s)]^2] = -2^{2H-1}(t^{2H} + s^{2H}) + (t+s)^{2H} + |t-s|^{2H}.$$

Thus, $\{S_H(t)\}_{t\geq 0}$ is nothing but the classical sub-fractional Brownian motion.

Proof. **(i):** From Definition 3.1, the sfGBm $(S_H(t))_{t\geq 0}$ is centred and we have

$$\begin{aligned}\mathbb{E}[S_H^2(at)] &= \overline{\sigma}^2(2-2^{2H-1})(at)^{2H}\\ &= a^{2H}\mathbb{E}[S_H^2(t)],\end{aligned}$$

and

$$\begin{aligned}-\mathbb{E}[-S_H^2(at)] &= \underline{\sigma}^2(2-2^{2H-1})(at)^{2H}\\ &= -a^{2H}\mathbb{E}[S_H^2(t)].\end{aligned}$$

Note that

$$S_H(at) \rightsquigarrow \mathcal{N}\left(0;[-\mathbb{E}[-S_H^2(at)],\mathbb{E}[S_H^2(at)]]\right).$$

It follows that

$$S_H(at) \rightsquigarrow \mathcal{N}\left(0;[-a^{2H}\mathbb{E}[-S_H^2(t)],a^{2H}\mathbb{E}[S_H^2(t)]]\right).$$

Since

$$a^H S_H(t) \rightsquigarrow \mathcal{N}\left(0;[-a^{2H}\mathbb{E}[-S_H^2(t)],a^{2H}\mathbb{E}[S_H^2(t)]]\right),$$

then

$$S_H(at) \overset{d}{=} a^H S_H(t),$$

thus the H-self-similarity.

(ii): By the representation (3.1), we have

$$\begin{aligned}\mathbb{E}[|S_H(t)-S_H(s)|^2] &\leq \frac{1}{2}\mathbb{E}[(|B_H(t)-B_H(s)|+|B_H(-t)-B_H(-s)|)^2]\\ &\leq 2\mathbb{E}[|B_H(t)-B_H(s)|^2+|B_H(-t)-B_H(-s)|^2]\\ &\leq 2\left(\mathbb{E}[|B_H(t)-B_H(s)|^2]+\mathbb{E}[|B_H(-t)-B_H(-s)|^2]\right),\end{aligned}$$

by stationary of increments and self-similarity properties of the fGBm $(B_H(t))_{t\in\mathbb{R}}$, yields

$$\begin{aligned}\mathbb{E}[|S_H(t)-S_H(s)|^2] &\leq 2\left(\mathbb{E}[|B_H(t-s)|^2]+\mathbb{E}[|B_H(s-t)|^2]\right)\\ &\leq 4\mathbb{E}[|B_H(1)|^2]|t-s|^{2H},\end{aligned}$$

hence by Theorem 36 in [9], we conclude.

(iii): Follows from the facts that

$$\begin{aligned}\mathbb{E}[(S_H(t)-S_H(s))^2] &= \overline{\sigma}^2(c_H')^2\int_{\mathbb{R}}(K_H(t,u)-K_H(s,u))^2\,du\\ &= \overline{\sigma}^2\mathbb{E}[(X_H(t)-X_H(s))^2]\end{aligned}$$

and

$$\begin{aligned}-\mathbb{E}[-(S_H(t)-S_H(s))^2] &= \underline{\sigma}^2(c_H')^2\int_{\mathbb{R}}(K_H(t,u)-K_H(s,u))^2\,du\\ &= \underline{\sigma}^2\mathbb{E}[(X_H(t)-X_H(s))^2].\end{aligned}$$

(iv): Already proved in the Theorem 3.5 □

Proposition 3.9. *For $0\leq s<t$ define $\overline{C}(s,t)$ and $\underline{C}(s,t)$ as follows*

$$\overline{C}(s,t) = \mathbb{E}[(S_H(t)-S_H(s))^2],$$

and

$$\underline{C}(s,t) = -\mathbb{E}[-(S_H(t)-S_H(s))^2].$$

Then, for $|t-s|\to 0$, we have

$$\overline{C}(s,t) \sim \overline{\sigma}^2|t-s|^{2H} \tag{3.5}$$

and

$$\underline{C}(s,t) \;\sim\; \underline{\sigma}^2|t-s|^{2H} \tag{3.6}$$

Proof. We put $h = t-s$. Then by substitution in (3.3), we obtain

$$\begin{aligned}
\overline{C}(s,t) &= -\overline{\sigma}^2 2^{2H-1}\left((s+h)^{2H}+s^{2H}\right)+\overline{\sigma}^2\left[(2s+h)^{2H}+h^{2H}\right] \\
&= -\overline{\sigma}^2 2^{2H-1}s^{2H}\left(\left(1+\frac{h}{s}\right)^{2H}+1\right) \\
&\quad +\overline{\sigma}^2 2^{2H}s^{2H}\left(1+\frac{h}{2s}\right)^{2H}+\overline{\sigma}^2 h^{2H}.
\end{aligned}$$

