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Research Article

On the Parallel Ruled Surfaces with B-Darboux Frame

Mustafa DEDE [1], Hatice TOZAK [2], Cumali EKİCİ [3]

Abstract
In this work, the geometric properties of parallel ruled surfaces associated with the B-Darboux frame (BDF) in E^3 are introduced. Firstly, these surfaces are presented, and their main characteristic properties, such as developability, stress points and dispersion parameter, are analyzed. The paper presents a comprehensive analysis of how these surfaces are constructed and describes the conditions under which they preserve certain geometrical properties with the B-Darboux frame in Euclidean 3-space. These results contribute to a much better understanding of the theory of parallel ruled surfaces (PRS) with regard to the B-Darboux frame and provide insights into potential applications in various fields of geometry and mathematical modeling.

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[1] *Department of Mathematics, Kilis 7 Aralık University, 79000, Kilis, Türkiye*
[2] *Department of Strategy, MEB Denizli Directorate of National Education, 20030, Denizli, Türkiye*
[3] *Department of Mathematics and Computer Science, Eskişehir Osmangazi University, 26040, Eskişehir, Türkiye*
[1] ✉mustafadede@kilis.edu.tr, [2] ✉hatice.tozak@gmail.com, [3] ✉cekici@ogu.edu.tr
Corresponding author: Cumali EKİCİ
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1. Introduction

In differential geometry, a ruled surface is conventionally defined by moving a line along a given curve and forms as one of the simplest parameterizable surfaces. This concept was first characterized by Gaspard Monge, who developed the partial differential equations governing these surfaces. Building on Monge's fundamental study, researchers such as V. Hlavaty [1] and J. Hoschek [2] have further investigated ruled surfaces generated by families of one-parameter lines.

Along with this, the classification of parallel ruled surfaces was developed, where points maintain a constant distance along the principal vector of a ruled surface [3]. This understanding not only enriched the fundamental theoretical foundations of differential geometry but also opened up new possibilities for practical applications. In particular, the concepts of ruled and parallel ruled surfaces bridged the gap between geometry and industry, enabling the control of surface forms in complex structures.

Nowadays, both ruled and parallel ruled surfaces are indispensable in industrial applications, especially in the machining of chipped surfaces and the creation of tool paths for rapid prototyping. In additive manufacturing, these surfaces facilitate the creation of objects and assemblies from CAD models by methods such as laminated object production and stereolithography [4]. Beyond manufacturing, ruled surfaces significantly influence the design and development of vehicles and various products. They are also crucial in areas such as motion analysis and rigid body simulation. In modern surface modeling systems, ruled surfaces and their variations are an integral part of to analysis of kinematic and positional mechanisms in three-dimensional Euclidean space (E^3) [5].

Although these surfaces can be parametrized using the Frenet frame in research, this approach encounters limitations at points where curvature vanishes such as at singular points or along straight segments of the curve [6]. As a result, alternative methods for surface definition have been explored [7]. One significant advancement was introduced by Klok, who developed sweep surfaces based on rotation-minimizing frames, while Wang later refined these concepts by formulating a robust algorithm for calculating such frames [8, 9]. For space curves, several adapted frames exist, including the Frenet frame [10] and the parallel transport frame [11]. The parallel transport frame, often employed in computer graphics for its minimal twist, presents computational complexity but remains advantageous for its properties [9]. Another alternative frame is the quasi-frame constructed with n_q quasi-normal vector found using a projection vector. This frame is also computable for curves when the second derivative is zero [12]. More recently, inspired by the application of the quasi-normal vector n_q in 3D offset curves, Dede et al. have proposed the quasi-frame as an innovative tool for surface modeling [12, 13].

Another fundamental frame in differential geometry, the Darboux frame, serves as a surface-based counterpart to the Frenet frame and represents a naturally occurring moving frame on surfaces, both in Euclidean and non-Euclidean spaces. Originally introduced by Jean Gaston Darboux in his seminal four-volume study published between 1887 and 1896, the Darboux frame has profoundly influenced the field and has been extensively studied [14, 15]. In particular, the properties of ruled surfaces described by the Darboux frame in three-dimensional Euclidean space (E^3) have garnered considerable attention, as analyzed in [16]. Some authors have investigated the ruled surfaces and examined various properties of these surfaces, which are constructed using Bishop frame vectors and quasi-frame vectors in both three- and four-dimensional spaces [17–19].

A new frame, called the B-Darboux frame (BDF), is introduced on a surface in E^3. It is well known that the parallel transportation frame is derived from the Frenet frame along a space curve. Similarly, the B-Darboux frame is derived from the Darboux frame on a surface in Euclidean space. Along a curve lying on the surface, a new $\{T, B_1, B_2\}$ frame consisting of three orthogonal vectors, called the B-Darboux frame, is constructed. The BDF is shown as an alternative to the Darboux frame on the surface [20, 21].

In [4], the Darboux frame was specifically employed to characterize parallel ruled surfaces in E^3, where normal curvature and geodesic torsion were computed, and additional geometric properties were outlined. Subsequent studies have continued to investigate the Darboux and B-Darboux frames, further expanding the theoretical frame for these surfaces [21–24].

In our work, we study ruled surfaces and parallel ruled surfaces using to the B-Darboux frame in Euclidean space. We also present some examples which support the results we put forward in this study. Our work has been consisted of two original sections which contain interesting and fruitful consequences.

2. Preliminaries

Consider a space curve with a non-zero second derivative, denoted by $\alpha(s)$. The Frenet frame is defined as follows:

$$T = \frac{\alpha'}{\|\alpha'\|},\ b = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|},\ n = b \wedge T.$$

The curvature κ and the torsion τ are defined as

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}.$$

The well-known Frenet formulas are defined as

$$\begin{bmatrix} T' \\ n' \\ b' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ n \\ b \end{bmatrix},$$

where $v = \|\alpha'(s)\|$.

