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Unveiling $f-$Biharmonic $\theta_\alpha-$Slant Curves in $\mathcal{S}-$ Space Forms

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ABSTRACT

In this paper, we firstly provide a concise overview of $\mathcal{S}-$manifolds, f-biharmonicity and θ_α-slant curves. We then derive a key equation and analyze it in detail to establish the necessary and sufficient conditions for θ_α-slant curves to be f-biharmonic. Finally, we present an example to support our findings.

Keywords: *$\theta_\alpha-$slant curve; f-biharmonic curve; Frenet curve; $\mathcal{S}$-space form.*
AMS Subject Classification (2020): *Primary: 53C25 ; Secondary: 53C40; 53A04.*

1. Introduction

In 1964, J. Eells and L. Lemaire explored key aspects of harmonic maps, later extending their ideas to k-harmonic maps [7]. Based on this, G. Y. Jiang studied the case $k = 2$, deriving the variational formulas for 2-harmonic maps [15]. B. Y. Chen, in a comprehensive survey, defined biharmonic submanifolds of Euclidean space, as those satisying the condition $\Delta H = 0$, with Δ being the Laplace operator and H representing the mean curvature vector field [5]. Notably, if the ambient space is Euclidean, the works of Jiang and Chen converge.

Expanding this field, J. T. Cho, J. Inoguchi, and J. E. Lee introduced the concept of slant curves in Sasakian manifolds, drawing a parallel to Lancret's theorem for Euclidean space curves [6]. Their findings highlighted that non-geodesic curves in Sasakian 3-manifolds are slant curves when the ratio of $(\tau \pm 1)$, , where τ is the torsion of the curve, to the geodesic curvature k remains constant.

Further advancements were made by D. Fetcu and C. Oniciuc, who proposed a method to generate biharmonic submanifolds in Sasakian space forms by utilizing the characteristic vector field flow ξ [9]. They demonstrated that this flow transforms biharmonic integral submanifolds into biharmonic anti-invariant submanifolds. Building on this work, the author and C. Özgür examined biharmonic slant curves in $\mathcal{S}-$space forms, aiming to generalize these findings [10].

In a recent study [13], the author introduced θ_α-slant curves, expanding the concept of slant curves within S-manifolds. This extension introduces an adjustable parameter, θ_α, allowing a broader classification of slant curves and providing examples in specific dimensional settings. The paper examines the biharmonicity of these curves in S-space forms, contributing new insights to the geometry of S-manifolds.

In this present study, we aim to extend the results for biharmonic slant curves by exploring $f-$biharmonic $\theta_\alpha-$slant curves in $\mathcal{S}-$space forms. The subsequent sections provide definitions, examine the necessary conditions for these curves to be proper $f-$biharmonic and conclude with an example that demonstrate these concepts.

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2. Background and Definitions

Let (M, g) be a $(2m+s)$-dimensional Riemannian manifold. The manifold M is called a *framed metric manifold* with a *framed metric structure* $(\phi, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1, ..., s\}$, if it satisfies:

$$\begin{gathered}\phi^2 X = -X + \textstyle\sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha, \quad \eta_\alpha(\phi(X)) = 0, \quad \eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}, \quad \phi(\xi_\alpha) = 0, \\ g(X, Y) = g(\phi X, \phi Y) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \\ \eta_\alpha(X) = g(X, \xi), \quad d\eta_\alpha(X, Y) = -d\eta_\alpha(Y, X) = g(X, \phi Y),\end{gathered} \tag{2.1}$$

where ϕ is a $(1, 1)$-type tensor field of rank $2m$; $\xi_1, ..., \xi_s$ are vector fields; $\eta_1, ..., \eta_s$ are 1-forms, and g is a Riemannian metric on M; $X, Y \in \mathfrak{X}(M)$ and $\alpha, \beta \in \{1, ..., s\}$ (see [17], [20]). The Nijenhuis tensor N_ϕ is defined by

$$N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y],$$

for all $X, Y \in \mathfrak{X}(M)$. The structure $(\phi, \xi_\alpha, \eta_\alpha, g)$ is said to be an *$\mathcal{S}$-structure* if the Nijenhuis tensor of ϕ is equal to $-2d\eta_\alpha \otimes \xi_\alpha$ for all $\alpha \in \{1, ..., s\}$ [2].

If $s = 1$, a framed metric structure is equivalent to an almost contact metric structure, and an $\mathcal{S}$-structure is equivalent to a Sasakian structure. For an $\mathcal{S}$-structure, we have the following equations [2]:

$$(\nabla_X \phi)Y = \sum_{\alpha=1}^{s} \left\{ g(\phi X, \phi Y)\xi_\alpha + \eta_\alpha(Y)\phi^2 X \right\}, \tag{2.2}$$

and

$$\nabla \xi_\alpha = -\phi, \tag{2.3}$$

for all $\alpha = 1, ..., s$. In the case of $s = 1$, (2.3) can be derived from (2.2).

Let $X \in \mathfrak{X}(M)$ be orthogonal to $\xi_1, ..., \xi_s$. The plane section spanned by $\{X, \phi X\}$ is called a *ϕ-section* in $\mathfrak{X}(M)$, and its sectional curvature is referred to as the *ϕ-sectional curvature*. Let $(M, \phi, \xi_\alpha, \eta_\alpha, g)$ be an $\mathcal{S}$-manifold. If M has constant ϕ-sectional curvature, its curvature tensor R is given by

$$\begin{aligned} R(X, Y)Z = &\sum_{\alpha,\beta} \Big\{ \eta_\alpha(X)\eta_\beta(Z)\phi^2 Y - \eta_\alpha(Y)\eta_\beta(Z)\phi^2 X - g(\phi X, \phi Z)\eta_\alpha(Y)\xi_\beta + g(\phi Y, \phi Z)\eta_\alpha(X)\xi_\beta \Big\} \\ &+ \frac{c + 3s}{4} \Big\{ - g(\phi Y, \phi Z)\phi^2 X + g(\phi X, \phi Z)\phi^2 Y \Big\} \\ &+ \frac{c - s}{4} \Big\{ g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z \Big\}, \end{aligned} \tag{2.4}$$

for $X, Y, Z \in \mathfrak{X}(M)$ [4]. In this case, M is called an *$\mathcal{S}$-space form* and is denoted by $M(c)$. If $s = 1$, an $\mathcal{S}$-space form is equivalent to a Sasakian space form [1].