By Taylor's expansion to second order when $h \to 0$, we derive that

$$\begin{aligned}
\left(1+\frac{h}{s}\right)^{2H} &= 1+2H\frac{h}{s}+\frac{2H\,(2H-1)}{2}\frac{h^2}{s^2}+o\left(\frac{h^2}{s^2}\right), \\
\left(1+\frac{h}{2s}\right)^{2H} &= 1+2H\frac{h}{2s}+\frac{2H\,(2H-1)}{2}\frac{h^2}{4s^2}+o\left(\frac{h^2}{s^2}\right),
\end{aligned}$$

thus

$$\mathbb{E}\left[(S_H(t)-S_H(s))^2\right] \;=\; \overline{\sigma}^2 h^{2H}+o\left(\frac{h}{s}\right).$$

Similarly, we prove (3.6). □

Remark 3.10. From Proposition 3.9, it turns out that the increments of $S_H(\cdot)$ are not stationary, but roughly stationary when the increments are very small. This is one of the main features of sfGBm.

Definition 3.11. A process $\{Y(t)\}_{t\geq 0}$, in the space $(\Omega, \mathcal{H}, \mathbb{E})$ is called long-memory of order $n \in \mathbb{N}$, if

$$\sum_k |\underline{r}(n,k)| \;=\; \sum_k |\overline{r}(n,k)| = \infty,$$

and short memory of order $n \in \mathbb{N}$, if

$$\sum_k |\underline{r}(n,k)| \;<\; \infty, \quad \sum_k |\overline{r}(n,k)| \;<\; \infty,$$

where

$$\overline{r}(n,k) \;=\; \mathbb{E}[Y(n)Y(n+k)], \quad \text{and} \quad \underline{r}(n,k) \;=\; -\mathbb{E}[-Y(n)Y(n+k)].$$

Remark 3.12. If the process $(Y(t))_{t\geq 0}$ has a stationary increments, then the long-memory (resp. short memory) of order n, is nothing but the long-memory (resp. short memory).

Theorem 3.13 (Long-range dependence)**.** *The process $\{S_H(t)\}_{t\geq 0}$ has a long-memory for $H \in (0, \frac{1}{2})$ and a short memory for $H \in (\frac{1}{2}, 1)$ for every order $n \in \mathbb{N}$.*

Proof. We have,

$$\begin{aligned}
\overline{r}(n,k) &= \overline{\sigma}^2\left(n^{2H}+(n+k)^{2H}\right)-\frac{1}{2}\overline{\sigma}^2\left((2n+k)^{2H}+|k|^{2H}\right) \\
&= k^{2H}\overline{\sigma}^2\left(\left(\frac{n}{k}\right)^{2H}+\left(1+\frac{n}{k}\right)^{2H}\right)-\frac{1}{2}k^{2H}\overline{\sigma}^2\left(\left(1+\frac{2n}{k}\right)^{2H}+1\right).
\end{aligned}$$

By Taylor's expansion when k large enough, we obtain

$$\overline{r}(n,k) \;\sim\; \overline{\sigma}^2 H\,(1-2H)\,n^2 k^{2H-2},$$

thus,

$$\sum_k |\overline{r}(n,k)| \quad = \quad \begin{cases} \text{is finite}, & H \in (0, \frac{1}{2}) \\ \text{infinite}, & H \in (\frac{1}{2}, 1) \end{cases}$$

Similarly, we can derive the second expression of $\underline{r}(n,k)$. Then, we conclude. □

4. Simulation of the sfGBm

To simulate the sfGBm process, we basically had to go through three main steps:

- Based on [18], and using the explicit Euler scheme, we have numerically solved the G-heat equation, which enables us to simulate the G-normal distribution and then the corresponding density as the following:

  The G-heat equation can be written as

$$\frac{\partial u}{\partial t} - G\left(\frac{\partial^2 u}{\partial x^2}\right) = 0,$$

  where $G\left(a\right) = \frac{1}{2} \sup\limits_{\sigma\in[\underline{\sigma},\overline{\sigma}]} \sigma^2 a$. For simplicity, this is often reformulated as

$$\frac{\partial u}{\partial t} = \sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]} \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}.$$

  The discrete form of the equation is given by

$$u_{i+1}^n = u_i^n + \Delta t G\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right)$$

  where, u_i^n is the value of u at spatial point i and time step n; Δt is the time step size; Δx is the spatial grid spacing.The operator G accounts for uncertainty in volatility. Numerically, this can be handled by evaluating the supremum at each spatial grid point i during every time step n,

$$G\left(\frac{\partial^2 u}{\partial x^2}\right) = \begin{cases} \frac{1}{2}\overline{\sigma}^2\frac{\partial^2 u}{\partial x^2}, & \text{if } \frac{\partial^2 u}{\partial x^2} > 0, \\ \frac{1}{2}\underline{\sigma}^2\frac{\partial^2 u}{\partial x^2}, & \text{if } \frac{\partial^2 u}{\partial x^2} < 0. \end{cases}$$

  The solution $u(x,t)$ is interpreted as the density of the G-normal distribution. Figure 1 shows the G-normal distribution with $\underline{\sigma}^2 = 0.3$ and $\overline{\sigma}^2$=0.6.

