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Research article

Some Singularities and Developable Surfaces in Galilean 3-Space

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Abstract

In this work we investigate singularities for the three types of developable surfaces, introduced by Izumiya and Takeuchi, in Galilean-3 space. Moreover we search the necessary conditions of being a geodesic for principal direction curves of the rectifying developable surface.

Keywords: *Singular points; developable surfaces; Galilean space.*
AMS Subject Classification (2020): *Primary: 53A35 ; Secondary: 53A40, 14H50.*

1. Introduction

The boundary scenario of pseudo-Euclidean spaces, wherein isotropic cones transform into planes, leads to the emergence of Galilean space. Described as the Klein geometry of a product space, this scenario enjoys wide usage in physics and geometry. Galilean geometry is notable for its straightforwardness, allowing students to engage with it comparatively effortlessly and without draining their intellectual reserves. Ultimately, this simplicity encourages broad comprehension, making it suitable for substantial comparisons with Euclidean geometry. Furthermore, notable advancements in Galilean geometry provide students with psychological reassurance regarding the coherence of the studied structures, and it also serves as an exemplary demonstration of the beneficial geometric principle of duality. For these reasons, I strongly support the incorporation of a mathematics curriculum in teacher training colleges that includes a comparative examination of the following three fundamental geometries: first, Euclidean geometry; second, the geometry related to the Galilean principle of relativity; and third, that associated with Einstein's principle of relativity. Moreover, introducing these students to the special theory of relativity would complement such a curriculum and merits serious consideration.

Intriguing qualities of curves and surfaces in Galilean 3-space were recently presented in the literature. These studies have encompassed investigations into curves such as helices and special curves on ruled surfaces [5,14].

A surface that may be developed into flat surfaces without changing the surface's metric is referred to as a developable surface. This is a useful application tool in cartographic projections and the production of flat materials. Numerous studies have been written about developable surfaces, some of which also incorporate the singularity theory. Zhao et al. investigated the geometric characteristics of surfaces with a single parameter and regular curves [15]. Murata and Umehara looked at flat surfaces with singularities' overall behavior in Euclidean three-space [12]. The research on the singularities of developable surfaces in Euclidean 3-space that Izumiya and Takeuchi introduced is what primarily inspired us to write this work. They considered three types of developable surfaces named as rectifying developable of a space curve, defined to be the envelope of the family of rectifying planes along the curve, Darboux developable of a space curve- whose singularities are given by the locus of the end points of modified Darboux vectors of the curve and the tangential Darboux developable of a space curve which is defined by the Darboux developable of the tangent indicatrix of the space curve. They have shown that these developable surfaces are locally diffeomorphic to the cuspidal edge, the swallowtail or the folded umbrella, also called cuspidal cross cap [8,9].

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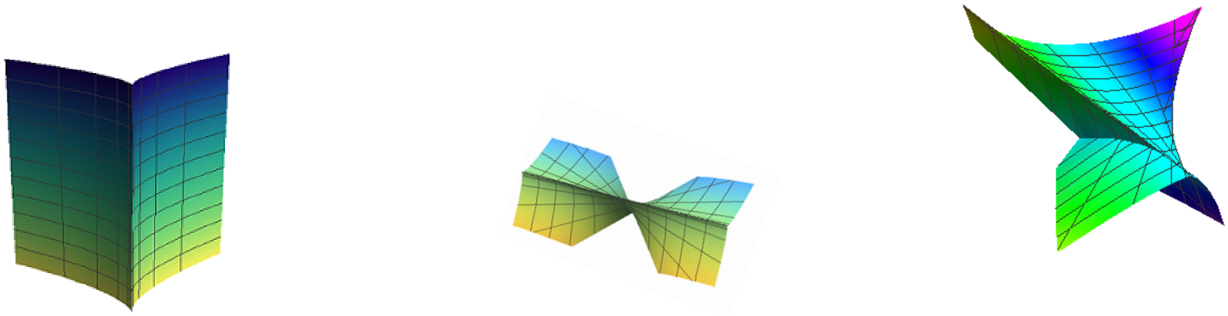
Moreover Ishikawa and Yamashita gave the complete solution to the local diffeomorphism classification problem in Euclidean 3-space and they give the following theorem;