Assume that M is an oriented surface in E^3 and the unit speed curve $\alpha(s) = M(u(s), v(s))$ lies on $M = M(u, v)$ in E^3. Due to the curve $\alpha(s)$ lying on the surface, there exists the Darboux frame, denoted by $\{T, G, N\}$. In the Darboux frame, T is the unit tangent vector of the curve, similar to that in the Frenet frame. N is the unit normal vector of the surface, and G is the unit vector which is defined by $G = N \wedge T$. Then the Darboux formulas are:

$$\begin{cases} T' &= \kappa_g G + \kappa_n N, \\ G' &= -\kappa_g T + \tau_g N, \\ N' &= -\kappa_n T - \tau_g G. \end{cases}$$

Let the angle between the surface normal N and the normal n of α be denoted by ϕ. According to that, $\kappa_n = \kappa \cos\phi$, $\kappa_g = \kappa \sin\phi$ and $\tau_g = \tau + \frac{d\phi}{ds}$ are called the normal curvature, geodesic curvature and the geodesic torsion of α, respectively [25]. The BDF is shown as an alternate adapted frame on the surface, which is an alternative to the Darboux frame on the surface [21]. The variation equations of BDF $\{T, B_1, B_2\}$ on the surface is therefore provided in matrix form as follows:

$$\frac{d}{ds}\begin{bmatrix} T \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & n_1 & n_2 \\ -n_1 & 0 & 0 \\ -n_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ B_1 \\ B_2 \end{bmatrix}, \tag{1}$$

where the BDF curvatures are

$$\begin{cases} n_1 &= \kappa_g \sin\theta + \kappa_n \cos\theta, \\ n_2 &= \kappa_n \sin\theta - \kappa_g \cos\theta, \end{cases} \tag{2}$$

where

$$\theta - \theta_0 = \int \tau_g ds \tag{3}$$

is used to determine the angle θ between N and B_1. Here θ_0 is an arbitrary integration constant. Also a relation matrix between the BDF and the Darboux frame may be expressed as (as appears straight away, the relation matrix may be written as)

$$\begin{bmatrix} T \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} T \\ G \\ N \end{bmatrix}.$$

Thus, from [12], we have

$$\begin{bmatrix} T \\ G \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & -\cos\theta \\ 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} T \\ B_1 \\ B_2 \end{bmatrix}.$$

Let us take the surface M as a regular surface defined in the special form $\varphi(s,v)$. If $X(s)$ is a generator vector of the curve $\alpha(s)$, then the parametric representation of the ruled surface $\varphi(s,v)$ is as follows:

$$\varphi(s,v) = \alpha(s) + vX(s),$$

that is, a surface formed by a straight line X moving along a curve α is called a ruled surface [26]. The striction point on a ruled surface is the foot of the common perpendicular to successive rulings along the principal ruling. The ruled surface's striction curve is made up of the grouping of these striction points [16]. It is given as

$$c(s) = \alpha(s) - \frac{\langle \alpha_s, X_s \rangle}{\langle X_s, X_s \rangle} X(s), \tag{4}$$

where α_s is the derivative of α with respect to parameter s. For a straight line X, the system $\{T,G\}$ stretches its unit direction vector. Thus we can write X as

$$X = T\cos\phi + G\sin\phi,$$

where the angle between vectors T and X is denoted by ϕ [16].

Theorem 1. *The ruled surface is developable if consecutive rulings intersect. The unit vector in the direction of X is the unit tangent vector of the striction curve on a developable ruled surface [5].*

In [27], the distribution of the ruled surface is characterized by

$$P_X = \frac{\det(\alpha_s, X, X_s)}{\langle X_s, X_s \rangle}.$$

Theorem 2. *For the surface to be developable, P_X must be equal to zero [5].*

The ruled surface is regarded as a non-cylindrical ruled surface if $\langle X_s, X_s \rangle \neq 0$.

Definition 3. *Let M and $\overline{M}$ be two surfaces in Euclidean space. Then the function*

$$\begin{aligned} f: \quad M &\rightarrow \overline{M} \\ P &\rightarrow f(P) = P + rN_P \end{aligned}$$

is referred to as the parallelization function between M and $\overline{M}$. Furthermore, $\overline{M}$ is called the parallel surface to M, where r is a specified real number and N is the unit normal vector field on M [26].

Theorem 4. *Let M and $\overline{M}$ be two parallel surfaces in Euclidean space, and let*

$$f: \quad M \quad \to \quad \overline{M}$$

represents the parallelization function. For $X \in \chi(M)$, the followings hold:

i. $f_*(X) = X - rS(X)$,

ii. $S^r(f_*(X)) = S(X)$,

iii. *the function f preserves the principal directions of curvature, meaning*

$$S^r(f_*(X)) = \frac{\kappa}{1-r\kappa} f_*(X),$$

where f_* is the derivative transformation of f, κ is a principal curvature of M at point P in the direction of X, and S^r is the shape operator on $\overline{M}$ [26].

Definition 5. *[28] Let M and $\overline{M}$ be two surfaces that are parallel. The image of a unit-speed curve on M, denoted by α, is on M such that $(f \circ \alpha) = \beta$. The Darboux frame of curve β with $\|\beta'\| = \|f_*(T)\| = \mu \neq 1$ on $\overline{M}$ is*

$$\left\{ \overline{T} = \frac{f_*(T)}{v},\ \overline{g} = \overline{T} \wedge \overline{N},\ \overline{N} = N \right\},$$

where $\overline{N}$ is unit normal vector of $\overline{M}$. Additionally, $\|\beta'\| = \sqrt{(1-r\kappa_n)^2 + r^2\tau_g^2} = \mu$ is the norm of β'.