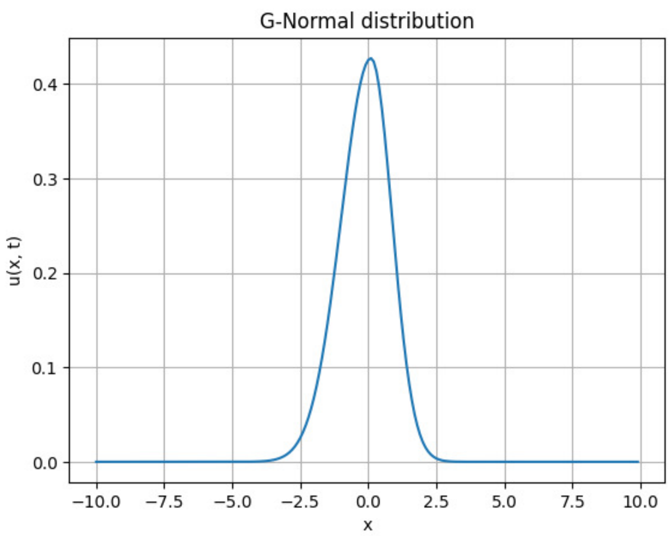

Figure 1. G-normal distribution, $\underline{\sigma}^2 = 0.3$ and $\overline{\sigma}^2$=0.6

- After solving the G-heat equation and deriving the G-normal distribution, we were able to simulate both the G-Brownian motion and the fractional G-Brownian motion. This is because the increments of both processes follow a G-normal distribution.
- In the last step, we simulated the sfGBm through the moving average representation.

The simulation method of sfGBm through the moving average representation used here is similar to the technique of [1], in the simulation of the classical sub-fractional Brownian motion, namely, since the increments of the sfGBm are non-stationary, then we use its integral representation with respect to the G-Brownian motion as given in Theorem 3.5. We have

$$S_H(t) \;=\; c'_H \int_{\mathbb{R}} [(t-u)_+^{H-1/2} + (-t-u)_+^{H-1/2} - 2(-u)_+^{H-1/2}]dB(u),$$

where $c'_H = \frac{(\sin(\pi H)\Gamma(2H+1))^{1/2}}{\Gamma(H+1/2)\sqrt{2}}$.
The idea is to approximate the path of the process S_H by a sequence $(S_H(n))_{1\leq n\leq N}$, $N \in \mathbb{N}$. To this end, we express $S_H(n)$ by its increments as follows:

$$S_H(n) \;=\; \sum_{k=1}^{n}(S_H(k) - S_H(k-1)) \;=\; \sum_{k=1}^{n} Z_H(k),$$

where

$$\begin{aligned} Z_H(k) \;=\; c'_H \int_{\mathbb{R}} \Big(& [(k-u)_+^{H-1/2} - (k-1-u)_+^{H-1/2}] \\ & + [(-k-u)_+^{H-1/2} - (-k-u+1)_+^{H-1/2}] \Big)\, dB(u). \end{aligned}$$

We make the change of variables in the above integral, $z = k - u$ and we obtain for all $1 \leq k \leq n$,

$$\begin{aligned} Z_H(k) \;&=\; -c'_H \int_{\mathbb{R}} \Big([(z)_+^{H-1/2} - (z-1)_+^{H-1/2}] \\ &\qquad + [\,(z-2k)_+^{H-1/2} - (z-2k+1)_+^{H-1/2}\,] \Big)\, dB(k-z) \\ &=\; -c'_H \int_{\mathbb{R}} h(z)dB(k-z), \end{aligned}$$

where

$$h(z) \;=\; [(z)_+^{H-1/2} - (z-1)_+^{H-1/2}] + [(z-2k)_+^{H-1/2} - (z-2k+1)_+^{H-1/2}].$$

Then, we approximate the increments of the G-Brownian motion as follows:
Let $m,\ M \in \mathbb{N}$:

$$\Delta B(mk-j) \;=\; -\int_{\frac{j-1}{m}}^{\frac{j}{m}} dB(k-z),$$

therefore, the approximation $Z_{m,M,H}(k)$ of $Z_H(k)$, for $1 \leq k \leq n$, can be written as the following,

$$Z_{m,M,H}(k) \;=\; c'_H \sum_{j=1}^{mM} h\left(\frac{j}{m}\right) \Delta B\,(mk-j)\,.$$

In the end, we compute the sample path $S_H(n)$, $1 \leq n \leq N$, by means of $Z_{m,M,H}(k)$, $1 \leq k \leq n$. Hereafter, we give a simple example of the simulation of sfGBm produced by Python. We set $\underline{\sigma}^2 = 0.3$, $\overline{\sigma}^2 = 0.6$, $H = 0.85$, and $n = 5$. The increments of the sfGBm can be generated as shown in Figure 2.