Theorem 1.1. *Let ∇ be a torsion free affine connection on a manifold M. Let $\gamma : I \longrightarrow M$ be a C^∞ curve from an open interval I.Let* $\dim(M) = 3$
1) *If $(\nabla_\gamma)(s_0)$,$(\nabla_\gamma^2)(s_0)$,$(\nabla_\gamma^3)(s_0)$ are linearly independent, then the $\nabla-$tangent surface is locally diffeomorphic to the cuspidal edge at $(s_0, 0)$.*

2) *If $(\nabla_\gamma)(s_0)$,$(\nabla_\gamma^2)(s_0)$,$(\nabla_\gamma^3)(s_0)$ are linearly dependent and $(\nabla_\gamma)(s_0)$,$(\nabla_\gamma^2)(s_0)$,$(\nabla_\gamma^4)(s_0)$ are linearly independent then $\nabla-$tangent surface is locally diffeomorphic to the cuspidal crosscap at $(s_0, 0)$.*

3) *If $(\nabla_\gamma)(s_0) = 0$ and $(\nabla_\gamma^2)(s_0)$,$(\nabla_\gamma^3)(s_0)$,$(\nabla_\gamma^4)(s_0)$ are linearly independent then $\nabla-$tangent surface is locally diffeomorphic to the swallowtail at $(s_0, 0)$ [7].*



(a) Cuspidaledge **(b)** Cuspidal crosscap **(c)** Swallowtail

Figure 1. Types of Singularities

There are articles about the singularities of surfaces in many spaces as well. In Lorentz 3-space, Brander studies singularities of surfaces with constant (non-zero) mean curvature [4]. Fujimori et al.demonstrate that cuspidal edges, swallowtails, and cuspidal singularities are the generic forms of singularities of spacelike maximum surfaces in Lorentz 3-space [6]. Kokubu et al. prove that generically flat fronts in hyperbolic 3-space admit only cuspidal edges and swallowtails [11].

In this work we generalize the developable surfaces and examine the geometric structure of these surfaces in Galilean 3-space. Then singularities of developable surfaces are investigated and characterizations of the singular points are determined by using the method given by Ishikawa and Yamashita [7].

2. Basic Concepts and Notions

For 3-dimensional Galilean space $\mathbb{G}_3$, the Galilean scalar product between two vectors $\xi = (\xi_1, \xi_2, \xi_3)$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ is defined by

$$\langle \xi, \zeta \rangle_{G_3} = \begin{cases} \xi_1\zeta_1, & \text{if } \xi_1 \text{ or } \zeta_1 \text{ is not zero,} \\ \xi_2\zeta_2 + \xi_3\zeta_3, & \text{if } \xi_1 \text{ and } \zeta_1 \text{ are zero} \end{cases}$$

and the Galilean cross product is given as

$$(\xi \times \zeta)_{G_3} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix}, & \text{if } \xi_1 \text{ or } \zeta_1 \text{ is not zero,} \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix}, & \text{if } \xi_1 \text{ and } \zeta_1 \text{ are zero.} \end{cases}$$

where e_1, e_2, e_3 are Euclidean standard basis [1]. Let $\gamma : I \subset \mathbb{R} \to \mathbb{G}_3$, $\gamma(s) = (x(s), y(s), z(s))$ be an arbitrary curve with the Galilean invariant parameter s. If $x(s)$ is considered as the arc length parameter of the curve, we get the curve as $\gamma(s) = (s, y(s), z(s))$. Then the curvature $\kappa(s)$ and torsion $\tau(s)$ of the curve γ are defined by