Let (M, g) and (N, h) be Riemannian manifolds, and $\phi : M \to N$ a differentiable map. A *harmonic map* is a critical point of the energy functional of ϕ, defined as

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \, \upsilon_g,$$

(see [8]). The *first tension field* of ϕ is defined by $\tau(\phi) = \text{trace}\nabla d\phi$. The Euler-Lagrange equation of the energy functional E is given by $\tau(\phi) = 0$, which characterizes harmonic maps. Furthermore, a *biharmonic map* is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, \upsilon_g.$$

Recall that the Laplacian (also called the rough Laplacian) of a section X of the pullback bundle $\phi^* TN$ is given by

$$\Delta^\phi X = \text{trace}\nabla^\phi \nabla^\phi X,$$

where ∇^ϕ is the pullback connection induced by the Levi-Civita connection ∇^N on N. Also, the *curvature tensor field* R^N of N is defined by

$$R^N(U, V)W = \nabla^N_U \nabla^N_V W - \nabla^N_V \nabla^N_U W - \nabla^N_{[U,V]} W.$$

Using these formulas, Jiang derived the biharmonic map equation [15]:

$$\tau_2(\phi) = -J^\phi(\tau(\phi)) = -\Delta\tau(\phi) - \text{trace}R^N(d\phi, \tau(\phi))d\phi = 0,$$

where J^ϕ denotes the Jacobi operator of ϕ. It is evident that harmonic maps are biharmonic; thus, non-harmonic biharmonic maps are referred to as *proper biharmonic*.

The *f-bi-energy functional* of a smooth map $\phi : (M, g) \to (N, h)$ is defined by

$$E_{2,f}(\phi) = \frac{1}{2}\int_\Omega f|\tau(\phi)|^2 \upsilon_g$$

for every compact domain $\Omega \subset M$. A map ϕ is called *f-biharmonic* if it is a critical point of the f-bi-energy functional [18].

In [18], Y.L. Ou proved the following lemma:

Lemma 2.1. *[18] A curve $\gamma : (a, b) \to (M, g)$ parametrized by arclength is an f-biharmonic curve with a function $f : (a, b) \to (0, \infty)$ if and only if*

$$f(\nabla_T\nabla_T\nabla_T T - R(T, \nabla_T T)T) + 2f'\nabla_T\nabla_T T + f''\nabla_T T = 0.$$

As a result, γ is a proper f-biharmonic curve if and only if

$$\tau_3 = \nabla_T\nabla_T\nabla_T T - R(T, \nabla_T T)T + 2\frac{f'}{f}\nabla_T\nabla_T T + \frac{f''}{f}\nabla_T T = 0,$$

where f is a non-constant function.

Let $\gamma : I \to M$ be a unit-speed curve in an m-dimensional Riemannian manifold (M, g). The curve γ is called a *Frenet curve of osculating order* r $(1 \leq r \leq m)$, if there exist g-orthonormal vector fields $V_1, V_2, ..., V_r$ along the curve satisfying the Frenet equations:

$$\begin{aligned} T &= V_1 = \gamma', \\ \nabla_T V_1 &= k_1 V_2, \\ \nabla_T V_j &= -k_{j-1}V_{j-1} + k_j V_{j+1}, \quad 1 < j < r, \\ \nabla_T V_r &= -k_{r-1}V_{r-1}. \end{aligned} \tag{2.5}$$

Here, $k_1, ..., k_{r-1}$ are positive functions known as the curvatures of γ. If $k_1 = 0$, then γ is referred to as a *geodesic*. If k_1 is a non-zero positive constant and $r = 2$, γ is called a *circle*. If $k_1, ..., k_{r-1}$ are all non-zero positive constants, then γ is called a *helix of order* r $(r \geq 3)$. When $r = 3$, it is simply referred to as a *helix*.

A submanifold of an $\mathcal{S}$-manifold is said to be an *integral submanifold* if $\eta_\alpha(X) = 0$, $\alpha \in \{1, ..., s\}$, where X is tangent to the submanifold [16]. A *Legendre curve* is a 1-dimensional integral submanifold of an $\mathcal{S}$-manifold $(M^{2m+s}, \phi, \xi_\alpha, \eta_\alpha, g)$. More precisely, a unit-speed curve $\gamma : I \to M$ is a Legendre curve if T is g-orthogonal to all ξ_α $(\alpha = 1, ..., s)$ [19].

3. $\theta_\alpha-$Slant Curves in $\mathcal{S}-$Manifolds

In this section, we review the concept of $\theta_\alpha-$slant curves in $\mathcal{S}$-manifolds:

Definition 3.1. [13] Let $M = (M^{2m+s}, \phi, \xi_\alpha, \eta_\alpha, g)$ be an $\mathcal{S}$-manifold, and let $\gamma : I \to M$ be a unit-speed curve. γ is called a *$\theta_\alpha-$slant curve* if there are constant angles θ_α $(\alpha = 1, \ldots, s)$ such that $\eta_\alpha(T) = \cos\theta_\alpha$. These angles θ_α are referred to as the *contact angles* of γ.

It is easy to observe that Definition 3.1 generalizes the family of slant curves to $\theta_\alpha-$slant curves. In particular, a $\theta_\alpha-$slant curve is called *slant* if all its contact angles are equal (see [10]).

For a $\theta_\alpha-$slant curve, it is known that $\eta_\alpha(V_2) = 0$ for all $\alpha = 1, \ldots, s$. The following notations are used similarly to [13]:

$$a = \sum_{\alpha=1}^{s}\cos^2\theta_\alpha, \quad b = \sum_{\alpha=1}^{s}\cos\theta_\alpha, \quad \mathcal{V} = \sum_{\alpha=1}^{s}\cos\theta_\alpha\xi_\alpha.$$

Corollary 3.1. *[13] If γ is a slant curve, then*

$$a = s\cos^2\theta, \quad b = s\cos\theta, \quad \mathcal{V} = \cos\theta\sum_{\alpha=1}^{s}\xi_\alpha,$$

where θ denotes the common contact angle of γ.

Let γ be a non-geodesic unit-speed $\theta_\alpha-$slant curve. Then, $g(\phi T, \phi T) = 1 - a \geq 0$. If $a = 1$, we have $\phi T = 0$, which implies $T = \mathcal{V}$. Hence, $\nabla_T T = \nabla_{\mathcal{V}}\mathcal{V} = 0$, meaning that γ is a geodesic as an integral curve of $\mathcal{V}$.