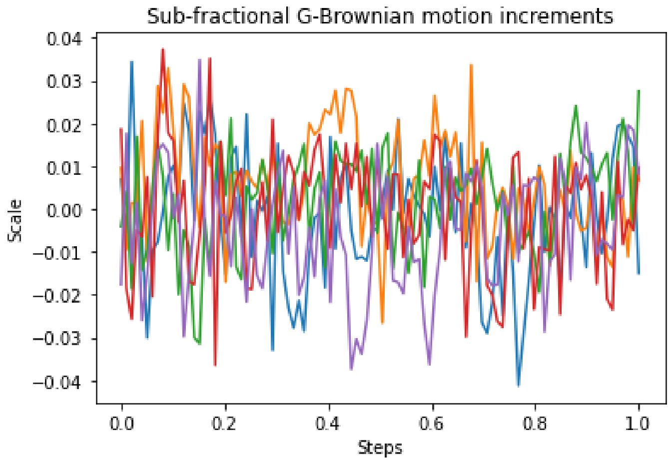

Figure 2. sfGBm increments, $n = 5$, $\underline{\sigma}^2 = 0.3$ and $\overline{\sigma}^2$=0.6

Remark 4.1. From Figure 2, it is clear that the increments of the sfGBm are non-stationary, which is one of the key features of this process.

Next, we represent the simulation of the sfGBm, see Figure 3:

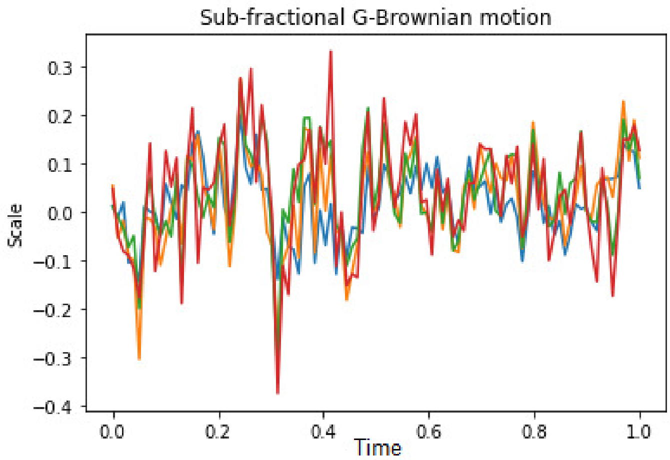

Figure 3. sfGBm, $n = 5$, $\underline{\sigma}^2 = 0.3$ and $\overline{\sigma}^2 = 0.6$

Now, we use this simulation to calculate an estimated value of

$$\mathbb{E}[(S_H(t) - S_H(s))^2].$$

We set $t = 1$, $s = \frac{t}{2}$, $\underline{\sigma}^2 = 0.3$, $\overline{\sigma}^2 = 0.6$, $n = 100$, thus

$$\mathbb{E}[(S_H(t) - S_H(s))^2] \approx 0.1140,$$

and the estimated value derived from Theorem 3.7 is 0.1053. Therefore, the difference between the two values is 0.0087, which exhibits good estimation.

5. Stochastic integral with respect to sfGBm

Since the sfGBm is neither a Markov processes nor a semi-martingale, and all we have is the Hölder's continuity of the paths. Then, it is possible to define the Riemann-Stieltjes integral w.r.t. sfGBm by using p-variation. Let $\Pi_N : 0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of $[0, T]$, $p > 0$. The p-variation of a function $f : [0, T] \to \mathbb{R}$, is defined by

$$V^p(f, \Pi_N) = \sum_{i=0}^{N-1} |f(t_{i+1}) - f(t_i)|^p.$$

Let $V^p(f) := \sup_{\Pi_N} V^p(f, \Pi_N)$ and $V_0^p(f) := \lim_{|\Pi_N| \to 0} V^p(f, \Pi_N)$, where $|\Pi_N|$ stands for the mesh of the subdivision Π_N> Then, f has a bounded p-variation if $V^p(f)$ is finite and a finite p-variation if $V_0^p(f)$ is finite. Consequently, we define the p-variation of the sfGBm by

$$V^p(S_H, \Pi) = \sum_{i=0}^{N-1} |S_H(t_{i+1}) - S_H(t_i)|^p.$$

Proposition 5.1. *For $p > 0$, we have*
(i)

$$V_0^p(S_H) = 0, \ V^p(S_H) < \infty, \quad \text{if } p > \frac{1}{H}.$$

(ii)

$$V_0^p(S_H) = V^p(S_H) = \mathbb{E}[|N(0, [\underline{\sigma}, \overline{\sigma}])|^p), \quad \text{if } p = \frac{1}{H}.$$

(iii)

$$V_0^p(S_H) = V^p(S_H) = \infty, \quad \text{if } p < \frac{1}{H},$$

where $N(0, [\underline{\sigma}, \overline{\sigma}])$ is G-Gaussian distribution.