Theorem 6. *[28] Let M and $\overline{M}$ be two parallel surfaces and let their Darboux frames be $\{T, G, N\}$ and $\{\overline{T}, \overline{G}, \overline{N}\}$ respectively. In this case, the frame vectors and their curvatures are equal to*

$$\begin{cases} \overline{T} = \dfrac{1}{\mu}\left[(1-r\kappa_n)T - r\tau_g G\right], \\ \overline{G} = \dfrac{1}{\mu}\left[(1-r\kappa_n)G + r\tau_g T\right], \\ \overline{N} = N \end{cases}$$

and

$$\begin{cases} \overline{\kappa}_g = \dfrac{\kappa_g}{\mu^3} - \dfrac{r}{\mu^3}\left[(\tau'_g + r(\tau_g \kappa'_n - \tau'_g \kappa_n))\right], \\ \overline{\kappa}_n = \dfrac{1}{\mu^2}\left[\kappa_n - r(\kappa_n^2 + \tau_g^2)\right], \\ \overline{\tau}_g = \dfrac{\tau_g}{\mu^2}. \end{cases} \tag{5}$$

Theorem 7. *[28] Let $\overline{M}$ and M be two parallel surfaces with the Frenet and Darboux frames $\{\overline{T}, \overline{n}, \overline{b}\}$ and $\{T, G, N\}$, respectively. In this case, the relations*

$$\begin{cases} \overline{T} = \dfrac{1}{\mu}\left[(1-r\kappa_n)T - r\tau_g G\right], \\ \overline{n} = \dfrac{1}{\mu\sqrt{(\overline{\kappa}_n)^2 + (\overline{\kappa}_g)^2}}\left[r\tau_g \overline{\kappa}_g T + (1-r\kappa_n)\overline{\kappa}_g G + \mu \overline{\kappa}_n N\right], \\ \overline{b} = \dfrac{1}{\mu^3\sqrt{(\overline{\kappa}_n)^2 + (\overline{\kappa}_g)^2}}\left[-r\tau_g \mu^2 \overline{\kappa}_n T - (1-r\kappa_n)\mu^2 \overline{\kappa}_n G + \mu^3 \overline{\kappa}_g N\right] \end{cases}$$

hold between the frame vectors and curvatures. Let $\overline{\kappa}$ be the curvature of the curve β at $f(P)$ on the surface $\overline{M}$, and assume that the angle between vectors $\overline{n}$ and $\overline{N}$ is Φ^r. Therefore, we have the relationship:

$$(\overline{\kappa})^2 = (\overline{\kappa}_n)^2 + (\overline{\kappa}_g)^2, \quad \cos\Phi^r = \frac{\overline{\kappa}_n}{\sqrt{(\overline{\kappa}_n)^2 + (\overline{\kappa}_g)^2}}.$$

3. On the ruled surfaces with the B-Darboux frame

The parametrization of a ruled surface is as follows:

$$\varphi(s,v) = \alpha(s) + vX(s). \tag{6}$$

Since the curve $\alpha(s)$ is referred as the basis curve of the ruled surface, there exists the moving Darboux frame $\{T,G,N\}$ along the curve and $\{T,\mathbf{B_1},B_2\}$ is a BDF at the point $\alpha(s)$.

The system $\{T,B_1\}$ may be used to define the unit direction vector of the straight line X as follows:

$$X = T\cos\phi + B_1\sin\phi, \tag{7}$$

where at $\alpha \in \varphi(s,v)$, ϕ is the angle between the direction vector X and the tangent vector T. The BDF is derived by rotating the Darboux frame around T by an angle $\theta = \theta(s)$, where θ is the angle between the unit vector B_1 and the normal vector field N of $\alpha(s)$. Differentiating equation (7) and using equation (1), we find

$$X' = -(\phi' + n_1)\sin\phi\ T + (\phi' + n_1)\cos\phi\ B_1 + n_2\cos\phi\ B_2. \tag{8}$$

Holding $v =$ constant, we obtain a curve $\beta(s)$ on a ruled surface whose the tangent vector field is

$$\beta(s)' = (\alpha(s) + vX(s))' = T + vX' = T^*.$$

Using equation (8), we get

$$T^* = (1 - v(\phi' + n_1)\sin\phi)T + v(\phi' + n_1)\cos\phi\ B_1 + vn_2\cos\phi\ B_2. \tag{9}$$

Moreover, the relation between the vectors X and T^* is:

$$< T^*, X >= \cos\phi. \tag{10}$$

By utilizing equations (7) and (8), with a BDF, the distribution parameter of the ruled surface can be found as

$$P_X = \frac{n_2\cos\phi\sin\phi}{(\phi' + n_1)^2 + n_2^2\cos^2\phi}. \tag{11}$$

Theorem 8. *A ruled surface that possesses a BDF is deemed developable if and only if*

$$n_2\cos\phi\sin\phi = 0.$$

Proof. $P_X = 0$ implies that the ruled surface with a BDF is developable, which leads to the equation

$$n_2\cos\phi\sin\phi = 0.$$

In this situation, we examine the following subcases concerning the vanishing of equation (7):

i. If $\sin\phi = 0$, then from equation (7), we obtain $X = T\cos\phi$. So, we get $T^* = T$ from equation (10). Along the main ruling, this indicates that the tangent plane remains constant.

ii. The tangent vector field T^* and the normal vector B_2 of the ruled surface with BDF are orthogonal vectors. Accordingly, if $n_2 = 0$, it can be easily shown from equation (9) that ($T^* \in Sp\{T,B_1\}$).

iii. If $\cos\phi = 0$, then from equation (7), we obtain $X = B_1\sin\phi$. So using equation (10) tangent vector field T^* and normal vector B_1 are orthogonal vectors.

Hence, the ruled surface equipped with a BDF is a developable surface.