$$\kappa(s) = \sqrt{(y'')^2(s) + (z'')^2(s)}$$

$$\tau(s) = \frac{\det(\gamma'(s), \gamma''(s), \gamma'''(s))}{(\kappa(s))^2}$$

and associated moving trihedron is given by

$$T(s) = \gamma'(s) = (1, y'(s), z'(s))$$

$$N(s) = \frac{1}{\kappa(s)}\gamma''(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s))$$

$$B(s) = \frac{1}{\kappa(s)}(0, -z''(s), y''(s)).$$

The vectors T, N, B are called the vectors of the tangent, principal normal and binormal line of γ, respectively. For their derivatives the following Frenet formulas are hold:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

[1]

Definition 2.1. Let $\gamma : I \subset \mathbb{R} \to \mathbb{G}^3$ be a unit speed curve with the Frenet frame apparatus $\{T, N, B, \kappa \neq 0, \tau\}$ in Galilean 3−Space. The harmonic curvature $\mathbf{H} : I \subset \mathbb{R} \to \mathbb{R}$ of the curve γ is defined by

$$\mathbf{H(s)} = \frac{\tau(s)}{\kappa(s)}$$

where κ and τ are curvature and torsion of the curve γ, respectively [13].

Theorem 2.1. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{G}^3$ be a unit speed curve with the Frenet frame apparatus $\{T, N, B, \kappa \neq 0, \tau\}$ in Galilean 3−Space. Then the curve γ is named as general helix if its harmonic curvature function is a constant function [13].*

Definition 2.2. Let $\gamma : I \subset \mathbb{R} \to \mathbb{G}^3$ be a unit speed curve with the Frenet frame apparatus $\{T, N, B, \kappa \neq 0, \tau\}$ in Galilean 3−Space. The curve γ is called slant helix which has the property that the principal normal vector of γ makes a constant angle with a fixed line [10].

Theorem 2.2. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{G}^3$ be a unit speed curve with the Frenet frame apparatus $\{T, N, B, \kappa \neq 0, \tau\}$ in Galilean 3−Space. The curve γ is a slant helix if and only if*

$$\sigma(s) = \pm\frac{\kappa(s)^2}{\tau(s)^3}\left(\frac{\tau(s)}{\kappa(s)}\right)'$$

is a constant function [10].

Definition 2.3. Let $\gamma(\mathbf{s}) : I \subset \mathbb{R} \to \mathbb{G}^3$ be a unit speed curve with the Frenet frame apparatus $\{T, N, B, \kappa \neq 0, \tau\}$ in Galilean 3−Space. Then the curve $W(s) = \tau(s)T(s) + \kappa(s)B(s)$ is named as Darboux vector of the curve γ.

Definition 2.4. Let $\gamma(\mathbf{s}) : I \subset \mathbb{R} \to \mathbb{G}^3$ be a unit speed curve with the Frenet frame apparatus $\{T, N, B, \kappa \neq 0, \tau\}$ in Galilean 3−Space. Then the curve $\tilde{W}(s) = \dfrac{\tau(s)}{\kappa(s)}T(s) + B(s)$ is named as modified Darboux vector of the curve γ .

Definition 2.5. Let $\gamma(\mathbf{s}) : I \subset \mathbb{R} \to \mathbb{G}^3$ be a unit speed curve with the Frenet frame apparatus $\{T, N, B, \kappa, \tau \neq 0\}$ in Galilean 3–Space. Then the curve $\bar{W}(s) = T(s) + \dfrac{\kappa(s)}{\tau(s)}B(s)$ is named as unit Darboux vector of the curve γ.

Definition 2.6. Let $F(s,u)$ be a surface in G^3 parameterized by the mapping $F(s,u) = (X(s,u), Y(s,u), Z(s,u))$ and the partial derivatives are $X_s = \frac{\partial X}{\partial s}$, $X_u = \frac{\partial X}{\partial u}$. Then the first fundamental form of the surface is given by

$$I = (g_s ds + g_u du)^2 + (h_{ss}ds^2 + 2h_{su}dsdu + h_{uu}du^2)$$

here

$$g_s = X_s, g_u = X_u, h_{su} = Y_sY_u + Z_sZ_u$$

.