The above proposition can be proved similarly to the proof of Proposition 2.3 in [15]. See also [16] for more further properties.

Remark 5.2. From the above proposition, it is it is straightforward to see that S_H has a p-bounded variation and a p-finite variation for $p \geq \frac{1}{H}$.

According to [19], if $u(\cdot)$ and $w(\cdot)$ be continuous paths such that $u(\cdot)$ has p-bounded variation and $w(\cdot)$ has q-bounded variation, where $\frac{1}{p} + \frac{1}{q} > 1$, then the integral $\int_0^t u(s)dw(s)$, can be defined as Riemann-Stieltjes sum. More precisely, Feyel and Pradelle in [5] proved that if $u(\cdot))$ (resp. $w(\cdot))$), is α-Hölder (resp. β-Hölder) with $\alpha + \beta > 1$, then the integral $\int_0^t u(s)dw(s)$ is well defined and is β-Hölder. In addition, we have for $0 < \varepsilon < \alpha + \beta - 1$

$$\left| \int_0^T u(s)dw(s) \right| \leq C_{\alpha,\beta} \|u\|_{[0,T],\alpha} \|w\|_{[0,T],\beta} T^{\varepsilon+1},$$

where

$$\|v\|_{[0,T],\alpha} = \sup_{t \neq s, 0 \leq s < t \leq T} \frac{|v(t) - v(s)|}{|t - s|^{\alpha}}.$$

Consequently, we define the stochastic integral w.r.t. sfGBm as follows:

Definition 5.3. Let $\{u(t)\}_{t \in [0,T]}$ be a process with p-bounded variation and $p < \frac{1}{1-H}$. The Riemann-Stieltjes integral $\int_0^t u(r) \, dS_H(r)$ is well-defined. Moreover, if u is α-Hölder for some $\alpha > 1 - H$, then the integral $\int_0^t u(r) \, dS_H(r)$ has β-Hölder paths for $\beta < H$.

5.1. Wiener integral and sfGBm

We also characterize the Wiener integral with respect to S_H in time interval $[0,T]$. Let Γ be the family of elementary deterministic functions, i.e.,

$$\Gamma = \Big\{f : [0,T] \to \mathbb{R},\ f = \sum_{i=0}^{N-1} a_i \mathbb{1}_{[t_i,t_{i+1})}\Big\},$$

where a_i is the value of f on the interval $[t_i,t_{i+1})$. For $f \in \Gamma$, we define the Wiener integral in the natural way by

$$\int_0^T f(s)dS_H(s) := \sum_{i=0}^{N-1} a_i(S_H(t_{i+1}) - S_H(t_i)).$$

The stochastic integral can be extended to bigger spaces of functions. Let $\mathcal{V}([0,T])$ be the space of continuous functions $f : [0,T] \to \mathbb{R}$, endowed with the norm:

$$\|f\|^2_{\mathcal{V}([0,T])} := \int_0^T |f(s)|^2\, ds < \infty,$$

and $L^2_G(\Omega)$ be the space of square-integrable random variables $X : \Omega \to \mathbb{R}$ such that

$$\|X\|^2_{L^2_G(\Omega)} := \mathbb{E}\,|X|^2 < \infty.$$

Now, we come up to define the stochastic integral for $f \in \mathcal{V}([0,T])$. Let the mapping $I : \mathcal{V}([0,T]) \to L^2_G(\Omega)$ defined by

$$I(f) := \int_0^T f(s)dS_H(s),$$

and we have the following remark.

Remark 5.4. For $f,\ g \in \mathcal{V}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f(s)dS_H(s) \rightsquigarrow \mathcal{N}\left(0, [\sigma_1,\sigma_2]\right),$$

where

$$\sigma_1 = \underline{\sigma}^2\left(2 - 2^{2H-1}\right)\int_{\mathbb{R}} f^2(s)ds \text{ and } \sigma_2 = \overline{\sigma}^2\left(2 - 2^{2H-1}\right)\int_{\mathbb{R}} f^2(s)ds.$$

In addition,

$$\mathbb{E}\Big[\left(\int_{\mathbb{R}} f(s)dS_H(s)\right)\left(\int_{\mathbb{R}} g(s)dS_H(s)\right)\Big] = \overline{\sigma}^2\left(2 - 2^{2H-1}\right)\int_{\mathbb{R}} f(s)g(s)ds,$$

and

$$-\mathbb{E}\Big[-\left(\int_{\mathbb{R}} f(s)dS_H(s)\right)\left(\int_{\mathbb{R}} g(s)dS_H(s)\right)\Big] = \underline{\sigma}^2\left(2 - 2^{2H-1}\right)\int_{\mathbb{R}} f(s)g(s)ds.$$

Let $f \in \mathcal{V}([0,T])$, then there exists a sequence of elementary functions $(f_n) \subset \Gamma$ such that $\|f_n - f\|_{\mathcal{V}([0,T])} \to 0$, as $n \to \infty$. For $n,k \in \mathbb{N}$, $n \geq k$, we have

$$\|I(f_n) - I(f_k)\|^2_{L^2_G(\Omega)} = \mathbb{E}\Big[\Big|\int_0^T (f_n(s) - f_k(s))dS_H(s)\Big|^2\Big].$$