In contrast, the distribution parameter becomes $P_X = 0$ if we substitute $n_2\cos\phi\sin\phi = 0$ into equation (11). Using equations (4), (8) and (9), the striction curve of the ruled surface equipped with a BDF can be expressed as follows:

$$c(s) = \alpha(s) + \frac{(\phi' + n_1)\sin\phi}{(\phi' + n_1)^2 + n_2^2\cos^2\phi}X.$$

∎

Theorem 9. *Let M be a ruled surface with a BDF as described in equation (6). The following value is the smallest distance between the rulings of the surface M along the orthogonal paths:*

$$v=\frac{(\phi'+n_1)\sin\phi}{(\phi'+n_1)^2+n_2^2\cos^2\phi}.$$

Proof. The distance between the two rulings along an orthogonal trajectory is given by

$$J(v)=\int_{s_1}^{s_2}\|T^*\|\,ds, \tag{12}$$

assuming that they meet at α_{s_1} and α_{s_2} where $s_1<s_2$. Substituting equation (9) into equation (12) gives

$$J(v)=\int_{s_1}^{s_2}\left(\left(1-v(\phi'+n_1)\sin\phi\right)^2+v^2(\phi'+n_1)^2\cos^2\phi+v^2n_2^2\cos^2\phi\right)^{\frac{1}{2}}ds. \tag{13}$$

Differentiating equation (13) with respect to the parameter v, the minimal value of $J(v)$ is obtained by using $J'(v)=0$ then we get

$$v=\frac{(\phi'+n_1)\sin\phi}{(\phi'+n_1)^2+n_2^2\cos^2\phi}.$$

■

Theorem 10. *Let M be a ruled surface with a BDF. The vector field X' is the unit normal to the tangent plane at the point $\varphi(s,v_0)$ if and only if the point $\varphi(s,v_0)$, $v_0\in E$, in the main ruling that goes through the point $\alpha(s)$, is a striction point.*

Proof. Assume that the main ruling at the point $\varphi(s,v_0)$ which goes via $\alpha(s)$, is a striction point. It is necessary to demonstrate that $\langle X',T^*\rangle=\langle X,X'\rangle=0$. We are aware that

$$\|X\|=1. \tag{14}$$

Differentiating equation (14), we obtain $\langle X,X'\rangle=0$. Furthermore, when we evaluate the value of v_0 in equation (9), we find that $\langle X',T^*\rangle=0$. This indicates that X' is orthogonal to both X and T^*. Conversely, assume that X' are functions as a unit normal vector field for the tangent plane at the point $\varphi(s,v_0)$. If v is taken constant at v_0, the tangent vector field of $\varphi(s,v_0)$ is

$$T^*=(1-v_0(\phi'+n_1)\sin\phi)T+v_0(\phi'+n_1)\cos\phi\,B_1+v_0n_2\cos\phi\,B_2.$$

Since X' is a unit normal to the tangent plane at the point $\varphi(s,v_0)$, we can write that $\langle X',T^*\rangle=0$. Thus we get

$$v_0=\frac{(\phi'+n_1)\sin\phi}{(\phi'+n_1)^2+n_2^2\cos^2\phi}.$$

Therefore, on the ruled surface with the BDF, the point $\varphi(s,v_0)$ is a striction point. ■

4. On the parallel ruled surfaces with the B-Darboux frame

The parametrization for a parallel ruled surface (PRS) is given as follows:

$$\overline{\varphi}(s,v)=\overline{\alpha}(s)+\overline{v}\overline{X}(s), \tag{15}$$

where $\overline{\alpha}(s)$ is referred to as the base curve of the PRS, and $\overline{X}$ represents the ruling. By applying the results from equation (1), the derivative formulas for the BDF of the PRS can be determined as follows:

$$\frac{d}{ds}\begin{bmatrix}\overline{T}\\ \overline{B}_1\\ \overline{B}_2\end{bmatrix}=\begin{bmatrix}0&\overline{n}_1&\overline{n}_2\\ -\overline{n}_1&0&0\\ -\overline{n}_2&0&0\end{bmatrix}\begin{bmatrix}\overline{T}\\ \overline{B}_1\\ \overline{B}_2\end{bmatrix}. \tag{16}$$

Rotating the Darboux frame around $\overline{T}$ until $\overline{\theta}=\overline{\theta}(s)$, where the angle $\overline{\theta}$ is between $\overline{N}$ and $\overline{B}_1$, yields the BDF.

Definition 11. *Let M and $\overline{M}$ be two parallel ruled surfaces. Let α be a unit-speed curve on M and image of α stands on $\overline{M}$ which $(f \circ \alpha) = \beta$. If β is non-unit speed curve then $\|\beta'\| = \|f_*(T)\| = \mu \neq 1$. In this case, the frame*

$$\left\{\overline{T},\overline{B}_1,\overline{B}_2\right\} \tag{17}$$

is called as the BDF of the curve β on $\overline{M}$.

Theorem 12. *Let $\left\{\overline{T},\overline{B}_1,\overline{B}_2\right\}$ be BDF of curve β at $f(\alpha(s)) = f(P)$ on $\overline{M}$, and let $\{T,B_1,B_2\}$ be BDF of curve α at the point $\alpha(s)$ on M. Then the relation between the B-Darboux frames $\left\{\overline{T},\overline{B}_1,\overline{B}_2\right\}$ and $\{T,B_1,B_2\}$ is written as*

$$\begin{bmatrix} \overline{T} \\ \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} = \begin{bmatrix} 1 & -rn_1 & -rn_2 \\ rn_1 & 1 & 0 \\ rn_2 & -r^2 n_1 n_2 & 1+r^2 n_1^2 \end{bmatrix} \begin{bmatrix} T \\ B_1 \\ B_2 \end{bmatrix}. \tag{18}$$

Proof. Using Theorem 4 and from equations (16) and (17), equation (18) can be obtained easily. ∎

Theorem 13. *Let α be a regular curve on surface M. The curvatures of the curve $(f \circ \alpha) = \beta$ are, respectively,*

$$\begin{cases} \overline{n}_1 = n_1 - r\cos\theta\left(k_n^2 - \tau_g^2\right) + r\sin\theta\left(rk_n^2\left(\dfrac{\tau_g}{k_n}\right)' - \tau_g'\right), \\ \\ \overline{n}_2 = n_2 - r\sin\theta\left(k_n^2 + \tau_g^2\right) + r\cos\theta\left(rk_n^2\left(\dfrac{\tau_g}{k_n}\right)' - \tau_g'\right) \end{cases} \tag{19}$$

for the point $f(\alpha(s))$ on the PRS $\overline{M}$.