Proposition 3.1. *[13] For a non-geodesic unit-speed $\theta_\alpha-$slant curve in an $\mathcal{S}$-manifold,*

$$a = \sum_{\alpha=1}^{s}\cos^2\theta_\alpha < 1.$$

Proposition 3.2. *[13] For a non-geodesic unit-speed $\theta_\alpha-$slant curve in an $\mathcal{S}$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha, g)$, we have*

$$\nabla_T\phi T = (1-a)\sum_{\alpha=1}^{s}\xi_\alpha + b(-T + \mathcal{V}) + k_1\phi V_2. \tag{3.1}$$

4. f-biharmonic θ_α-Slant Curves in $\mathcal{S}$-Space Forms

In this section, we consider proper f-biharmonic θ_α-slant curves in $\mathcal{S}$-space forms. Let γ be a unit-speed θ_α-slant curve in an $\mathcal{S}$-space form $(M, \phi, \xi_\alpha, \eta_\alpha, g)$. In [13], it is shown that

$$R(T, \nabla_T T)T = -k_1\left[b^2 + \frac{c+3s}{4}(1-a)\right]V_2 - 3k_1\frac{c-s}{4}g(\phi T, V_2)\phi T,$$

$$\tau_2(\gamma) = \nabla_T\nabla_T\nabla_T T - R(T, \nabla_T T)T.$$

Thus, we can calculate

$$\begin{aligned}
\tau_3 &= \nabla_T\nabla_T\nabla_T T - R(T, \nabla_T T)T + 2\frac{f'}{f}\nabla_T\nabla_T T + \frac{f''}{f}\nabla_T T \\
&= \left(-3k_1k_1' - 2k_1^2\frac{f'}{f}\right)T \\
&\quad + \left[k_1'' - k_1^3 - k_1k_2^2 + k_1\left[b^2 + \frac{c+3s}{4}(1-a)\right] + 2\frac{f'}{f}k_1' + \frac{f''}{f}k_1\right]V_2 \\
&\quad + \left(2k_1'k_2 + k_1k_2' + 2\frac{f'}{f}k_1k_2\right)V_3 + k_1k_2k_3V_4 + 3k_1\frac{c-s}{4}g(\phi T, V_2)\phi T.
\end{aligned} \tag{4.1}$$

As a result, we can state the following theorem:

Theorem 4.1. *γ is a proper f-biharmonic θ_α-slant curve in an $\mathcal{S}$-space form $(M, \phi, \xi_\alpha, \eta_\alpha, g)$ if and only if*

$$3\frac{k_1'}{k_1} + 2\frac{f'}{f} = 0, \tag{4.2}$$

$$k_1^2 + k_2^2 = \frac{k_1''}{k_1} + \frac{f''}{f} + 2\frac{f'}{f}\frac{k_1'}{k_1} + b^2 + \frac{c+3s}{4}(1-a) + 3\frac{c-s}{4}g(\phi T, V_2)^2, \tag{4.3}$$

$$k_2' + 2k_2\frac{k_1'}{k_1} + 2k_2\frac{f'}{f} + 3\frac{c-s}{4}g(\phi T, V_2)g(\phi T, V_3) = 0, \tag{4.4}$$

$$k_2k_3 + 3\frac{c-s}{4}g(\phi T, V_2)g(\phi T, V_4) = 0 \tag{4.5}$$

and $g(\tau_3, \phi T) = 0$.

Proof. Let γ be a proper f-biharmonic θ_α-slant curve. Then $\tau_3 = 0$. From equation 4.1, if we apply T, V_2, V_3, V_4, and ϕT, we obtain equations (4.2), (4.3), (4.4), (4.5), and $g(\tau_3, \phi T) = 0$. Conversely, if the given equations are satisfied for a θ_α-slant curve, it can be easily shown that $\tau_3 = 0$. Therefore, γ is a proper f-biharmonic curve. □

The following Lemma will be crucial for the reader to understand the results of next cases:

Lemma 4.1. *Let $y = y(x)$ be a real valued function. Furthermore, let $c_2 \geqslant 0$, $\lambda \geqslant 0$ be real constants, $\varepsilon \in \{-1, 0, +1\}$ and $u = u(x) = 2\lambda x + c_4$ for an arbitrary constant c_4. Then the autonomous ODE*

$$3\left(y'\right)^2 - 2yy'' = 4y^2\left[\left(1 + c_2^2\right)y^2 - \varepsilon\lambda^2\right]$$

has the general solution of the form

$$y = \frac{\pm\sqrt{N} + M}{D}, \tag{4.6}$$

where the functions N, M and D denotes

(i) *for $\varepsilon = +1$ and $\lambda > 0$:*

$$N = \lambda^2 \sec^2 u.\left[-\left(1 + c_2^2 + c_3^2\right)\sec^2 u + \left(1 + c_2^2 - c_3^2\right)\right], \tag{4.7}$$

$$M = \lambda c_3 \sec^2 u,\ D = \left(1 + c_2^2\right)\sec^2 u - \left(1 + c_2^2 - c_3^2\right); \tag{4.8}$$

(ii) *for $\varepsilon = -1$ and $\lambda > 0$:*

$$N = \lambda^2 \operatorname{sech}^2 u.\left[\left(1 + c_2^2 + c_3^2\right)\operatorname{sech}^2 u - \left(1 + c_2^2 + c_3^2\right)\right], \tag{4.9}$$

$$M = \lambda c_3 \operatorname{sech}^2 u,\ D = \left(1 + c_2^2\right)\operatorname{sech}^2 u - \left(1 + c_2^2 + c_3^2\right); \tag{4.10}$$

(iii) *for $\varepsilon = 0$ or $\lambda = 0$:*

$$N = 0,\ M = 4c_3, \tag{4.11}$$

$$D = c_3^2x^2 + 2c_3^2c_4x + c_3^2c_4^2 + 16c_2^2 + 16. \tag{4.12}$$

Note that the arbitrary constants of the general solutions in Lemma 4.1 are denoted by c_3 and c_4 to distinguish from c_1 and c_2, which will be used in some other equations later.

Now, we are all set to consider the equation $g(\tau_3, \phi T) = 0$ from all points of view. Notice that if the coefficient of ϕT vanishes, then $g(\tau_3, \phi T) = 0$ is satisfied directly. The cases are:

Case I. $c = s$.

Case II. $c \neq s$ and $g(\phi T, V_2) = 0$.