According to the above Remark, we can write

$$\begin{aligned}
\|I(f_n) - I(f_k)\|^2_{L^2_G(\Omega)} &\leq C\int_0^T |f_n(s) - f_k(s)|^2\, ds \\
&= C\,\|f_n - f_k\|^2_{\mathcal{V}([0,T])} \\
&\leq 2C\left(\|f_n - f\|^2_{\mathcal{V}([0,T])} + \|f - f_k\|^2_{\mathcal{V}([0,T])}\right),
\end{aligned}$$

where $C = \overline{\sigma}^2(2-2^{2H-1})$. It is clear that the right-hand side of the above inequality tends to zero when $n,k\to\infty$. This implies that $(\int_0^T f_n(s)dS_H(s))_{n\geq 0}$, is a Cauchy sequence in $L_G^2(\Omega)$, which is a complete space, then we conclude that $(\int_0^T f_n(s)dS_H(s))_{n\geq 0}$ converges to a limit in $L_G^2(\Omega)$. Therefor, we put

$$I(f) \;=\; \int_0^T f(s)dS_H(s) \;=\; \lim_{n\to\infty}\int_0^T f_n(s)dS_H(s).$$

5.2. Quadratic variation process of sfGBm:

Let Π_t^N, $N\geq 1$, be a sequence of partitions of $[0,t]$. It is clear that

$$\begin{aligned} S_H^2(t) &= \sum_{i=0}^{N-1}\left(S_H^2(t_{i+1}^N)-S_H^2(t_i^N)\right)\\ &= \sum_{i=0}^{N-1}2S_H(t_i^N)\left(S_H(t_{i+1}^N)-S_H(t_i^N)\right)+\sum_{i=0}^{N-1}(S_H(t_{i+1}^N)-S_H(t_i^N))^2. \end{aligned}$$

When the Mesh$(\Pi_t^N)\to 0$, the term

$$\sum_{i=0}^{N-1}2S_H(t_i^N)\left(S_H(t_{i+1}^N)-S_H(t_i^N)\right)$$

converges to

$$2\int_0^t S_H(s)dS_H(s), \text{ in } L_G^2(\Omega),$$

and the second term

$$\sum_{i=0}^{N-1}(S_H(t_{i+1}^N)-S_H(t_i^N))^2.$$

converges to the limit denoted by $\langle S_H\rangle_t$,

$$\langle S_H\rangle_t : \;=\; S_H^2(t)-2\int_0^t S_H(s)dS_H(s).$$

Note that from the above construction, the quadratic variation process of the sfGBm $(\langle S_H\rangle_t)_{t\geq 0}$ is an increasing process with $\langle S_H\rangle_0 = 0$. Now, we define the stochastic integral of a process with respect to $\langle S_H\rangle$.

Definition 5.5. For each $\varphi\in M_G^{1,0}(0,T)$ of the form

$$\varphi_t \;=\; \sum_{i=0}^{N-1}\xi_i 1\!\!1_{[t_i,t_{i+1})},$$

we define

$$Q(\varphi) \;=\; \int_0^T \varphi_t d\left\langle S_H\right\rangle_t : \;=\; \sum_{i=0}^{N-1}\xi_i\left(\left\langle S_H\right\rangle_{t_{i+1}}-\left\langle S_H\right\rangle_{t_i}\right).$$

The mapping $Q: M_G^{1,0}(0,T)\to L_G^1(\Omega)$ can be extended continuously to $M_G^1(0,T)$.

Lemma 5.6. *For each $\varphi\in M_G^1(0,T)$, we have*

$$\mathbb{E}\left[\,|Q(\varphi)|\,\right] \;\leq\; \overline{\sigma}^2\mathbb{E}\Big[\int_0^T |\varphi_t|^{2H}\,dt\Big].$$

The proof of the above Lemma is omitted because it can be performed similarly to the proof of Lemma 3.4.3 in [13].

5.3. Stochastic differential equations driven by sfGBm:

Consider the following G-SDE driven by a sfGBm for $t \in [0,T]$

$$X_t = X_0 + \int_0^t f(s,X_s)ds + \int_0^t g(s,X_s)dS_H(s) + \int_0^t h(s,X_s)d\langle S_H\rangle_s, \tag{5.1}$$

where the initial condition $X_0 \in \mathbb{R}$, f,g,h are given functions such that $f(\cdot,x)$, $g(\cdot,x)$, $h(\cdot,x) \in M_G^2(0,T;\mathbb{R})$ and satisfying the following hypotheses: for each $t \in [0,T]$, $x,y \in \mathbb{R}$ and $J = f,g,h$.