Proof. Assume that $v = 1$ and using equations (2) and (5), we can find the results easily.
Let us denote the point at $\overline{\alpha} \in \overline{\varphi}(s,\overline{v})$ with the angle $\overline{\phi}$ which is between the tangent vector $\overline{T}$ and the direction vector $\overline{X}$. If we define the direction vector $\overline{X}$ as:

$$\overline{X} = \overline{T}\cos\overline{\phi} + \overline{B}_1\sin\overline{\phi}, \tag{20}$$

then we obtain:

$$\left\langle \overline{X},\overline{T}\right\rangle = \cos\overline{\phi} = \cos\phi,$$

where $\|\overline{X}\| = 1$. ∎

Corollary 14. *In the case of parallel ruled surfaces that utilize a BDF, the angle ϕ remains constant.*

According to Corollary 14, equation (20) becomes as

$$\overline{X} = \overline{T}\cos\phi + \overline{B}_1\sin\phi. \tag{21}$$

Differentiating equation (21) and using equation (16), we find

$$\overline{X}' = -(\phi' + \overline{n}_1)\sin\phi\overline{T} + (\phi' + \overline{n}_1)\cos\phi\overline{B}_1 + \overline{n}_2\cos\phi\overline{B}_2. \tag{22}$$

If we take $\overline{v}$ constant, a curve $\beta^r(s)$ on a ruled surface with a tangent vector field of $\overline{v}$ is obtained as:

$$(\beta^r)' = (f(\alpha(s)))' + \overline{v}(f_*(\alpha(s)))' = \overline{T} + \overline{v}\overline{X}' = \overline{T}^*. \tag{23}$$

Using equations (22) and (23), we get

$$\overline{T}^* = (1 - \overline{v}(\phi' + \overline{n}_1)\sin\phi)\overline{T} + \overline{v}(\phi' + \overline{n}_1)\cos\phi\overline{B}_1 + \overline{v}.\overline{n}_2\cos\phi\overline{B}_2. \tag{24}$$

The vectors $\overline{X}$ and $\overline{T}^*$ have the following relationship:

$$< \overline{T}^*,\overline{X} >= \cos\phi. \tag{25}$$

If we put equations (21), (22), and (24) into equation (7), the distribution parameter of the PRS using the BDF is obtained as:

$$\overline{P}_{\overline{X}}=\frac{\left[\phi'+n_1-r\cos\theta(\kappa_n^2-\tau_g^2)+\Delta\right]\cos\phi\sin\phi}{\left[\phi'+n_1-r\cos\theta(\kappa_n^2-\tau_g^2)+\Delta\right]^2+\left[n_2-r\sin\theta(\kappa_n^2+\tau_g^2)+\dfrac{d\Delta}{d\theta}\right]^2\cos^2\phi}, \tag{26}$$

where $\Delta=r\cos\theta\left(r\kappa_n^2\left(\dfrac{\tau_g}{\kappa_n}\right)'-\tau_g'\right)$.

Theorem 15. *A PRS with BDF is a developable surface if and only if*

$$\left[\phi'+n_1-r\cos\theta(\kappa_n^2-\tau_g^2)+r\cos\theta\left(r\kappa_n^2\left(\frac{\tau_g}{\kappa_n}\right)'-\tau_g'\right)\right]\cos\phi\sin\phi=0. \tag{27}$$

Proof. If the PRS with BDF is a developable surface, then $\overline{P}_{\overline{X}}=0$, that is,

$$\left[\phi'+n_1-r\cos\theta(\kappa_n^2-\tau_g^2)+r\cos\theta\left(r\kappa_n^2\left(\frac{\tau_g}{\kappa_n}\right)'-\tau_g'\right)\right]\cos\phi\sin\phi=0.$$

Then, let's examine the following subcases in relation to the disappearance of equation (27):

i. In the case $\cos\phi=0$, from the equation (21), we get $\overline{X}=\overline{B}_1\sin\phi$. So using the equation (25) the tangent vector field $\overline{t}^*$ and the normal vector $\overline{B}_1$ are orthogonal vectors.

ii. In the case $\sin\phi=0$, from equation (21), we get $\overline{X}=\overline{T}\cos\phi$. So $\overline{T}^*=\overline{T}$ from the equation (25). This result indicates that along the main ruling, the tangent plane remains constant.

iii. If $\phi'+n_1-r\cos\theta(\kappa_n^2-\tau_g^2)+r\cos\theta\left(r\kappa_n^2\left(\dfrac{\tau_g}{\kappa_n}\right)'-\tau_g'\right)=0$. Consequently, the parallel ruled surface with BDF is a developable surface.