When the coefficient of ϕT does not vanish, we will investigate two more cases:

Case III. $c \neq s$ and $\phi T \parallel V_2$.

Case IV. $c \neq s$ and $g(\phi T, V_2) \neq 0$ or $\pm\sqrt{1-a}$.

One might ask what happens if ϕT itself vanishes as a vector field. In this case, as shown in [13], we have $T = \mathcal{V} = \sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha$ and $k_1 = 0$. Thus, equation (4.2) implies that f is a constant. Consequently, γ cannot be a proper f-biharmonic curve. Note that being proper f-biharmonic means f-biharmonic but not biharmonic. Likewise, being proper biharmonic means biharmonic but not harmonic.

Case I. $c = s$.

In this case, equations (4.2-4.5) become

$$3\frac{k_1'}{k_1} + 2\frac{f'}{f} = 0, \tag{4.13}$$

$$k_1^2 + k_2^2 = \frac{k_1''}{k_1} + \frac{f''}{f} + 2\frac{f'}{f}\frac{k_1'}{k_1} + b^2 + s(1-a), \tag{4.14}$$

$$k_2' + 2k_2\frac{k_1'}{k_1} + 2k_2\frac{f'}{f} = 0, \tag{4.15}$$

$$k_2k_3 = 0, \tag{4.16}$$

By solving these equations, we can state the following theorem:

Theorem 4.2. *Under the assumption $c = s$, γ is a proper f-biharmonic θ_α-slant curve in $(M, \phi, \xi_\alpha, \eta_\alpha, g)$ if and only if:*
(i) γ is of osculating order $r = 3$ with $f = c_1 k_1^{-3/2}$, $\frac{k_2}{k_1} = c_2$, and $k_1 = k_1(t)$ is of the form

$$k_1(t) = \frac{\pm\sqrt{N} + M}{D},$$

where the functions N, M and D are denoted by equations (4.7) and (4.8) and

$$\lambda_1 = \sqrt{b^2 + s(1-a)} > 0, \ u = u(t) = 2\lambda_1 t + c_4,$$

$c_1 > 0$, $c_2 > 0$, c_3 and c_4 are arbitrary constants, t is the arc-length parameter; or
(ii) γ is of osculating order $r = 2$ and satisfies the above equations and inequalities, except for the inequality $c_2 > 0$, where instead $c_2 = 0$.

Proof. Let $k_1 = k_1(t)$, where t denotes the arc-length parameter. From equation (4.13), it is easy to see that $f = c_1 k_1^{-3/2}$ for an arbitrary constant $c_1 > 0$. Thus, we obtain

$$\frac{f'}{f} = \frac{-3}{2}\frac{k_1'}{k_1}, \quad \frac{f''}{f} = \frac{15}{4}\left(\frac{k_1'}{k_1}\right)^2 - \frac{3}{2}\frac{k_1''}{k_1}. \tag{4.17}$$

If $k_2 = 0$, then γ is of osculating order $r = 2$, and equations (4.13) and (4.14) must be satisfied. Therefore, the second equation combined with (4.17) yields the ODE

$$3(k_1')^2 - 2k_1k_1'' = 4k_1^2\left[k_1^2 - \left(b^2 + s(1-a)\right)\right]. \tag{4.18}$$

This ODE can be solved using Lemma 4.1. Firstly, $1 + c_2^2 = 1$ gives us $c_2 = 0$. Since $s \geqslant 1$, $1 - a > 0$ and $b^2 \geqslant 0$, we have $b^2 + s(1-a) > 0$. We can write

$$\begin{aligned} b^2 + s(1-a) &= sgn\left(b^2 + s(1-a)\right).\left(\sqrt{b^2 + s(1-a)}\right)^2 \\ &= \varepsilon\lambda_1^2, \end{aligned}$$

where we denote $\lambda_1 = \sqrt{b^2 + s(1-a)}$. Now that $\varepsilon = +1$ and $\lambda_1 > 0$, using Lemma 4.1 (i), we find that $k_1(t)$ is of the form (4.6) where N, M and D are as in equations (4.7) and (4.8) (with $c_2 = 0$). Notice that $c_2 = 0$ is equivalent to $k_2 = 0 = c_2k_1$ when $r = 2$.

If $k_2 =$ constant $\neq 0$, we find that f is a constant, indicating that γ is not proper f-biharmonic in this case. For $k_2 \neq$ constant, from equation (4.16), we have $k_3 = 0$. Thus, γ is of osculating order $r = 3$. Using equation (4.17), equation (4.15) gives us $\frac{k_2}{k_1} = c_2$, where $c_2 > 0$ is a constant. Substituting these results into equation (4.14), we obtain the ODE

$$3(k_1')^2 - 2k_1k_1'' = 4k_1^2[(1 + c_2^2)k_1^2 - \left(b^2 + s(1-a)\right)]$$

which gives the general solution using Lemma 4.1 and the proof is completed. □

Case II. $c \neq s$ and $\phi T \perp V_2$.
In this case, $g(\phi T, V_2) = 0$. From Theorem 4.1, we obtain

$$\begin{gathered} 3\frac{k_1'}{k_1} + 2\frac{f'}{f} = 0, \\ k_1^2 + k_2^2 = \frac{k_1''}{k_1} + \frac{f''}{f} + 2\frac{k_1'}{k_1}\frac{f'}{f} + b^2 + \frac{c+3s}{4}(1-a), \\ k_2' + 2k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} = 0, \\ k_2k_3 = 0. \end{gathered}$$

Firstly, we need the following Lemma from [13]:

Lemma 4.2. *[13] Let γ be a $\theta_\alpha-$slant curve of order $r = 3$ in an $\mathcal{S}$-space form $(M, \phi, \xi_\alpha, \eta_\alpha, g)$ and $\phi T \perp V_2$. Then, $\{T, V_2, V_3, \phi T, \nabla_T\phi T, \xi_1, ..., \xi_s\}$ is linearly independent. So $dimM \geqslant 5 + s$.*

Now we have the following theorem:

Theorem 4.3. *Let γ be a $\theta_\alpha-$slant curve in an $\mathcal{S}$-space form $(M^{2m+s}, \phi, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1, ..., s\}$, $c \neq s$ and $\phi T \perp V_2$. Then γ is proper f-biharmonic if and only if*
(1) *γ is of osculating order $r = 3$ with $f = c_1 k_1^{-3/2}$, $\frac{k_2}{k_1} = c_2$, $m \geq 3$,*

$$\{T = V_1, V_2, V_3, \phi T, \nabla_T \phi T, \xi_1, ..., \xi_s\}$$

is linearly independent and $k_1 = k_1(t)$ is of the form

$$k_1 = \frac{\pm\sqrt{N} + M}{D}$$

where the functions N, M and D are denoted by
(*a*) *equations* (4.7) *and* (4.8)*, for $b^2 + [(c + 3s)/4](1 - a) > 0$;*
(*b*) *equations* (4.9) *and* (4.10)*, for $b^2 + [(c + 3s)/4](1 - a) < 0$;*
(*c*) *equations* (4.11) *and* (4.12)*, for $b^2 + [(c + 3s)/4](1 - a) = 0$;*
with

$$\lambda_2 = \sqrt{\left|b^2 + \frac{(c+3s)}{4}(1-a)\right|},\ u = u(t) = 2\lambda_2 t + c_4,$$

for arbitrary constants $c_1 > 0$, $c_2 > 0$, c_3 and c_4; or
(2) *γ is of osculating order $r = 2$ and satisfies the above equations and inequalities, except for $m \geq 3$, $c_2 > 0$,*

$$\{T = V_1, V_2, V_3, \phi T, \nabla_T \phi T, \xi_1, ..., \xi_s\}$$

is linearly independent, where instead $m \geq 2, c_2 = 0$,

$$\{T = V_1, V_2, \phi T, \nabla_T \phi T, \xi_1, ..., \xi_s\}$$

is linearly independent.

Proof. The proof is similar to the proof of Theorem 4.2. In this case, we have the ODE

$$3(k_1')^2 - 2k_1 k_1'' = 4k_1^2[(1 + c_2^2)k_1^2 - \left[b^2 + \frac{(c+3s)}{4}(1-a)\right].$$

We use Lemma 4.1 using the fact that

$$\begin{aligned} b^2 + \frac{c+3s}{4}(1-a) &= sgn\left(b^2 + \frac{c+3s}{4}(1-a)\right)\left(\sqrt{\left|b^2 + \frac{c+3s}{4}(1-a)\right|}\right)^2 \\ &= \varepsilon\lambda_2^2. \end{aligned}$$

For $b^2 + [(c + 3s)/4](1 - a) > 0$, we have $\varepsilon = +1$ and $\lambda_2 > 0$. For $b^2 + [(c + 3s)/4](1 - a) < 0$, we find $\varepsilon = -1$ and $\lambda_2 > 0$. Finally $b^2 + [(c + 3s)/4](1 - a) = 0$ gives $\varepsilon = 0$ and $\lambda_2 = 0$. □

Case III. $c \neq s$, $\phi T \parallel V_2$.
As a result of the assumptions of this case, we have $\phi T = \epsilon\sqrt{1-a}V_2$, $g(\phi T, V_2) = \epsilon\sqrt{1-a}$, $g(\phi T, V_3) = 0$ and $g(\phi T, V_4) = 0$, where $\epsilon = sgn(g(\phi T, V_2)) = \pm 1$. From equations (4.4) and (4.17), we can write

$$k_2' + 2k_2\left(\frac{-3}{2}\frac{k_1'}{k_1}\right) + 2k_2\frac{k_1'}{k_1} = 0. \tag{4.19}$$

Integrating (4.19), we have

$$\frac{k_2}{k_1} = c_2,$$

for some constant $c_2 > 0$. Likewise in [13], for non-constant k_1, one can show that if $\phi T = \epsilon\sqrt{1-a}V_2$, then

$$k_2 = \sqrt{ad^2 - as + b^2 + 2\epsilon bd + s},$$

where $d = k_1/\sqrt{1-a}$. So we get

$$\frac{\sqrt{ad^2 - as + b^2 + 2\epsilon bd + s}}{k_1} = c_2,$$

which is equivalent to

$$\left(c_2^2 + \frac{a}{a-1}\right)k_1^2 + \frac{2\epsilon b}{a-1}k_1 + \left(as - b^2 - s\right) = 0.$$

If this equation is quadratic or linear in terms of k_1, then k_1 becomes a constant and γ cannot be proper f-biharmonic. Let us assume

$$c_2^2 + \frac{a}{a-1} = 0,$$
$$\frac{2\epsilon b}{a-1} = 0,$$
$$as - b^2 - s = 0.$$

The second equation above gives that $b = 0$. Then the last equation reduces to $as - s = 0$, that is, $a = 1$. But in this case γ becomes a geodesic as an integral curve of $\mathcal{V}$ and cannot be proper f-biharmonic. Hence, we give the following result:

Theorem 4.4. *There does not exist any proper f-biharmonic $\theta_\alpha-$slant curve in an $\mathcal{S}$-space form $(M^{2m+s}, \phi, \xi_\alpha, \eta_\alpha, g)$ with $c \neq s$ and $\phi T \parallel V_2$.*

Case IV. $c \neq s$ and $g(\phi T, V_2)$ is not 0 or $\pm\sqrt{1-a}$.

In this final case, let $(M^{2m+s}, \phi, \xi_\alpha, \eta_\alpha, g)$ be an $\mathcal{S}$-space form and $\gamma : I \to M$ a θ_α-slant curve of osculating order r. Note that $r = 2$ corresponds to $\phi T \in$span$\{V_2\}$, which was investigated in Case III. So let $3 \leq r \leq 2m+s$. If γ is f-biharmonic, then $\phi T \in \text{span}\{V_2, V_3, V_4\}$. Let $\beta(t)$ denote the angle function between ϕT and V_2, that is, $g(\phi T, V_2) = \sqrt{1-a}\cos\beta(t)$. If we differentiate $g(\phi T, V_2)$ along γ and use equations (2.5) and (3.1), we get

$$\begin{aligned} -\sqrt{1-a}\beta'(t)\sin\beta(t) &= \nabla_T g(\phi T, V_2) = g(\nabla_T \phi T, V_2) + g(\phi T, \nabla_T V_2) \\ &= g\left((1-a)\sum_{\alpha=1}^{s}\xi_\alpha + b\left(-T + \mathcal{V}\right) + k_1\phi V_2, V_2\right) + g(\phi T, -k_1 T + k_2 V_3) \\ &= k_2 g(\phi T, V_3). \end{aligned} \tag{4.20}$$