(H1) There exists a positive constant L such that

$$|J(t,x) - J(t,y)| \quad \leq \quad L\,|x-y|$$

(H2) There exists a positive constant K such that

$$|J(t,x)| \quad \leq \quad K\,|x|\,.$$

In order to prove the existence and uniqueness of the solution, we first introduce the following mapping on $[0,T]$:

$$\theta : M_G^2\,(0,T;\mathbb{R}) \to M_G^2\,(0,T;\mathbb{R})$$

such that

$$\theta(X)_t \quad = \quad X_0 + \int_0^t f(s,X_s)ds + \int_0^t g(s,X_s)dS_H(s) + \int_0^t h(s,X_s)d\langle S_H\rangle_s.$$

The following Lemma proves that the mapping θ is well-defined.

Lemma 5.7. *Under the hypothesis* (**H2**)*, we have*

$$\mathbb{E}\big[\sup_{0\leq t\leq T}|X_t|^2\,\big] \quad < \quad \infty.$$

Proof. We have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leq t\leq T}|X_t|^2\Big] \quad \leq \quad & 4\,|X_0|^2 + 4\mathbb{E}\Big[\Big|\int_0^T f(s,X_s)ds\Big|^2\Big] \\ & +4\mathbb{E}\Big[\Big|\int_0^T g(s,X_s)dS_H(s)\Big|^2\Big] + 4\mathbb{E}\Big[\Big|\int_0^T h(s,X_s)d\,\langle S_H\rangle_s\Big|^2\Big],\end{aligned}$$

we use the hypothesis (**H2**), we obtain

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|X_t|^2\Big] \quad \leq \quad 4\,|X_0|^2 + C\int_0^T \mathbb{E}\Big[\sup_{0\leq t\leq s}|X_t|^2\Big]ds,$$

where C is a positive constant. By the Grönwall's inequality

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|X_t|^2\Big] \quad \leq \quad 4\,|X_0|^2\,e^{CT} \;<\; \infty.$$

The proof is complete. □

Now, we can state the existence-uniqueness theorem.

Theorem 5.8. *The G-SDE* (5.1) *admits a unique solution* $(X_t)_{0\leq t\leq T} \in M_G^2\,(0,T;\mathbb{R})$.

Proof. It suffices to show that θ is a contraction mapping: Let $X, Y \in M_G^2(0,T;\mathbb{R})$ with $X_0 = Y_0$, then we have

$$\begin{aligned}\mathbb{E}[\,|\theta(X)_t - \theta(Y)_t|^2\,] &\leq 3\mathbb{E}\Big[\Big|\int_0^t (f(s,X_s) - f(s,Y_s))\,ds\Big|^2\Big] \\ &\quad +3\mathbb{E}\Big[\Big|\int_0^T (g(s,X_s) - g(s,Y_s))\,dS_H(s)\Big|^2\Big] \\ &\quad +3\mathbb{E}\Big[\Big|\int_0^T (h(s,X_s) - h(s,Y_s))\,d\left\langle S_H\right\rangle_s\Big|^2\Big]\end{aligned}$$

We use the hypothesis (**H1**), we can write

$$\mathbb{E}[\,|\theta(X)_t - \theta(Y)_t|^2\,] \leq C\int_0^T \mathbb{E}[\,|X_s - Y_s|^2\,]ds$$

where C is a positive constant. We multiply the both sides of the above inequality by e^{-2Ct}, and integrate them on $[0,T]$, we obtain

$$\begin{aligned}\int_0^T \mathbb{E}[\,|\theta(X)_t - \theta(Y)_t|^2\,]e^{-2Ct}dt &\leq C\int_0^T e^{-2Ct}\int_0^t \mathbb{E}[\,|X_s - Y_s|^2\,]dsdt \\ &\leq C\int_0^T\int_s^T e^{-2Ct}dt\mathbb{E}[\,|X_s - Y_s|^2\,]ds \\ &= \frac{1}{2}\int_0^T \left(e^{-2Cs} - e^{-2CT}\right)\mathbb{E}[\,|X_s - Y_s|^2\,]ds\end{aligned}$$

thus

$$\int_0^T \mathbb{E}[\,|\theta(X)_t - \theta(Y)_t|^2\,]e^{-2Ct}dt \leq \frac{1}{2}\int_0^T e^{-2Ct}\mathbb{E}[\,|X_t - Y_t|^2\,]dt.$$

Note that, the norms

$$\left(\int_0^T \mathbb{E}[\,|X_t|^2\,]e^{-2Ct}dt\right)^{\frac{1}{2}} \text{ and } \left(\int_0^T \mathbb{E}[\,|X_t|^2\,]dt\right)^{\frac{1}{2}}$$

are equivalent in $M_G^2(0,T;\mathbb{R})$. Consequently, θ is a contraction mapping and its fixed point is the unique solution of the G-SDE (5.1). ☐

5.4. Numerical simulation of SDEs driven by sfGBm

In this part, we simulate the solution of linear SDE driven by sfGBm by using the Euler-Maruyama scheme. Given the following linear G-SDE

$$\begin{cases} dX_t = X_t dt + X_t dS_H(t) + X_t d\left\langle S_H\right\rangle_t \;\; t\in[0,1] \\ X_0 = 1 \end{cases}, \tag{5.2}$$