And the second fundamental form is

$$II = L_{ss}ds^2 + 2L_{su}dsdu + L_{uu}du^2$$

where

$$L_{ss} = \frac{1}{g_u}(g_u(0, Y_{ss}, Z_{ss}) - g_{uu}(0, Y_u, Z_u)).N$$

$$L_{su} = \frac{1}{g_u}(g_u(0, Y_{su}, Z_{su}) - g_{su}(0, Y_u, Z_u)).N$$

$$L_{uu} = \frac{1}{g_u}(g_u(0, Y_{uu}, Z_{uu}) - g_{uu}(0, Y_u, Z_u)).N$$

Here the dot "." denotes the scalar product and the normal vector field N is defined by

$$N = \frac{1}{W}(0, -X_sZ_u + X_uZ_s, X_sY_u - X_uY_s)$$

with the function

$$W = \sqrt{(-X_sZ_u + X_uZ_s)^2 + (X_sY_u - X_uY_s)^2}$$

[3].

Definition 2.7. Let $F(s,u)$ be a surface in G^3 parameterized by the mapping $F(s,u) = (X(s,u), Y(s,u), Z(s,u))$. The Gaussian curvature of the surface is defined as

$$K = \frac{L_{ss}L_{uu} - L_{su}^2}{W^2}$$

[3].

3. Developable Surfaces and Singularities in Galilean 3-Space

A developable surface is a ruled surface and a ruled surface in G^3 is locally the map $F_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s)$, where $\gamma : I \longrightarrow G^3, \delta : I \longrightarrow G^3\backslash\{0\}$ are smooth mappings and I is an open interval. γ is called the base curve and δ is called the director curve of the surface. The straight lines $u \longrightarrow \gamma(s) + u\delta(s)$ are called rulings.

Theorem 3.1. *Let γ be an arc-length parametrized, differentiable curve with the Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$ in G^3. We consider three developable surfaces associated to a space curve in Galilean 3-space.*

1) A ruled surface $F_{(\gamma,\tilde{W})}(s,u) = \gamma(s) + u\tilde{W}(s)$ is called the rectifying developable of γ.

2) A ruled surface $F_{(B,T)}(s,u) = B(s) + uT(s)$ is called the Darboux developable of γ .

3) A ruled surface $F_{(\bar{W},N)}(s,u) = \bar{W}(s) + uN(s)$ is called the tangential Darboux developable of γ .

Here $\tilde{W}(s) = \dfrac{\tau}{\kappa}(s)T(s) + B(s)$ is the modified Darboux vector field of γ , under the condition that $\kappa(s) \neq 0$ and $\bar{W}(s)$ is the unit Darboux vector field of γ.

Proof. Now let's see that these surfaces are developable surfaces in Galilean space. For a developable surface, the Gaussian curvature must be zero. If we investigate the Gaussian curvature for these surfaces:

i) $F_{(\gamma,\tilde{W})}(s,u) = \gamma(s) + u\tilde{W}(s)$ is the rectifying developable surface of γ. Here $\gamma(s) = (s, y(s), z(s))$ and

$$F_{(\gamma,\tilde{W})}(s,u) = \gamma(s) + u\tilde{W}(s) = (s + u\tfrac{\tau(s)}{\kappa(s)}, y(s) + u\tfrac{\tau(s)}{\kappa(s)}y'(s) - \tfrac{u}{\kappa(s)}z''(s), z(s) + u\tfrac{\tau(s)}{\kappa(s)}z'(s) + \tfrac{u}{\kappa(s)}y''(s))$$