On the other hand, the distribution parameter is $\overline{P}_{\overline{X}}=0$ if we substitute equation (27) into equation (26). Considering equations (4), (22), and (24), the following formula determines the striction curve of the PRS with BDF

$$\overline{c}(s)=\overline{\alpha}(s)-\frac{\left[\phi'+n_1-r\cos\theta(\kappa_n^2-\tau_g^2)+\Delta\right]\sin\phi}{\left[\phi'+n_1-r\cos\theta(\kappa_n^2-\tau_g^2)+\Delta\right]^2+\left[n_2-r\sin\theta(\kappa_n^2+\tau_g^2)+\dfrac{d\Delta}{d\theta}\right]^2\cos^2\phi}\overline{X},$$

where $\Delta=r\cos\theta\left(r\kappa_n^2\left(\dfrac{\tau_g}{\kappa_n}\right)'-\tau_g'\right)$. ■

Theorem 16. *With a BDF, let M be a PRS, as shown in (15). As a result, the minimum distance among the rulings of $\overline{M}$ along the orthogonal trajectories is given by:*

$$\overline{v}=\frac{\left[\phi'+n_1-r\cos\theta(\kappa_n^2-\tau_g^2)+\Delta\right]\sin\phi}{\left[\phi'+n_1-r\cos\theta(\kappa_n^2-\tau_g^2)+\Delta\right]^2+\left[n_2-r\sin\theta(\kappa_n^2+\tau_g^2)+\dfrac{d\Delta}{d\theta}\right]^2\cos^2\phi}, \tag{28}$$

where $\Delta=r\cos\theta\left(r\kappa_n^2\left(\dfrac{\tau_g}{\kappa_n}\right)'-\tau_g'\right)$.

Proof. Suppose that the two rulings pass the points $\overline{\alpha}_{s_1}$ and $\overline{\alpha}_{s_2}$ where $s_1<s_2$, the distance between these rulings along an orthogonal trajectory is given by:

$$\overline{J}(\overline{v})=\int_{s_1}^{s_2}\left\|\overline{T}^*\right\|ds. \tag{29}$$

Substituting equation (24) into equation (29) gives

$$\overline{J}(\overline{v})=\int_{s_1}^{s_2}\left((1-\overline{v}(\phi'+\overline{n}_1)\sin\phi)^2+\overline{v}^2(\phi'+\overline{n}_1)^2\cos^2\phi+\overline{v}^2\overline{n}_2^2\cos^2\phi\right)^{\frac{1}{2}}ds. \tag{30}$$

Differentiating equation (30) the minimal value of $\overline{J}(\overline{v})$ with regard to the parameter $\overline{v}$ is found by using $\overline{J}'(\overline{v})=0$ then we et which satisfies

$$\overline{v}=\frac{(\phi'+\overline{n}_1)\sin\phi}{(\phi'+\overline{n}_1)^2+\overline{n}_2^2\cos^2\phi}. \tag{31}$$

Using equations (19) and (31), the parameter $\overline{v}$ can be obtained as given in equation (28). ■

Theorem 17. *Let $\overline{M}$ be a ruled surface with a BDF. The vector field X' is the unit normal to the tangent plane at the point $\overline{\varphi}(s,\overline{v}_0)$ if and only if the point $\overline{\varphi}(s,\overline{v}_0)$, $\overline{v}_0\in E$, in the main ruling that goes through the point $\overline{\alpha}(s)$, is a striction point.*

Proof. Assume that the point $\overline{\varphi}(s,\overline{v}_0)$, which passes through $\overline{\alpha}(s)$, is a striction point. We need to demonstrate that $\left\langle\overline{X},\overline{T}^*\right\rangle=\left\langle\overline{X},\overline{X}'\right\rangle=0$. We know that

$$\left\|\overline{X}\right\|=1. \tag{32}$$

By differentiating equation (32), we arrive at $\left\langle\overline{X},\overline{X}'\right\rangle=0$. Additionally, when we substituting $\overline{v}_0$ from equation (31), we find $\left\langle\overline{X}',\overline{T}^*\right\rangle=0$. This indicates that $\overline{X}'$ is orthogonal to the vectors $\overline{T}$ and $\overline{T}^*$. Conversely, for the tangent plane at the point $\overline{\varphi}(s,\overline{v}_0)$ assume that $\overline{X}'$ serves as a unit normal vector field.

At $\overline{\varphi}(s,\overline{v}_0)$, $\left\langle\overline{X}',\overline{T}^*\right\rangle=0$ is obtained since $\overline{X}'$ is a unit normal to the tangent plane. That is we obtain:

$$\overline{v}_0=\frac{\left[\phi'+n_1-r\cos\theta(\kappa_n^2-\tau_g^2)+\Delta\right]\sin\phi}{\left[\phi'+n_1-r\cos\theta(\kappa_n^2-\tau_g^2)+\Delta\right]^2+\left[n_2-r\sin\theta(\kappa_n^2+\tau_g^2)+\dfrac{d\Delta}{d\theta}\right]^2\cos^2\phi},$$

where $\Delta=r\cos\theta\left(r\kappa_n^2\left(\dfrac{\tau_g}{\kappa_n}\right)'-\tau_g'\right)$. Therefore, the point $\overline{\varphi}(s,\overline{v}_0)$ on the PRS with the BDF is a striction point. ■

Example 18. *Let us examine a surface $M=M(s,v)$ that is provided by*

$$M(s,v)=\left(s+5v,2+\frac{s^2}{2},s+v\right).$$

It is easy to see that the normal vector vector of the surface is that

$$N(s,v)=\left(\frac{s}{\sqrt{16+26s^2}},\frac{4}{\sqrt{16+26s^2}},-\frac{5s}{\sqrt{16+26s^2}}\right).$$

The curve

$$\alpha(s)=M(s,-1)=\left(s-5,2+\frac{s^2}{2},s-1\right)$$

lying on the surface $M=M(s,v)$ is clear to discern. The s-parameter curve $\alpha(s)$ is the curve shown in red in Figure 1. As a result, the tangent vector of the curve is as follows:

$$T(s)=\left(\frac{1}{\sqrt{2+s^2}},\frac{s}{\sqrt{2+s^2}},\frac{1}{\sqrt{2+s^2}}\right).$$