If we write $\phi T = g(\phi T, V_2)V_2 + g(\phi T, V_3)V_3 + g(\phi T, V_4)V_4$, Theorem 4.1 gives us

$$3\frac{k_1'}{k_1} + 2\frac{f'}{f} = 0, \tag{4.21}$$

$$k_1^2 + k_2^2 = \frac{k_1''}{k_1} + \frac{f''}{f} + 2\frac{f'}{f}\frac{k_1'}{k_1} + \left[b^2 + \frac{c+3s}{4}(1-a) + 3\frac{c-s}{4}(1-a)\cos^2\beta\right], \tag{4.22}$$

$$k_2' + 2k_2\frac{k_1'}{k_1} + 2k_2\frac{f'}{f} + 3\frac{c-s}{4}\sqrt{1-a}\cos\beta g(\phi T, V_3) = 0, \tag{4.23}$$

$$k_2 k_3 + 3\frac{c-s}{4}\sqrt{1-a}\cos\beta g(\phi T, V_4) = 0. \tag{4.24}$$

If we substitute (4.17) into (4.22) and (4.23), we find

$$k_1^2 + k_2^2 = b^2 + \frac{c+3s}{4}(1-a) + 3\frac{c-s}{4}(1-a)\cos^2\beta - \frac{k_1''}{2k_1} + \frac{3}{4}\left(\frac{k_1'}{k_1}\right)^2, \tag{4.25}$$

$$k_2' - \frac{k_1'}{k_1}k_2 + 3\frac{c-s}{4}\sqrt{1-a}\cos\beta g(\phi T, V_3) = 0. \tag{4.26}$$

If we multiply (4.26) by $2k_2$ and use (4.20), we obtain

$$2k_2 k_2' - 2\frac{k_1'}{k_1}k_2^2 + \frac{3(c-s)}{4}(1-a)(-2\beta'\cos\beta\sin\beta) = 0. \tag{4.27}$$

Let us denote $v(t) = k_2^2(t)$, where t is the arc-length parameter. Then (4.27) becomes

$$\upsilon' - 2\frac{k_1'}{k_1}\upsilon = -\frac{3(c-s)}{4}(1-a)(-2\beta'\cos\beta\sin\beta), \tag{4.28}$$

which is a linear ODE. If we solve (4.28), we get the following results:
$i)$ If β is a constant, then

$$\frac{k_2}{k_1} = c_2, \tag{4.29}$$

where $c_2 > 0$ is an arbitrary constant. From (4.20) and (4.30), we find $g(\phi T, V_3) = 0$. Since $\|\phi T\| = \sqrt{1-a}$ and $\phi T = \sqrt{1-a}\cos\beta V_2 + g(\phi T, V_4)V_4$, we obtain $g(\phi T, V_4) = \pm\sqrt{1-a}\sin\beta$. Using (4.22) and (4.29), we have

$$3(k_1')^2 - 2k_1k_1'' = 4k_1^2\left[(1+c_2^2)k_1^2 - b^2 - \frac{c+3s+3(c-s)\cos^2\beta}{4}(1-a)\right].$$

$ii)$ If $\beta = \beta(t)$ is a non-constant function, then

$$k_2^2 = -\frac{3(c-s)}{4}(1-a)\cos^2\beta + \mu(t)k_1^2, \tag{4.30}$$

where

$$\mu(t) = -\frac{3(c-s)}{2}(1-a)\int\frac{\cos^2\beta k_1'}{k_1^3}dt. \tag{4.31}$$

If we substitute (4.30) into (4.25), we find

$$[1+\mu(t)]\,k_1^2 = b^2 + \frac{c+3s}{4}(1-a) + \frac{3(c-s)}{2}(1-a)\cos^2\beta - \frac{k_1''}{2k_1} + \frac{3}{4}\left(\frac{k_1'}{k_1}\right)^2.$$

Hence, we can state the following final theorem:

Theorem 4.5. *Let $\gamma : I \to M$ be a θ_α–slant curve of osculating order r in an $\mathcal{S}$-space form $(M^{2m+s}, \phi, \xi_\alpha, \eta_\alpha, g)$, where $r \geq 3$, $c \neq s$, $g(\phi T, V_2) = \sqrt{1-a}\cos\beta(t)$ is not 0 or $\pm\sqrt{1-a}$. Then γ is proper f-biharmonic if and only if $f = c_1k_1^{-3/2}$ and*
(i) if β is a constant,

$$\frac{k_2}{k_1} = c_2,$$

$$3(k_1')^2 - 2k_1k_1'' = 4k_1^2\left\{(1+c_2^2)k_1^2 - \left[b^2 + \frac{c+3s+3(c-s)\cos^2\beta}{4}(1-a)\right]\right\}$$

$$k_2k_3 = \pm\frac{3(c-s)\sin 2\beta}{8}(1-a),$$

(ii) if β is a non-constant function,

$$k_2^2 = -\frac{3(c-s)}{4}(1-a)\cos^2\beta + \mu(t).k_1^2,$$

$$3(k_1')^2 - 2k_1k_1'' = 4k_1^2\left\{(1+\mu(t))k_1^2 - \left[b^2 + \frac{c+3s+3(c-s)\cos^2\beta}{4}(1-a)\right]\right\},$$

$$k_2k_3 = \pm\frac{3(c-s)\sin 2\beta\sin w}{8}(1-a),$$

where c_1 and c_2 are positive constants,

$$\phi T = \sqrt{1-a}\left(\cos\beta V_2 \pm \sin\beta\cos wV_3 \pm \sin\beta\sin wV_4\right), \tag{4.32}$$

$w = w(t)$ is the angle function between V_3 and the orthogonal projection of ϕT onto span $\{V_3, V_4\}$. w is related to β by $\cos w = \mp\beta'/k_2$ and $\mu(t)$ is given by

$$\mu(t) = -\frac{3(c-s)}{2}(1-a)\int\frac{\cos^2\beta k_1'}{k_1^3}dt.$$

Proof. The proof follows directly from the preceding calculations. The equation $\cos w = \mp\frac{\beta'}{k_2}$ is derived using equations (4.20) and (4.32). □

In case β is a constant, we can give the following direct corollary of Theorem 4.5:

Corollary 4.1. *Let $\gamma : I \to M$ be a $\theta_\alpha-$slant curve of osculating order $r \geq 3$ in an $\mathcal{S}$-space form $(M^{2m+s}, \phi, \xi_\alpha, \eta_\alpha, g)$, where $c \neq s$, $g(\phi T, V_2) = \sqrt{1-a}\cos\beta$ is a constant and $\beta \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. Then γ is proper f-biharmonic if and only if $f = c_1 k_1^{-3/2}$, $\frac{k_2}{k_1} = c_2$ and $k_1 = k_1(t)$ is of the form*

$$k_1 = \frac{\pm\sqrt{N} + M}{D}$$

where the functions N, M and D are denoted by
(a) equations (4.7) and (4.8), for $b^2 + \left\{\left[c + 3s + 3(c-s)\cos^2\beta\right]/4\right\}(1-a) > 0$;
(b) equations (4.9) and (4.10), for $b^2 + \left\{\left[c + 3s + 3(c-s)\cos^2\beta\right]/4\right\}(1-a) < 0$;
(c) equations (4.11) and (4.12), for $b^2 + \left\{\left[c + 3s + 3(c-s)\cos^2\beta\right]/4\right\}(1-a) = 0$;
with

$$\lambda_4 = \sqrt{\left|b^2 + \frac{c + 3s + 3(c-s)\cos^2\beta}{4}(1-a)\right|}, u = u(t) = 2\lambda_4 t + c_4,$$

for arbitrary constants $c_1 > 0$, $c_2 > 0$, c_3 and c_4;

$$k_2 k_3 = \pm\frac{3(c-s)\sin 2\beta}{8}(1-a)$$

and $\phi T = \sqrt{1-a}\,(\cos\beta V_2 \pm \sin\beta V_4)$.

Proof. In this case, we have the ODE

$$3(k_1')^2 - 2k_1 k_1'' = 4k_1^2[(1+c_2^2)k_1^2 - \left[b^2 + \frac{c + 3s + 3(c-s)\cos^2\beta}{4}(1-a)\right].$$

Using Lemma 4.1 and Theorem 4.5, the proof is clear. □

5. Construction of an Example in $\mathbb{R}^6(-6)$

Firstly, let us recall the structures defined on a special $\mathcal{S}$-manifold. Consider $M = \mathbb{R}^{2m+s}$ with the coordinate functions $\{x_1, ..., x_m, y_1, ..., y_m, z_1, ..., z_s\}$ and the following structures:

$$\xi_\alpha = 2\frac{\partial}{\partial z_\alpha}, \quad \alpha = 1, ..., s,$$

$$\eta^\alpha = \frac{1}{2}\left(dz_\alpha - \sum_{i=1}^{m} y_i dx_i\right), \quad \alpha = 1, ..., s,$$

$$\phi X = \sum_{i=1}^{m} Y_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{m} X_i \frac{\partial}{\partial y_i} + \left(\sum_{i=1}^{m} Y_i y_i\right)\left(\sum_{\alpha=1}^{s} \frac{\partial}{\partial z_\alpha}\right),$$

$$g = \sum_{\alpha=1}^{s} \eta^\alpha \otimes \eta^\alpha + \frac{1}{4}\sum_{i=1}^{m}(dx_i \otimes dx_i + dy_i \otimes dy_i).$$

Here,

$$X = \sum_{i=1}^{m}\left(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}\right) + \sum_{\alpha=1}^{s}\left(Z_\alpha \frac{\partial}{\partial z_\alpha}\right) \in \chi(M).$$

Then, $\left(\mathbb{R}^{2m+s}, \phi, \xi_\alpha, \eta^\alpha, g\right)$ becomes an $\mathcal{S}$-space form with constant ϕ-sectional curvature $-3s$. This manifold is denoted by $\mathbb{R}^{2n+s}(-3s)$ [14]. The following vector fields

$$X_i = 2\frac{\partial}{\partial y_i}, \quad X_{m+i} = \phi X_i = 2\left(\frac{\partial}{\partial x_i} + y_i \sum_{\alpha=1}^{s}\frac{\partial}{\partial z_\alpha}\right), \quad \xi_\alpha = 2\frac{\partial}{\partial z_\alpha}$$

form a g-orthonormal basis of $\chi(M)$, and the Riemannian connection is given by

$$\nabla_{X_i}X_j = \nabla_{X_{m+i}}X_{m+j} = 0, \quad \nabla_{X_i}X_{m+j} = \delta_{ij}\sum_{\alpha=1}^{s}\xi_\alpha, \quad \nabla_{X_{m+i}}X_j = -\delta_{ij}\sum_{\alpha=1}^{s}\xi_\alpha,$$

$$\nabla_{X_i}\xi_\alpha = \nabla_{\xi_\alpha}X_i = -X_{m+i}, \quad \nabla_{X_{m+i}}\xi_\alpha = \nabla_{\xi_\alpha}X_{m+i} = X_i,$$

[14]. Let us choose $m = 2$ and $s = 2$. Now, let $\gamma : I \to \mathbb{R}^6(-6), \gamma = (\gamma_1, ..., \gamma_6)$ be a unit-speed θ_α-slant curve. We can calculate

$$T = \frac{1}{2}\left[\gamma_3' X_1 + \gamma_4' X_2 + \gamma_1' X_3 + \gamma_2' X_4 + \xi_2\right],$$

where we take $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{3}$. Then, $g(T,T) = 1$ gives us

$$(\gamma_1')^2 + (\gamma_2')^2 + (\gamma_3')^2 + (\gamma_4')^2 = 3. \tag{5.1}$$

We also have

$$\phi T = \frac{1}{2}\left[-\gamma_1' X_1 - \gamma_2' X_2 + \gamma_3' X_3 + \gamma_4' X_4\right]. \tag{5.2}$$

After calculations, we obtain

$$\nabla_T T = \frac{1}{2}\left[\begin{array}{c}(\gamma_3'' + \gamma_1')X_1 + (\gamma_4'' + \gamma_2')X_2 \\ +(\gamma_1'' - \gamma_3')X_3 + (\gamma_2'' - \gamma_4')X_4\end{array}\right]. \tag{5.3}$$

Let us select

$$k_1 = \frac{1}{2+t^2}, \tag{5.4}$$

$$\begin{aligned} k_2 &= \frac{1}{2+t^2}, \\ k_3 &= \frac{\sqrt{17}}{4}\left(2+t^2\right), \\ g(\phi T, V_2) &= \frac{\sqrt{3}}{2}\cos\beta = \text{constant}, \\ cos\beta &= \pm\frac{\sqrt{2}}{6}\ (\beta \approx 1.3329 \text{ or } 1.8087), \\ f &= (2+t^2)^{3/2}, \end{aligned} \tag{5.5}$$