For this surface

$$g_u=\frac{\tau(s)}{\kappa(s)},\ g_s=1+u(\frac{\tau(s)}{\kappa(s)})',\ g_{su}=(\frac{\tau(s)}{\kappa(s)})',\ g_{uu}=0,\ Y_u=\frac{\tau(s)}{\kappa(s)}y'(s)\text{ and }Z_u=\frac{\tau(s)}{\kappa(s)}z'(s)$$

and if we use the equations given in the Definition 2.6., we have

$$L_{uu}=0\text{ and }L_{su}=0 \qquad .$$

Via Definition 2.7. the Gaussian curvature is zero and $F_{(\gamma,\tilde{W})}(s,u)=\gamma(s)+u\tilde{W}(s)$ is a developable surface.

ii) $F_{(B,T)}(s,u)=B(s)+uT(s)$ is the Darboux developable of γ. Here $\gamma(s)=(s,y(s),z(s))$ and

$$F_{(B,T)}(s,u)=B(s)+uT(s)=(u,uy'(s)-\frac{z''(s)}{\kappa(s)},uz'(s)+\frac{y''(s)}{\kappa(s)})$$

For this surface

$$g_u=1,\ g_s=0,\ g_{su}=0,\ g_{uu}=0,\ Y_u=y'(s)\text{ and }Z_u=z'(s)$$

and if we use the equations given in the Definition 2.6., we have

$$L_{uu}=0\text{ and }L_{su}=0 \qquad .$$

Via Definition 2.7. the Gaussian curvature is zero and $F_{(B,T)}(s,u)=B(s)+uT(s)$ is a developable surface.

iii) $F_{(\bar{W},N)}(s,u)=\bar{W}(s)+uN(s)$ is the tangential Darboux developable of γ. Here $\gamma(s)=(s,y(s),z(s))$ and

$$F_{(\bar{W},N)}(s,u)=(1,y'(s)-\frac{z''(s)}{\tau(s)}+\frac{u}{\kappa(s)}y''(s),z'(s)+\frac{y''(s)}{\tau(s)}+\frac{u}{\kappa(s)}z''(s))$$

Since the parametrization constrains the surface to the x=1 plane, it defines a planar surface. A direct computation of the first and second fundamental forms confirms that the Gaussian curvature vanishes everywhere on the surface, as is characteristic of planar geometries. Thus $F_{(\bar{W},N)}(s,u)=\bar{W}(s)+uN(s)$ is a developable surface. □

Theorem 3.2. *Let γ be an arc-length parametrized, differentiable curve with the Frenet frame apparatus $\{T,N,B,\kappa,\tau\}$ in G^3.*
i) *The Darboux developable of γ is locally diffeomorphic to the cuspidal edge at $F_{(B,T)}(s_0,u_0)$ if and only if $\tau(s_0)\neq 0$, $(\frac{\tau}{\kappa})'(s_0)\neq 0$ and $u_0=\frac{\tau}{\kappa}(s_0)$.*

ii) *The Darboux developable of γ is locally diffeomorphic to the swallowtail at $F_{(B,T)}(s_0,u_0)$ if and only if $\tau(s_0)\neq 0$, $(\frac{\tau}{\kappa})'(s_0)=0$, $(\frac{\tau}{\kappa})''(s_0)\neq 0$ and $u_0=\frac{\tau}{\kappa}(s_0)$.*

iii) *The Darboux developable of γ is locally diffeomorphic to the cuspidal crosscap at $F_{(B,T)}(s_0,u_0)$ if and only if $\tau(s_0)=0$, $(\frac{\tau}{\kappa})'(s_0)=0$, $(\frac{\tau}{\kappa})''(s_0)\neq 0$ and $u_0=0$.*