In the following simulations, we assume the parameters values are $H = 0.6$, $\underline{\sigma}^2 = 0.2$ and $\overline{\sigma}^2 = 0.5$. We compute the approximate solution as follows: Let $N \in \mathbb{N}$, and $0 = t_0 < t_1 < t_2 < \ldots < t_N = 1$ be a grid points of the interval $[0,1]$ such that $t_n = \frac{n}{N}$. For $n = \{0,1,2,\ldots,N-1\}$, we have

$$\begin{aligned}X_{t_0} &= X_0 \\ X_{t_{n+1}} &= X_{t_n} + X_{t_n}(t_{n+1} - t_n) + X_{t_n}(S_H(t_{n+1}) - S_H(t_n)) \\ &\quad + X_{t_n}(\left\langle S_H\right\rangle_{t_{n+1}} - \left\langle S_H\right\rangle_{t_n}).\end{aligned}$$

We implement the above algorithm; then we obtain Figure 4.

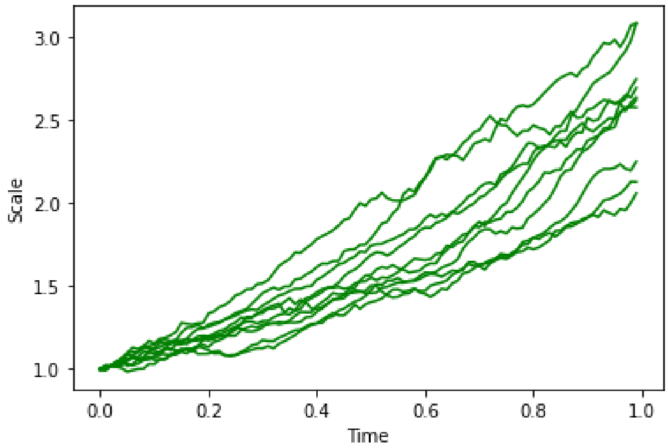

Figure 4. Approximate solutions of equation (5.2).

The expected value of X_t at $t = 1$, is approximately equal to 2.710. Now, let us take the above G-SDE and omit the quadratic variation term, namely

$$\begin{cases} dX_t &= X_t dt + X_t dS_H(t), \quad t \in]0, 1] \\ X_0 &= 1 \end{cases} \tag{5.3}$$

Our goal is to simulate this G-SDE, then we compare it with the exact solution, which we will inspire by the classical fractional SDE. Figure 5, represents the approximate solutions of G-SDE (5.3)

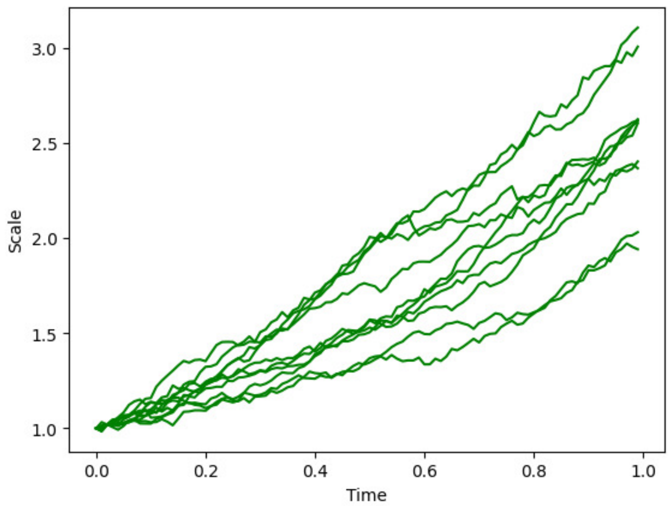

Figure 5. Approximate solutions of equation (5.3).

The expected value of X_t at the final time $t = 1$, is approximately equal to 2.621. We claim that the exact solution of the equation (5.3) is given by

$$X_t = e^{t+S_H(t)}, \quad t \in [0, 1].$$

Indeed, this is inspired by the exact solutions of linear SDEs driven by a fractional Brownian motion. Next, using the exact solution, we calculate the expected value of X_t at the final time $t = 1$, obtaining 2.675. This results in an error of less than 0.06, which supports the credibility of our claim.

6. Conclusion

In this article, we introduced a new stochastic process under volatility uncertainty, termed sub-fractional G-Brownian motion (sfGBm). The process is defined through a fractional G-Brownian motion, and we explored its key properties such as self-similarity, Hölder continuity, and long-memory. A notable feature of sfGBm is that its increments are non-stationary, which, together with volatility uncertainty, makes simulation more complex. To address this, the simulation is based on the moving average representation of sfGBm. We also defined the stochastic integral with respect to this process and its quadratic variation, enabling the simulation of solutions to linear stochastic differential equations (SDEs) driven by sfGBm. This process holds promise for modeling situations that require capturing both variability and uncertainty in systems.

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Data availability. The data presented in this research are fully displayed in the article.

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