The Darboux vector G is written as

$$G(s)=N(s)\times T(s)=\left(\frac{5s^2+4}{\sqrt{16+26s^2}\sqrt{2+s^2}},-\frac{6s}{\sqrt{16+26s^2}\sqrt{2+s^2}},\frac{s^2-4}{\sqrt{16+26s^2}\sqrt{2+s^2}}\right).$$

From equation (3), the angle is obtained as:

$$\theta = -\arctan\left(\frac{3s}{2\sqrt{2+s^2}}\right)$$

where $\theta_0 = 0$. *Consequently,*

$$B_1(s) = \left(-\frac{s}{\sqrt{4+2s^2}}, \frac{2}{\sqrt{4+2s^2}}, -\frac{s}{\sqrt{4+2s^2}}\right)$$

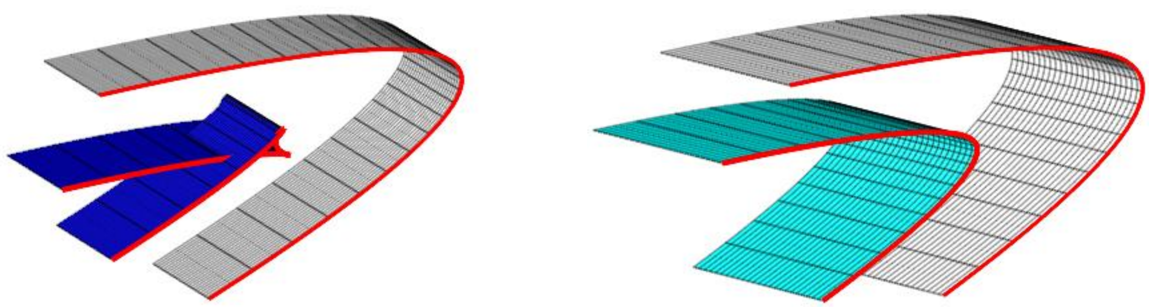
and

$$B_2(s) = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

are used to compute the new normal vectors B_1 *and* B_2 *of the surface. Thus, for* $r = 1$, *the parallel ruled surface (PRS)* $\overline{M}$ *of the surface* M *is parametrized by:*

$$\overline{M}(s,v) = \left(s+5v-\frac{s}{\sqrt{4+2s^2}}, 2+\frac{s^2}{2}+\frac{2}{\sqrt{4+2s^2}}, s+v-\frac{s}{\sqrt{4+2s^2}}\right).$$

Observe that the cuspidal edge occurs in construction of the parallel surface, shown in Figure 1.



(a) (b)

Fig. 1. The generator surface (gray), (a) The parallel surface (blue), (b) The B-Darboux parallel ruled surface (cyan).

Example 19. *Let the surface* $M = M(s,v)$ *be parameterized as*

$$M(s,v) = \left(s-3, sv+1, \frac{1}{s}-5\right).$$

It is easy to see tat the normal vector vector of the surface is that

$$N(s,v) = \left(\frac{1}{\sqrt{s^4+1}}, 0, \frac{s^2}{\sqrt{s^4+1}}\right).$$

The curve

$$\alpha(s) = M(s,-1) = \left(s-3, -s+1, \frac{1}{s}-5\right)$$

lying on the surface $M = M(s,v)$ *is clear to discern. The s-parameter curve* $\alpha(s)$ *is the curve shown in green in Figure 2. Also, the s-parameter curve* $\overline{\alpha}(s)$, *which is referred to as the base curve of the PRS, is the curve shown in red in Figure 2. As a result, the tangent vector of the curve is as follows:*

$$T(s) = \left(\frac{s^2}{\sqrt{2s^4+1}}, \frac{-s^2}{\sqrt{2s^4+1}}, \frac{-1}{\sqrt{2s^4+1}}\right).$$

The Darboux vector G *is written as*

$$G(s) = N(s) \times T(s) = \left(\frac{s^4}{\sqrt{s^4+1}\sqrt{2s^4+1}}, \frac{1+s^4}{\sqrt{s^4+1}\sqrt{2s^4+1}}, \frac{-s^2}{\sqrt{s^4+1}\sqrt{2s^4+1}}\right).$$

From equation (3), the angle is obtained as:

$$\theta = -\arctan(\sqrt{2s^4+1})$$

where $\theta_0 = 0$. *Consequently,*

$$B_1(s) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

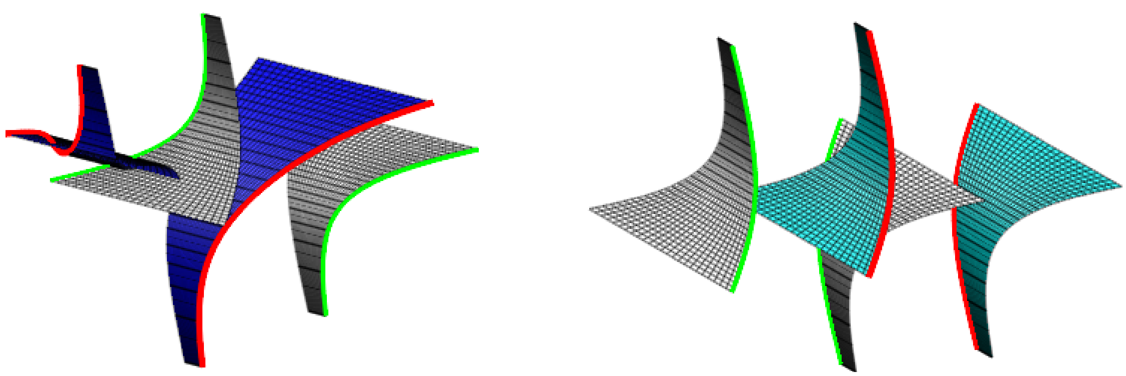
and

$$B_2(s) = \left(\frac{1}{\sqrt{4s^4+2}}, -\frac{1}{\sqrt{4s^4+2}}, \frac{2s^2}{\sqrt{4s^4+2}}\right)$$

are used to compute the new normal vectors B_1 *and* B_2 *of the surface. Thus, for* $r=1$, *the parallel ruled surface (PRS)* $\overline{M}$ *of the surface M is parametrized by:*

$$\overline{M}(s,v) = \left(s-3+\frac{1}{\sqrt{2}}, sv+1+\frac{1}{\sqrt{2}}, \frac{1}{s}-5\right).$$

Observe that the cuspidal edge occurs in construction of the parallel surface, shown in Figure 2.