Proof. Because of other cases are similar we only give the first proof.
Let γ be an arc-length parametrized, differentiable curve with the Frenet frame apparatus $\{T,N,B,\kappa,\tau\}$ in G^3. The Darboux developable of γ is $F_{(B,T)}(s,u)=B(s)+uT(s)$ and the partial derivatives of this surface are

$$F_s=(u\kappa-\tau)N$$

and

$$F_u=T$$

and the Galilean cross product of these derivatives is

$$F_s\times F_u=(0,0,-u\kappa+\tau)$$

This indicates that $F_{(B,T)}(s,u)=B(s)+uT(s)$ is a ∇-tangent surface and the singular point of the Darboux developable is $u_0=\frac{\tau}{\kappa}(s_0)$. The cuspidal edge singularities are obtained along points where $\{\gamma',\gamma'',\gamma'''\}$ are linearly independent. So if we do the necessary computations with the value of $u_0=\frac{\tau}{\kappa}(s_0)$, we get

$$\gamma'=(\tfrac{\tau}{\kappa})'T$$

$$\gamma''=(\tfrac{\tau}{\kappa})''T+(\tfrac{\tau}{\kappa})'\kappa N$$

$\gamma''' = (\frac{\tau}{\kappa})'''T + 2(\frac{\tau}{\kappa})''\kappa N + (\frac{\tau}{\kappa})'(\kappa' N + \kappa\tau B).$

A straightforward computation gives us $\det(\gamma', \gamma'', \gamma''') = (\frac{\tau}{\kappa})'^3\kappa^2\tau$. In order to ensure linear independence, this determinant must be nonzero. Thus there is a diffeomorphism between the Darboux developable of γ and the cuspidal edge when $\tau(s_0) \neq 0$, $(\frac{\tau}{\kappa})'(s_0) \neq 0$ and $u_0 = \frac{\tau}{\kappa}(s_0)$. □

Theorem 3.3. *Let γ be an arc-length parametrized, differentiable curve with the Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$ in G^3.*

i) *The tangential Darboux developable of γ doesn't have cuspidal edge singularities in G^3.*

ii) *The tangential Darboux developable of γ doesn't have swallowtail singularities in G^3.*

iii) *The tangential Darboux developable of γ doesn't have cuspidal crosscap singularities in G^3.*

Proof. Because of other cases are similar we only give the first proof.

Let γ be an arc-length parametrized, differentiable curve with the Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$ in G^3. The tangential Darboux developable of γ is $F_{(\bar{W},N)}(s,u) = \bar{W}(s) + uN(s)$ and the partial derivatives of this surface are

$$F_s = (u\tau + (\tfrac{\kappa}{\tau})')B$$

and

$$F_u = N$$

and the Galilean cross product of these derivatives is

$$F_s \times F_u = (-u\tau - (\tfrac{\kappa}{\tau})', 0, 0)$$

Thus the singular point of the tangential Darboux developable is $u_0 = -\dfrac{(\frac{\kappa}{\tau})'}{\tau}(s_0) = \sigma(s_0)$. The cuspidal edge singularities are seen along points where $\{\gamma', \gamma'', \gamma'''\}$ are linearly independent. So if we do the necessary computations with the value of $u_0 = \sigma(s_0)$, we have

$\gamma = \bar{W}(s_0) + \sigma(s_0)N(s_0)$

$\gamma' = \sigma' N + ((\frac{\kappa}{\tau})' + \sigma\tau)B$

$\gamma'' = (\sigma'' - (\frac{\kappa}{\tau})'\tau - \sigma\tau^2)N + (2\sigma'\tau + \sigma\tau' + (\frac{\kappa}{\tau})'')B$

$\gamma''' = (\sigma''' - 2(\frac{\kappa}{\tau})''\tau - (\frac{\kappa}{\tau})'\tau' - 3\sigma'\tau^2 - 3\sigma\tau\tau' + 3\sigma''\tau + 3\sigma'\tau' + \sigma\tau'' - \sigma\tau^3 - \frac{\kappa}{\tau}'\tau^2 + (\frac{\kappa}{\tau})''')B$

One can easily see that $\det(\gamma', \gamma'', \gamma''') = 0$. Thus $\{\gamma', \gamma'', \gamma'''\}$ are linearly dependent and there are not any cuspidal edge singularities. □