From (5.3) and (5.4), we get

$$\left[\begin{array}{c}(\gamma_3'' + \gamma_1')^2 + (\gamma_4'' + \gamma_2')^2 \\ +(\gamma_1'' - \gamma_3')^2 + (\gamma_2'' - \gamma_4')^2\end{array}\right] = \frac{4}{(2+t^2)^2}.$$

We also have $\eta^1(T) = \cos\theta_1 = 0$ and $\eta^2(T) = \cos\theta_2 = 1/2$, which leads to

$$\gamma_5' = \gamma_1'\gamma_3 + \gamma_2'\gamma_4,$$

$$\gamma_6' = 1 + \gamma_1'\gamma_3 + \gamma_2'\gamma_4.$$

Using $\nabla_T T = k_1 V_2$, we get

$$\begin{aligned} g(\phi T, V_2) &= \frac{1}{k_1} g(\phi T, \nabla_T T) \\ &= \frac{2+t^2}{4}\left(-3 + \gamma_1''\gamma_3' - \gamma_1'\gamma_3'' + \gamma_2''\gamma_4' - \gamma_2'\gamma_4''\right). \end{aligned}$$

Notice that for $c_1 = 1$, $c_2 = 1$, $c_3 = 4$, $c_4 = 0$, Corollary 4.1 (c) is satisfied. In fact, since $b = 1/2$, $c = -6$, $s = 2$, $\sin 2\beta = \pm\sqrt{17}/9$ and $a = 1/4$, we have

$$b^2 + \frac{c + 3s + 3(c-s)\cos^2\beta}{4}(1-a) = 0,$$

$$k_1 = \frac{1}{2+t^2} = \frac{4c_3}{c_3^2x^2 + 2c_3^2c_4x + c_3^2c_4^2 + 16c_2^2 + 16},$$
$$k_2 = \frac{1}{2+t^2} = c_2k_1,$$
$$k_2k_3 = \pm\frac{\sqrt{17}}{4} = \pm\frac{3(c-s)\sin 2\beta}{8}(1-a).$$

As a result, under these circumstances, γ becomes a proper f-biharmonic $\theta_\alpha-$slant curve for $f = (2+t^2)^{3/2}$ in $\mathbb{R}^6(-6)$.

6. Conclusions and Future Work

In this study, the properties of f-biharmonic $\theta_\alpha-$slant curves defined on $\mathcal{S}$-manifolds have been investigated. In the future, it will be important to explore broader classes of these structures and conduct studies on applications of $\theta_\alpha-$slant curves and their implications in various systems. Recall that $\theta_\alpha-$slant curves are defined as all the contact angles are constant seperately. As an idea, two possible generalizations can be given as

$$a = \sum_{i=1}^{s} \cos^2\theta_\alpha = constant$$

or

$$b = \sum_{i=1}^{s} \cos\theta_\alpha = constant,$$

where the contact angles do not need to be constant but their sum or squared sum to be constant. Then $\theta_\alpha-$slant curves will be a subclass and the results on these curves will be corollaries of those future studies.

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References

[1] Baikoussis, C., Blair, D. E.: *On Legendre curves in contact 3-manifolds*. Geom. Dedicata. **49**, 135-142 (1994).
[2] Blair, D. E.: *Geometry of manifolds with structural group $U(n) \times O(s)$*. J. Differential Geometry. **4**, 155-167 (1970).
[3] Blair, D. E.: *Riemannian geometry of contact and symplectic manifolds*, Second edition. Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA (2010).
[4] Cabrerizo, J. L., Fernandez, L. M., Fernandez, M.: *The curvature of submanifolds of an $\mathcal{S}$-space form*. Acta Math. Hungar. **62**, 373-383 (1993).
[5] Chen, B.Y.: *A report on submanifolds of finite type*. Soochow J. Math. **22**, 117-337 (1996).
[6] Cho, J. T., Inoguchi, J., Lee, J. E.: *On slant curves in Sasakian 3-manifolds*. Bull. Austral. Math. Soc. **74**, 359-367 (2006).
[7] Eells, Jr. J., Lemaire, L.: *Selected topics in harmonic maps*. Amer. Math. Soc., Providence, R.I. (1983).
[8] Eells, Jr. J., Sampson, J. H.: *Harmonic mappings of Riemannian manifolds*. Amer. J. Math. **86**, 109-160 (1964).
[9] Fetcu, D., Oniciuc, C.: *Explicit formulas for biharmonic submanifolds in Sasakian space forms*. Pacific J. Math. **240**, 85-107 (2009).
[10] Güvenç, Ş., Özgür, C.: *On slant curves in $\mathcal{S}$-manifolds*. Commun. Korean Math. Soc. **33**(1), 293-303 (2018).
[11] Güvenç, Ş., Özgür, C.: *C-parallel and C-proper Slant Curves of $\mathcal{S}$-manifolds*. Filomat **33**(19), 6305-6313 (2019).
[12] Güvenç, Ş.: *A Note on f-biharmonic Legendre Curves in $\mathcal{S}$-space forms*. International Electronic Journal of Geometry. **12**(2), 260-267 (2019).
[13] Güvenç, Ş.: *An Extended Family of Slant Curves in $\mathcal{S}$-manifolds*. Mathematical Sciences and Applications E-Notes. **8**(1), 69-77 (2020).
[14] Hasegawa, I., Okuyama, Y., Abe, T.: *On p-th Sasakian manifolds*. J. Hokkaido Univ. Ed. Sect. II A. **37**(1), 1-16 (1986).

[15] Jiang, G. Y.: 2-*harmonic maps and their first and second variational formulas*. Chinese Ann. Math. Ser. A. **7**, 389-402 (1986).
[16] Kim, J. S., Dwivedi, M. K., Tripathi, M. M.: *Ricci curvature of integral submanifolds of an $\mathcal{S}$-space form*. Bull. Korean Math. Soc. **44**, 395-406 (2007).
[17] Nakagawa, H.: *On framed f-manifolds*. Kodai Math. Sem. Rep. **18**, 293-306 (1966).
[18] Ou, Y. L.: *On f-biharmonic maps and f-biharmonic submanifolds*. Pacific Journal of Mathematics. **271**(2), 461-477.
[19] Özgür, C., Güvenç, Ş.: *On biharmonic Legendre curves in $\mathcal{S}$-space forms*. Turkish J. Math. **38**(3), 454-461 (2014).
[20] Yano, K., Kon, M.: *Structures on Manifolds*. Series in Pure Mathematics. **3**. World Scientific Publishing Co., Singapore (1984).

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