(a) (b)

Fig. 2. The generator surface (gray) and, (a) The parallel surface (blue), (b) The B-Darboux parallel ruled surface (cyan).

Example 20. *Let us consider a bit more complex example. Suppose that*

$$M(s,v) = \left(s+5v, 2+\frac{s^3}{2}, s+v^3\right)$$

parametrizes the surface $M = M(s,v)$. *It is easy to see that the normal vector vector of the surface is that*

$$N(s,v) = \frac{1}{\sqrt{81s^4v^4+225s^4+36v^4-120v^2+100}}(9s^2v^2, 10-6s^2, -15s^2).$$

The curve

$$\alpha(s) = M(s,-1) = \left(s-5, 2+\frac{s^3}{2}, s-1\right)$$

lying on the surface $M = M(s,v)$ *is clear to discern. The s-parameter curve* $\alpha(s)$ *is the curve shown in red in Figure 3. As a result, the tangent vector of the curve is as follows:*

$$T(s) = \left(\frac{2}{\sqrt{9s^4+8}}, \frac{3s^2}{\sqrt{9s^4+8}}, \frac{2}{\sqrt{9s^4+8}}\right).$$

The Darboux vector G is written as

$$G(s) = N(s) \times T(s) = \left(\frac{45s^4 - 12v^2 + 20}{\sqrt{\triangle}\sqrt{9s^4+8}}, \frac{-18s^2v^2 - 30s^2}{\sqrt{\triangle}\sqrt{9s^4+8}}, \frac{27s^4v^2 + 12v^2 - 20}{\sqrt{\triangle}\sqrt{9s^4+8}} \right).$$

where $\triangle = 81s^4v^4 + 225s^4 + 36v^4 - 120v^2 + 100$. *From equation (3), the angle is obtained as:*

$$\theta = \int \frac{24s(9v^2 - 25)}{\triangle\sqrt{9s^4+8}} ds$$

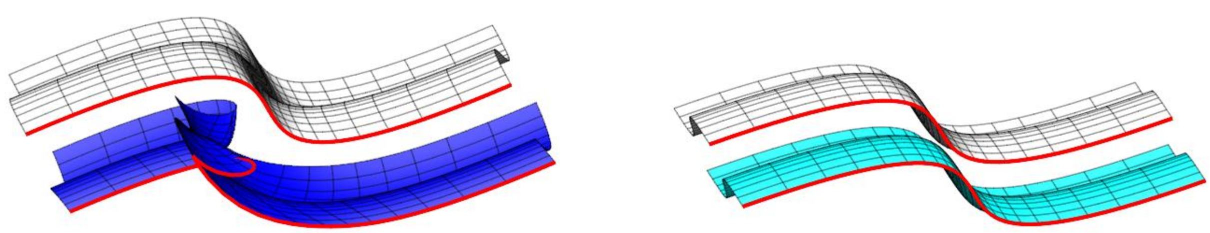
where $\theta_0 = 0$. *Consequently,*

$$B_1(s) = \frac{1}{\sqrt{\triangle}} \left(9\cos\theta s^2 v^2 + \frac{\sin\theta(45s^4 - 12v^2 + 20)}{\sqrt{9s^4+8}}, \cos\theta(10 - 6s^2) - \frac{\sin\theta(3v^2+5)}{\sqrt{\frac{1}{4} + \frac{2s^{-4}}{9}}}, \right.$$
$$\left. -15\cos\theta s^2 + \frac{\sin\theta(27s^4v^2 + 12v^2 - 20)}{\sqrt{9s^4+8}} \right)$$

is used to compute the new normal vectors B_1 *of the surface. Thus, for* $r = 1$, *the parallel surface* $\overline{M}$ *of the surface M is parametrized by:*

$$\overline{M}(s,v) = \left(s - 5 + \frac{1}{\sqrt{\triangle}} \left(9\cos\theta s^2 v^2 + \frac{\sin\theta(45s^4 - 12v^2 + 20)}{\sqrt{9s^4+8}} \right), \right.$$
$$2 + \frac{s^3}{2} + \frac{1}{\sqrt{\triangle}} \left(\cos\theta(10 - 6s^2) - \frac{\sin\theta(18s^2v^2 + 30s^2)}{\sqrt{9s^4+8}} \right),$$
$$\left. s - 1 + \frac{1}{\sqrt{\triangle}} \left(-15\cos\theta s^2 + \frac{\sin\theta(27s^4v^2 + 12v^2 - 20)}{\sqrt{9s^4+8}} \right) \right).$$

This new frame also allow us to obtain nice results. For $r = 1$, *the parallel surface with BDF and parallel surface of the surface* $M = M(s,v)$ *are illustrated in Figure 3.*



(a) (b)

Fig. 3. The generator surface (gray), and (a) The parallel surface (blue), (b) The B-Darboux parallel surface (cyan).

5. Conclusions

In this study, the geometric properties of ruled, and parallel ruled surfaces associated with the B-Darboux frame in Euclidean 3-space have been thoroughly investigated. The analysis presented herein provides a comprehensive understanding of how these surfaces are constructed and highlights their fundamental characteristic properties, including developability, striction point, and distribution parameter. The conditions under which these surfaces retain specific geometric properties within the B-Darboux frame have been carefully delineated, offering valuable insights into their theoretical structure. Additionally, three examples are also given, supported by figures.

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