Theorem 3.4. *Let γ be an arc-length parametrized, differentiable curve with the Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$ in G^3.*
i) *There is a diffeomorphism between rectifying developable of γ and the cuspidal edge at $F_{(\gamma,\tilde{W})}(s_0, u_0)$ if and only if $(\frac{\tau}{\kappa})'(s_0) \neq 0$, $(\frac{\tau}{\kappa})''(s_0) \neq 0$ and $u_0 = -\dfrac{1}{(\frac{\tau}{\kappa})'(s_0)}$.*
ii) *The rectifying developable of γ is diffeomorphic to the swallowtail at $F_{(\gamma,\tilde{W})}(s_0, u_0)$ if and only if $(\frac{\tau}{\kappa})'(s_0) \neq 0$, $(\frac{\tau}{\kappa})''(s_0) = 0$ and $u_0 = -\dfrac{1}{(\frac{\tau}{\kappa})'(s_0)}$.*

Proof. Because of the other case is similar we only give the second proof.

Let γ be an arc-length parametrized, differentiable curve with the Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$ in G^3. The rectifying developable of γ is

$$F_{(\gamma,\tilde{W})}(s,u) = \gamma(s) + u\tilde{W}(s)$$

Partial derivatives of this surface are

$$F_s = (1 + u(\tfrac{\tau}{\kappa})')T$$

and

$$F_u = (\tfrac{\tau}{\kappa})T + B$$

and the Galilean cross product of these derivatives is

$$F_s \times F_u = (0, -1 - u(\tfrac{\tau}{\kappa})', 0)$$

Thus $F_{(\gamma,\tilde{W})}$ is a ∇-tangent surface and the singular point of the rectifying developable is $u_0 = -\dfrac{1}{(\frac{\tau}{\kappa})'(s_0)}$.

The swallowtail singularities are obtained along points where $\{\gamma'', \gamma''', \gamma'^{v}\}$ are linearly independent and $\gamma' = 0$. So if we do the necessary computations with the value of $u_0 = -\dfrac{1}{(\frac{\tau}{\kappa})'(s_0)}$ we have

$\gamma' = u_0' \frac{\tau}{\kappa} T + u_0' B$

$\gamma'' = (u_0'' \frac{\tau}{\kappa} + u_0'(\frac{\tau}{\kappa})')T + u_0'' B$

$\gamma''' = (u_0''' \frac{\tau}{\kappa} + 2u_0''(\frac{\tau}{\kappa})' + u_0'(\frac{\tau}{\kappa})'')T + u_0'(\frac{\tau}{\kappa})' + \kappa N + u_0''' B$

For the swallowtail singularity γ' must be zero. Thus $u_0' = \frac{(\frac{\tau}{\kappa})''}{(\frac{\tau}{\kappa})'^2}(s_0) = 0$ and $(\frac{\tau}{\kappa})''(s_0) = 0$. With a brief calculation we have $\det(\gamma'', \gamma''', \gamma'^{v}) = 6u_0''^3(\frac{\tau}{\kappa})'^2\kappa$. In order to ensure linear independence we have $(\frac{\tau}{\kappa})'(s_0) \neq 0$ and this completes the proof.

□

Theorem 3.5. *Assume that γ be an arc-length parametrized, differentiable curve with the Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$ in G^3. Then the following cases are equivalent.*
i) *γ is a conical geodesic curve.*
ii) *The rectifying developable of γ is a conical surface.*

Proof. The singular locus of the rectifying developable of γ is defined by $\beta(s) = \gamma(s) + u_0\tilde{W}$. From Theorem 3.4. $u_0 = -\dfrac{1}{(\frac{\tau}{\kappa})(s_0)}$. Thus $\beta(s) = \gamma - \dfrac{1}{(\frac{\tau}{\kappa})}\tilde{W}$.

Let $F_{(\gamma,\tilde{W})}(s,u) = \gamma(s) + u\tilde{W}(s)$ be a conical surface. Via the Frenet frame formulas in G^3 we have $\beta'(s) = \frac{(\frac{\tau}{\kappa})''}{(\frac{\tau}{\kappa})'^2}$. Therefore $\beta' = 0$ if and only if $(\frac{\tau}{\kappa})'' = 0$. This completes the proof. □

Example 3.1. Let $\gamma(s) = (s, -\dfrac{3}{2}(\dfrac{\cos 6s}{36} + \dfrac{\cos 2s}{4}), -\dfrac{3}{2}(\dfrac{\sin 6s}{36} + \dfrac{\sin 2s}{4}) \in \mathbb{G}^3$ be an arc-length parametrized curve in $\mathbb{G}^3$. Then the developable surfaces of γ and corresponding singular points are:

i) The Darboux developable surface of γ, $F_{(B,T)} = B + uT$ and its singularities (highlighted in red) can be seen in Figure 2.

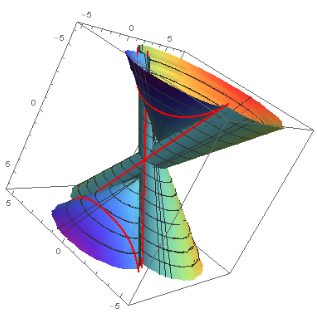


Figure 2. Darboux developable surface of γ

Here $\tau = 4$ and $\left(\frac{\tau}{\kappa}\right)'(s_0) = \frac{8}{3}\sec 2s_0 \tan 2s_0 \neq 0$ for all s_0 and $F_{(B,T)}$ is locally diffeomorphic to the cuspidal edge at the points $u_0 = \frac{4}{\cos 2s_0}$.

ii) The Tangential Darboux developable surface of γ is $F_{(\overline{W},N)} = \overline{W} + uN$ can be seen in Figure 3. This surface doesn't have any cuspidal edge, swallowtail or cuspidal crosscap singularities at $u_0 = \sigma(s_0) = \frac{3}{8}\sin 2s_0$.

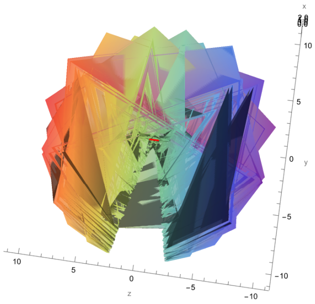


Figure 3. Tangential Darboux developable surface of γ

iii) The Rectifying developable surface of γ is $F_{(\gamma,\tilde{W})} = \gamma + u\tilde{W}$ determined with the modified Darboux vector $\tilde{W} = (\frac{4}{3\cos 2s}(1, \frac{3}{2}(\frac{\sin 6s}{6} + \frac{\sin 2s}{2}), -\frac{3}{2}(\frac{\cos 6s}{6} + \frac{\cos 2s}{2})) + \frac{1}{3\cos 2s}(0, -\frac{3}{2}(\sin 6s + \sin 2s), \frac{3}{2}(\cos 6s + \cos 2s)$ and its singularities (highlighted in red) can be seen in Figure 4.

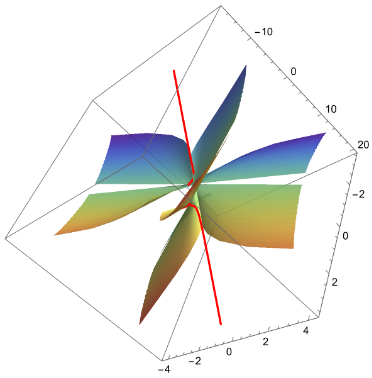


Figure 4. Rectifying developable surface of γ

Here $\left(\frac{\tau}{\kappa}\right)'(s_0) \neq 0, (\frac{\tau}{\kappa})''(s_0) = 0$ and $u_0 = \frac{3}{8\sec 2s_0 \tan 2s_0}$.

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Author's contributions

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